Mixed-Jet Rank Floors for Point-Span Higher Osculating Absorption

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Abstract

Let X be a smooth projective integral d-fold over an algebraically closed field, let H be very ample, and use the complete embedding defined by H^m. Fix $1 \le s \le m$. A nonempty finite reduced set $Z \subset X$ is point-span s-osculating-absorbing if the span S_Z of its value evaluations contains the affine osculating space of order s at every support. Write

$$m + 1 = q(s + 1) + r, \qquad 0 \le r \le s,$$

and put

$$B_{d,s}(m) = q\binom{d+s}{d} + \begin{cases} 0, & r = 0, \\ \binom{d+r-1}{d}, & r > 0. \end{cases}$$

We prove in arbitrary characteristic that

$$\dim S_Z,\ |Z| \ge B_{d,s}(m).$$

The proof is scheme-theoretic. Products of point separators isolate arbitrary mixed jets of total weight at most $m+1$, and discrete convexity selects full order-s blocks plus one residual block. The resulting heterogeneous rank certificate also applies to arbitrary jet-ample polarizations. For $s = 1$ the formula recovers the earlier growing first-jet bound. Equality holds in every characteristic on rational normal curves, for every s, and on $\mathbf{P}^d$ in the top-order regime $s = m$. No implication from tangent absorption to the higher-order hypothesis is used or claimed. No classification or exhaustive priority claim is made.

1 Statement and scope

Let $\mathbf{k}$ be algebraically closed, let X be smooth, projective, integral, and of dimension $d \ge 1$, and let H be very ample. Write

$$\varphi_m : X \longrightarrow \mathbf{P}(H^0(X, H^m)^*)$$

for the complete embedding. For a nonempty finite reduced set $Z \subset X$, set

$$S_Z = \operatorname{Im}\Big(H^0(Z, H^m|_Z)^* \longrightarrow H^0(X, H^m)^*\Big).$$

Equivalently, S_Z is the span of the evaluation lines at the supports; this formulation makes no choice of fiber trivializations. For $p \in X$ and $a \ge 0$, put

$$(a+1)p = \operatorname{Spec}(\mathcal{O}_{X,p}/\mathfrak{m}_p^{a+1}), \qquad \lambda_d(a) = \binom{d+a}{d},$$

and set $\lambda_d(-1) = 0$.

The order-a principal-parts evaluation is

$$j_p^a : H^0(X, H^m) \longrightarrow H^0\Big((a+1)p, H^m|_{(a+1)p}\Big).$$

We define the affine order-a osculating space intrinsically by

$$\widehat{\mathrm{Osc}}_p^a(H^m) = \mathrm{Im}((j_p^a)^*) \subset H^0(X, H^m)^*.$$

This convention uses no ordinary derivatives or factorials.

Definition 1.1. Fix $1 \le s \le m$. The set Z is *point-span s-osculating-absorbing* for H^m if

$$\widehat{\mathrm{Osc}}_p^s(H^m) \subseteq S_Z \qquad (p \in Z).$$

Write uniquely

$$m + 1 = q(s + 1) + r, \qquad q \ge 1, \quad 0 \le r \le s,$$

and define

$$B_{d,s}(m) = q\lambda_d(s) + \lambda_d(r - 1). \tag{1}$$

Theorem 1.2 (Universal mixed-jet floor)**.** *Over an algebraically closed field of arbitrary characteristic, every nonempty finite reduced point-span s-osculating-absorbing set for the complete H^m-embedding satisfies*

$$\dim S_Z \ge B_{d,s}(m), \qquad |Z| \ge B_{d,s}(m).$$

Corollary 1.3 (First-order specialization)**.** *For $s = 1$,*

$$B_{d,1}(m) = \left\lfloor \frac{m+1}{2} \right\rfloor (d + 1) + \begin{cases} 1, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases}$$

Thus the formula recovers the growing first-jet bound of [5].

For fixed d, s,

$$B_{d,s}(m) = \frac{1}{s+1}\binom{d+s}{d} m + O_{d,s}(1).$$

For $d \ge 2$ the leading coefficient strictly increases with s, because

$$\frac{\binom{d+s+1}{d}/(s+2)}{\binom{d+s}{d}/(s+1)} = \frac{d+s+1}{s+2} > 1.$$

The hierarchy is therefore asymptotically stronger under the correspondingly stronger osculating hypothesis.

2 The annihilator criterion

Let $(s + 1)Z = \coprod_{p \in Z}(s + 1)p$. Finite-scheme duality gives

$$S_Z^\perp = H^0(X, \mathcal{I}_Z \otimes H^m)$$

and

$$\left(\sum_{p \in Z} \widehat{\mathrm{Osc}}_p^s(H^m)\right)^\perp = H^0(X, \mathcal{I}_{(s+1)Z} \otimes H^m).$$

Lemma 2.1 (Higher-order kernel criterion)**.** *The set Z is point-span s-osculating-absorbing if and only if*

$$H^0(X, \mathcal{I}_Z \otimes H^m) = H^0(X, \mathcal{I}_{(s+1)Z} \otimes H^m).$$

Proof. The inclusion from right to left always holds because $\mathcal{I}_{(s+1)Z} \subseteq \mathcal{I}_Z$. Absorption is equivalent to

$$\sum_{p \in Z} \widehat{\mathrm{Osc}}_p^s(H^m) \subseteq S_Z.$$

Taking annihilators reverses this inclusion and supplies the converse inclusion between the displayed spaces of sections. $\square$

The principal-parts filtration gives $\widehat{\mathrm{Osc}}_p^a(H^m) \subseteq \widehat{\mathrm{Osc}}_p^s(H^m)$ for $a \leq s$. Thus order-s absorption contains every lower-order jet block. The converse is not asserted.

3 Mixed-jet interpolation

Lemma 3.1 (Full jets from powers)**.** *If $p \in X$ and $0 \leq a \leq k$, then*

$$H^0(X, H^k) \longrightarrow H^0\Big((a+1)p, H^k|_{(a+1)p}\Big)$$

is surjective. Hence $\dim \widehat{\mathrm{Osc}}_p^a(H^k) = \lambda_d(a)$.

Proof. Choose $u_0 \in H^0(X, H)$ nonzero at p. Very ampleness and smoothness give $u_1, \ldots, u_d$, vanishing at p, such that $x_i = u_i/u_0$ form a regular system of parameters. For every multi-index α with $|\alpha| \leq a$,

$$u_0^{k-|\alpha|} u_1^{\alpha_1} \cdots u_d^{\alpha_d}$$

has local representative x^α after trivializing by u_0^k. These monomials form a basis modulo $\mathfrak{m}_p^{a+1}$. No differentiation is used. $\square$

Lemma 3.2 (Mixed-jet interpolation)**.** *Let $t \geq 1$, let $p_1, \ldots, p_t$ be distinct points, let $a_i \geq 0$, and put*

$$K = \sum_{i=1}^{t} (a_i + 1) - 1.$$

Then restriction is surjective:

$$H^0(X, H^K) \longrightarrow \bigoplus_{i=1}^{t} H^0\Big((a_i+1)p_i, H^K|_{(a_i+1)p_i}\Big).$$

The same holds for H^k in every degree $k \geq K$.

Proof. For each $i \neq j$, choose $\ell_{ij} \in H^0(X, H)$ with $\ell_{ij}(p_j) = 0$ and $\ell_{ij}(p_i) \neq 0$, and set

$$P_i = \prod_{j \neq i} \ell_{ij}^{a_j+1}.$$

It is a unit at p_i and vanishes to the prescribed order at every other support. Lemma 3.1 realizes arbitrary order-a_i data using H^{a_i}. Multiplication by the unit P_i is an automorphism of the truncated local algebra, so degree-K sections realize arbitrary data at p_i and zero data elsewhere. Sum over i.

For $k > K$, multiply by a section of H^{k-K} nonzero at every support. It exists because $\mathbf{k}$ is infinite and a finite union of proper linear subspaces cannot exhaust the section space. $\square$

Over the complex numbers this is also the line-bundle specialization of [3, Proposition 2.3]. The direct proof records the exact specialization and is characteristic-free. The interpolation statement itself is not claimed as new; the point here is its rank consequence under the absorption constraint and the ensuing optimization.

4 Convex packing and the rank proof

The interpolation lemma first gives a heterogeneous certificate, before optimization.

Proposition 4.1 (Mixed-block rank certificate)**.** *Under the hypotheses of Theorem 1.2, choose $t \geq 1$ distinct supports $p_1, \dots, p_t \in Z$ and orders $0 \leq a_i \leq s$. If*

$$\sum_{i=1}^{t} (a_i + 1) \leq m + 1,$$

then

$$\dim S_Z \geq \sum_{i=1}^{t} \lambda_d(a_i).$$

Proof. Put $K = \sum_i (a_i + 1) - 1 \leq m$. Lemma 3.2, followed by degree raising when $K < m$, says that the selected infinitesimal neighborhoods impose independent conditions on H^m. Dually, their osculating blocks form a direct sum of dimension $\sum_i \lambda_d(a_i)$. Each block lies in S_Z because $a_i \leq s$ and Z is order-s absorbing. □

Lemma 4.2 (Discrete convex packing)**.** *If $1 \leq w_i \leq s + 1$ and $\sum_i w_i \leq m + 1$, then*

$$\sum_i \binom{d + w_i - 1}{d} \leq B_{d,s}(m).$$

Equality is attained by q weights $s + 1$ and, if $r > 0$, one weight r.

Proof. Put $f(0) = 0$ and $f(w) = \binom{d+w-1}{d}$ for $w \geq 1$. Its forward differences

$$f(w + 1) - f(w) = \binom{d + w - 1}{d - 1}$$

are nondecreasing for $w \geq 1$, while $f(1) - f(0) = 1$. Moving one unit from a smaller block to a larger block below the cap $s + 1$ does not decrease the total; a weight-one block disappears when it loses its unit. Repetition packs the budget into full blocks and at most one residual block. If the original total weight is smaller than $m + 1$, first enlarge a nonfull block or append a weight-one block; either operation strictly increases the objective. Thus an optimum uses the whole budget, and the packed pattern gives (1). □

Proof of Theorem 1.2. Put $n = |Z|$. If $r = 0$ and $q = 1$, nonemptiness gives $n \geq 1 = q$. Now suppose $r = 0$, $q \geq 2$, and $n \leq q - 1$. Then $n \geq 1$, and the union of the full order-s neighborhoods at all supports is independent in degree

$$(s + 1)n - 1 \leq (s + 1)(q - 1) - 1 < m$$

by Lemma 3.2, and remains so after raising degree to m. Its dual has dimension $n\lambda_d(s)$ and lies in S_Z by absorption. This contradicts $\dim S_Z \leq n$, since $\lambda_d(s) > 1$. Hence $n \geq q$.

If $r > 0$ and $n \leq q$, the same contradiction applies because

$$(s + 1)n - 1 \leq q(s + 1) - 1 = m - r < m.$$

Thus $n \geq q + 1$.

Choose q supports with full order-s blocks and, if $r > 0$, one further support with an order-$(r - 1)$ block. Their mixed interpolation degree is

$$q(s + 1) + r - 1 = m.$$

Their independent dual spaces lie in S_Z and have total dimension $B_{d,s}(m)$. Therefore $\dim S_Z \geq B_{d,s}(m)$, and $|Z| \geq \dim S_Z$ because S_Z is generated by $|Z|$ value lines. □

The certificate and packing argument do not intrinsically require a tensor power. The following form records the more general input precisely.

Corollary 4.3 (Jet-ample polarization)**.** *Let L be an M-jet ample line bundle on X, meaning that for every set of distinct points p_i and positive integers b_i with $\sum_i b_i = M+1$, the restriction map*

$$H^0(X,L) \longrightarrow \bigoplus_i H^0(b_i p_i, L|_{b_i p_i})$$

is surjective. Fix $1 \le s \le M$, write $M+1 = Q(s+1)+R$ with $0 \le R \le s$, and define S_Z and the osculating spaces using L in place of H^m. If a nonempty finite reduced set Z is point-span s-osculating-absorbing, then

$$\dim S_Z,\ |Z| \ge Q\binom{d+s}{d} + \binom{d+R-1}{d},$$

where the last summand is interpreted as zero when $R=0$. Over the complex numbers, repeated application of [3, Proposition 2.3] gives this conclusion with $M = \ell a$ for $L = A^\ell$ whenever A is a-jet ample and $1 \le s \le \ell a$. Here $a \ge 1$ and $\ell \ge 1$ are integers.

Proof. The stated jet-ampleness also gives surjectivity when $\sum_i b_i < M+1$: add one point outside the selected finite set with the missing positive weight, and then project away its jet target. Such a point exists because $d \ge 1$ and the algebraically closed field is infinite. Thus every selected collection with $b_i = a_i + 1 \le s+1$ and total weight at most $M+1$ is independent. The support-threshold proof above applies verbatim with M in place of m, including the separate case $Q=1, R=0$. Selecting Q full blocks and, when $R>0$, one residual block gives the displayed rank. The last assertion is the tensor-product theorem cited above, iterated $\ell-1$ times. ☐

Proof of Corollary 1.3. Divide $m+1$ by two. An odd m has no residual block; an even m has one residual simple value. Substitution in (1) gives the formula. ☐

5 Sharpness on the rational normal curve

Theorem 5.1 (Exact curve minimum)**.** *Let $X = \mathbf{P}^1$, $H = \mathcal{O}_{\mathbf{P}^1}(1)$, and $1 \le s \le m$. The minimum size and minimum span dimension of a point-span s-osculating-absorbing set for the complete mth Veronese embedding are both $m+1$.*

Proof. Since $\lambda_1(a) = a+1$, Theorem 1.2 gives $B_{1,s}(m) = q(s+1)+r = m+1$.

Conversely, choose any $m+1$ distinct points $Z \subset \mathbf{P}^1$. A nonzero binary form of degree m cannot vanish at all of them, so evaluation

$$H^0(\mathbf{P}^1, \mathcal{O}(m)) \longrightarrow H^0(Z, \mathcal{O}_Z(m))$$

is an isomorphism. Hence $S_Z = H^0(\mathbf{P}^1, \mathcal{O}(m))^*$, which contains every osculating space, and $\dim S_Z = |Z| = m+1$. ☐

There is a second exact regime in every dimension.

Theorem 5.2 (Exact top-order minimum)**.** *Let $X = \mathbf{P}^d$, $H = \mathcal{O}_{\mathbf{P}^d}(1)$, and $s = m \ge 1$. The minimum size and minimum span dimension of a point-span m-osculating-absorbing set are both $\binom{d+m}{d}$, over every algebraically closed field.*

Proof. Theorem 1.2 gives the lower bound because $q = 1$, $r = 0$, and $B_{d,m}(m) = \lambda_d(m) = \binom{d+m}{d}$. Conversely, the evaluation lines of all closed points of $\mathbf{P}^d$ span $H^0(\mathbf{P}^d, \mathcal{O}(m))^*$: their annihilator would otherwise contain a nonzero homogeneous form vanishing identically on $\mathbf{P}^d$. Select a basis of evaluation lines and let Z be its supports. Then $|Z| = \dim S_Z = \binom{d+m}{d}$ and S_Z is the whole dual space, so it contains every order-m osculating space. □

The exact replay uses rational Vandermonde matrices for $(m, s) = (3, 1), (5, 2), (8, 3), (10, 5)$. It also exhaustively checks 250 finite convex-packing cases and two simplex-lattice fixtures in the top-order regime. These computations certify only the displayed finite fixtures and integer identities, not the universal theorem or priority.

6 Context and limitations

Jet ampleness encodes simultaneous interpolation on fat points. A tensor-product generation result over the complex numbers appears in [3]. Higher osculating spaces and fundamental forms are formulated through principal-parts sheaves over algebraically closed fields in [7]; that is the intrinsic language used here.

Osculating spaces, higher Gauss maps, and secant varieties have a substantial literature [1, 2, 4]. The Veronese literature in particular relates osculating secant defectivity to Hilbert functions of fat points. These works provide context; this selective comparison does not establish priority for the value-span absorption floor (1).

The authorial note B215 contains the order-two mixed interpolation pattern [6]. It is an antecedent, not independent validation. The present argument removes its unrelated conditional framework, states the arbitrary-order absorption hypothesis, works in arbitrary characteristic, optimizes the closed formula, and proves curve sharpness.

The limitations are material.

- The higher-order absorption hypothesis is explicit. No implication from first-order tangent absorption to second- or higher-order absorption is asserted or used.
- The restriction $s \leq m$ ensures the full local rank in Lemma 3.1. No formula for $s > m$ is asserted.
- Completeness of the H^m system is used because all separator products must be available. Arbitrary subsystems are outside the theorem.
- Equality is settled for the numerical minimum in dimension one and for $s = m$ on projective space. No equality classification in the remaining higher-dimensional regimes is claimed; in particular, no nontrivial sharpness example is supplied here for $d \geq 2$ and $s < m$.
- The literature search is selective, the replay is not a proof checker, and no independent human peer review has yet been performed.

7 Conclusion

Once the higher-order hypothesis is stated explicitly, point-span osculating absorption has a uniform characteristic-free rank consequence. Very ampleness isolates mixed fat-point blocks of total weight at most $m + 1$, and convex packing gives the explicit floor $B_{d,s}(m)$. The same certificate yields a bound for arbitrary jet-ample polarizations. The first-order result is recovered exactly, while the full hierarchy is sharp on rational normal curves and the top-order endpoint is sharp on projective space in every dimension. Equality away from these regimes and the geometry of special higher Gauss fibers remain open.

References

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