Exact Floors and Proper-Span Extremizers for Higher Osculating Absorption

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Abstract

Let X be a smooth projective integral d-fold over an algebraically closed field, let H be very ample, and use the complete embedding defined by H^m. Fix $1 \le s \le m$. A nonempty finite reduced set $Z \subset X$ is point-span s-osculating-absorbing if the span S_Z of its value evaluations contains the affine osculating space of order s at every support. We prove the exact characteristic-free floor

$$\dim S_Z,\ |Z| \ge \binom{d+m}{d}.$$

The estimate follows from the exact tangent-absorption theorem, but its sharpness under the stronger higher-order hypothesis is not formal. For every (d, m, s) and every algebraically closed field we construct a smooth integral hypersurface and a proper-span equality set with $\dim S_Z = |Z| = \binom{d+m}{d}$. All supports lie in one hyperplane whose normal coordinate vanishes to order $s+1$ on the hypersurface. The proof uses two explicit incidence estimates and is valid in positive characteristic.

We also prove a rank-sensitive refinement. If $r_1(Z)$ is the evaluation rank in H and $m \ge 2s+1$, then

$$\dim S_Z,\ |Z| \ge \max\left\{\binom{d+m}{d}, \binom{d+s}{d} r_1(Z)\right\}.$$

The degree threshold is necessary for a uniform statement, as rational normal curves show. The earlier mixed-jet certificate is retained for jet-ample polarizations, but its numerical floor is explicitly identified as nonoptimal for tensor powers after the exact tangent theorem. No equality classification, minimal hypersurface degree, or exhaustive priority claim is made.

1 Statement and scope

Let $\mathbf{k}$ be algebraically closed, let X be smooth, projective, integral, and of dimension $d \ge 1$, and let H be very ample. Write

$$\varphi_m : X \longrightarrow \mathbf{P}(H^0(X, H^m)^*)$$

for the complete embedding. For a nonempty finite reduced set $Z \subset X$, set

$$S_Z = \operatorname{Im}\Big(H^0(Z, H^m|_Z)^* \longrightarrow H^0(X, H^m)^*\Big).$$

Equivalently, S_Z is the span of the evaluation lines at the supports; this formulation makes no choice of fiber trivializations. For $p \in X$ and $a \ge 0$, put

$$(a+1)p = \operatorname{Spec}(\mathcal{O}_{X,p}/\mathfrak{m}_p^{a+1}), \qquad \lambda_d(a) = \binom{d+a}{d},$$

and set $\lambda_d(-1) = 0$.

The order-a principal-parts evaluation is

$$j_p^a : H^0(X, H^m) \longrightarrow H^0\Big((a+1)p, H^m|_{(a+1)p}\Big) .$$

We define the affine order-a osculating space intrinsically by

$$\widehat{\mathrm{Osc}}_p^a(H^m) = \mathrm{Im}((j_p^a)^*) \subset H^0(X, H^m)^*.$$

This convention uses no ordinary derivatives or factorials.

Definition 1.1. Fix $1 \leq s \leq m$. The set Z is *point-span s-osculating-absorbing* for H^m if

$$\widehat{\mathrm{Osc}}_p^s(H^m) \subseteq S_Z \qquad (p \in Z).$$

Put

$$N_{d,m} := \binom{d+m}{d}.$$

Theorem 1.2 (Exact higher-osculating floor)**.** *Over an algebraically closed field of arbitrary characteristic, every nonempty finite reduced point-span s-osculating-absorbing set for the complete H^m-embedding satisfies*

$$\dim S_Z \geq N_{d,m}, \qquad |Z| \geq N_{d,m}.$$

Theorem 1.3 (Rank-sensitive higher blocks)**.** *Let*

$$r_1(Z) = \mathrm{rk}\big(H^0(X, H) \longrightarrow H^0(Z, H|_Z)\big).$$

If $m \geq 2s+1$, then

$$\dim S_Z,\ |Z| \geq \max\left\{N_{d,m}, \lambda_d(s) r_1(Z)\right\}.$$

The threshold $m \geq 2s+1$ cannot be weakened in a uniform theorem with the same second term.

Theorem 1.4 (Characteristic-free proper-span equality)**.** *For every algebraically closed field, every $d, m \geq 1$, and every $1 \leq s \leq m$, there exist a smooth integral hypersurface $X^d \subset \mathbf{P}^{d+1}$, a nonempty reduced set $Z \subset X$, and $H = \mathcal{O}_X(1)$ such that*

$$|Z| = \dim S_Z = N_{d,m} < h^0(X, H^m) = \binom{d+m+1}{d+1},$$

and Z is point-span s-osculating-absorbing for H^m. All supports may lie in one hyperplane whose defining coordinate belongs to $\mathfrak{m}_{X,p}^{s+1}$ for every $p \in Z$.

Theorem 1.2 supersedes the smaller mixed floor published in the first version of this record. That earlier inequality remains correct, as does its heterogeneous interpolation certificate, but higher-order absorption contains tangent absorption and the latter now has the exact binomial floor [6]. The contributions of this revision are the higher-block refinement and proper-span sharpness in every order and characteristic.

2 The annihilator criterion

Let $(s+1)Z = \coprod_{p\in Z}(s+1)p$. Finite-scheme duality gives

$$S_Z^{\perp} = H^0(X, \mathcal{I}_Z \otimes H^m)$$

and

$$\left(\sum_{p\in Z} \widehat{\operatorname{Osc}}_p^s(H^m)\right)^{\perp} = H^0(X, \mathcal{I}_{(s+1)Z} \otimes H^m).$$

Lemma 2.1 (Higher-order kernel criterion)**.** *The set Z is point-span s-osculating-absorbing if and only if*

$$H^0(X, \mathcal{I}_Z \otimes H^m) = H^0(X, \mathcal{I}_{(s+1)Z} \otimes H^m).$$

Proof. The inclusion from right to left always holds because $\mathcal{I}_{(s+1)Z} \subseteq \mathcal{I}_Z$. Absorption is equivalent to

$$\sum_{p\in Z} \widehat{\operatorname{Osc}}_p^s(H^m) \subseteq S_Z.$$

Taking annihilators reverses this inclusion and supplies the converse inclusion between the displayed spaces of sections. □

The principal-parts filtration gives $\widehat{\operatorname{Osc}}_p^a(H^m) \subseteq \widehat{\operatorname{Osc}}_p^s(H^m)$ for $a \le s$. Thus order-s absorption contains every lower-order jet block. The converse is not asserted.

Proof of Theorem 1.2. Because $s \ge 1$, every affine tangent space $\widehat{\operatorname{Osc}}_p^1(H^m)$ is contained in $\widehat{\operatorname{Osc}}_p^s(H^m)$ and hence in S_Z. Thus Z is point-span tangent-absorbing. The exact tangent-absorption theorem [6, Theorem 1.1] gives

$$\dim S_Z,\ |Z| \ge \binom{d+m}{d} = N_{d,m}.$$

No characteristic-zero derivative or factorial is introduced by this reduction. □

Lemma 2.2 (Downward propagation)**.** *If Z is point-span s-osculating-absorbing for H^m, then it is point-span s-osculating-absorbing for H^k for every $s \le k \le m$.*

Proof. Fix $a \in H^0(X, \mathcal{I}_Z \otimes H^k)$ and $p \in Z$. Choose $b_p \in H^0(X, H^{m-k})$ with $b_p(p) \ne 0$; take $b_p = 1$ if $k = m$. Then ab_p vanishes on Z, so Lemma 2.1 in degree m makes it vanish on $(s+1)p$. The germ of b_p is a unit, hence the germ of a belongs to $\mathfrak{m}_p^{s+1}$. This holds at every support, so the two kernels in degree k agree and Lemma 2.1 applies. □

3 Mixed-jet interpolation

Lemma 3.1 (Full jets from powers)**.** *If $p \in X$ and $0 \le a \le k$, then*

$$H^0(X, H^k) \longrightarrow H^0\Big((a+1)p, H^k|_{(a+1)p}\Big)$$

is surjective. Hence $\dim \widehat{\operatorname{Osc}}_p^a(H^k) = \lambda_d(a)$.

Proof. Choose $u_0 \in H^0(X, H)$ nonzero at p. Very ampleness and smoothness give $u_1, \dots, u_d$, vanishing at p, such that $x_i = u_i/u_0$ form a regular system of parameters. For every multi-index α with $|\alpha| \le a$,

$$u_0^{k-|\alpha|} u_1^{\alpha_1} \cdots u_d^{\alpha_d}$$

has local representative x^α after trivializing by u_0^k. These monomials form a basis modulo $\mathfrak{m}_p^{a+1}$. No differentiation is used. □

Lemma 3.2 (Higher-block extension). *Assume $m \geq 2s+1$. Suppose the schemes $(s+1)p_1, \dots, (s+1)p_t$ impose independent conditions on H^m. If the H-evaluation at $x \in X$ is not in the span of the evaluations at $p_1, \dots, p_t$, then adjoining $(s+1)x$ preserves independence.*

Proof. Choose $e \in H^0(X, H)$ vanishing at every p_i and nonzero at x. Then e^{s+1} vanishes on all old $(s+1)$-neighbourhoods and is a unit on $(s+1)x$. Put $k = m - s - 1$. The numerical hypothesis gives $k \geq s$, so Lemma 3.1 says that H^k realizes arbitrary order-s data at x. Multiplication by the local unit e^{s+1} preserves that jet target and kills all old targets. Correcting residual data at x proves surjectivity on the enlarged disjoint union. □

Proof of Theorem 1.3. Choose $r_1(Z)$ supports whose degree-one evaluation lines form a basis, ordered so that each new line escapes the preceding span. A single $(s+1)$-neighbourhood imposes $\lambda_d(s)$ independent conditions on H^m by Lemma 3.1. Repeated use of Lemma 3.2 makes the $r_1(Z)$ selected neighbourhoods independent. Their dual order-s osculating blocks form a direct sum of dimension $\lambda_d(s)r_1(Z)$ inside S_Z. Combine this with Theorem 1.2 and $|Z| \geq \dim S_Z$.

To see that the degree threshold is necessary for this uniform second term, take $X = \mathbf{P}^1$, $H = \mathcal{O}_{\mathbf{P}^1}(1)$, and $m+1$ distinct points. Their H^m-evaluation span is the whole $(m+1)$-dimensional ambient vector space, so they absorb every order $s \leq m$, while $r_1(Z) = 2$. If $m \leq 2s$, then $m + 1 < 2(s+1) = \lambda_1(s)r_1(Z)$. □

Lemma 3.3 (Mixed-jet interpolation). *Let $t \geq 1$, let $p_1, \dots, p_t$ be distinct points, let $a_i \geq 0$, and put*

$$K = \sum_{i=1}^{t}(a_i + 1) - 1.$$

Then restriction is surjective:

$$H^0(X, H^K) \longrightarrow \bigoplus_{i=1}^{t} H^0\Big((a_i+1)p_i, H^K|_{(a_i+1)p_i}\Big).$$

The same holds for H^k in every degree $k \geq K$.

Proof. For each $i \neq j$, choose $\ell_{ij} \in H^0(X, H)$ with $\ell_{ij}(p_j) = 0$ and $\ell_{ij}(p_i) \neq 0$, and set

$$P_i = \prod_{j \neq i} \ell_{ij}^{a_j+1}.$$

It is a unit at p_i and vanishes to the prescribed order at every other support. Lemma 3.1 realizes arbitrary order-a_i data using H^{a_i}. Multiplication by the unit P_i is an automorphism of the truncated local algebra, so degree-K sections realize arbitrary data at p_i and zero data elsewhere. Sum over i.

For $k > K$, multiply by a section of H^{k-K} nonzero at every support. It exists because $\mathbf{k}$ is infinite and a finite union of proper linear subspaces cannot exhaust the section space. □

Over the complex numbers this is also the line-bundle specialization of [4, Proposition 2.3]. The direct proof records the exact specialization and is characteristic-free. The interpolation statement itself is not claimed as new; the point here is its rank consequence under the absorption constraint and the ensuing optimization.

4 The retained mixed-jet certificate

Write uniquely

$$m + 1 = q(s + 1) + r, \qquad q \geq 1, \quad 0 \leq r \leq s,$$

and put

$$B_{d,s}(m) = q\lambda_d(s) + \lambda_d(r - 1). \tag{1}$$

Theorem 4.1 (Mixed-jet floor from version one)**.** *Under the hypotheses of Theorem 1.2,*

$$\dim S_Z, \ |Z| \geq B_{d,s}(m).$$

For $d \geq 2$ and $s < m$, this valid inequality is strictly weaker than the exact floor $N_{d,m}$ of Theorem 1.2; it is retained because the heterogeneous certificate below also applies to abstract jet-ample polarizations.

The interpolation lemma first gives a heterogeneous certificate, before optimization.

Proposition 4.2 (Mixed-block rank certificate)**.** *Under the hypotheses of Theorem 1.2, choose $t \geq 1$ distinct supports $p_1, \ldots, p_t \in Z$ and orders $0 \leq a_i \leq s$. If*

$$\sum_{i=1}^{t} (a_i + 1) \leq m + 1,$$

then

$$\dim S_Z \geq \sum_{i=1}^{t} \lambda_d(a_i).$$

Proof. Put $K = \sum_i (a_i + 1) - 1 \leq m$. Lemma 3.3, followed by degree raising when $K < m$, says that the selected infinitesimal neighborhoods impose independent conditions on H^m. Dually, their osculating blocks form a direct sum of dimension $\sum_i \lambda_d(a_i)$. Each block lies in S_Z because $a_i \leq s$ and Z is order-s absorbing. □

Lemma 4.3 (Discrete convex packing)**.** *If $1 \leq w_i \leq s + 1$ and $\sum_i w_i \leq m + 1$, then*

$$\sum_i \binom{d + w_i - 1}{d} \leq B_{d,s}(m).$$

Equality is attained by q weights $s + 1$ and, if $r > 0$, one weight r.

Proof. Put $f(0) = 0$ and $f(w) = \binom{d+w-1}{d}$ for $w \geq 1$. Its forward differences

$$f(w + 1) - f(w) = \binom{d + w - 1}{d - 1}$$

are nondecreasing for $w \geq 1$, while $f(1) - f(0) = 1$. Moving one unit from a smaller block to a larger block below the cap $s + 1$ does not decrease the total; a weight-one block disappears when it loses its unit. Repetition packs the budget into full blocks and at most one residual block. If the original total weight is smaller than $m + 1$, first enlarge a nonfull block or append a weight-one block; either operation strictly increases the objective. Thus an optimum uses the whole budget, and the packed pattern gives (1). □

Proof of Theorem 4.1. Put $n = |Z|$. If $r = 0$ and $q = 1$, nonemptiness gives $n \geq 1 = q$. Now suppose $r = 0$, $q \geq 2$, and $n \leq q - 1$. Then $n \geq 1$, and the union of the full order-s neighborhoods at all supports is independent in degree

$$(s+1)n - 1 \leq (s+1)(q-1) - 1 < m$$

by Lemma 3.3, and remains so after raising degree to m. Its dual has dimension $n\lambda_d(s)$ and lies in S_Z by absorption. This contradicts $\dim S_Z \leq n$, since $\lambda_d(s) > 1$. Hence $n \geq q$.

If $r > 0$ and $n \leq q$, the same contradiction applies because

$$(s+1)n - 1 \leq q(s+1) - 1 = m - r < m.$$

Thus $n \geq q + 1$.

Choose q supports with full order-s blocks and, if $r > 0$, one further support with an order-$(r-1)$ block. Their mixed interpolation degree is

$$q(s+1) + r - 1 = m.$$

Their independent dual spaces lie in S_Z and have total dimension $B_{d,s}(m)$. Therefore $\dim S_Z \geq B_{d,s}(m)$, and $|Z| \geq \dim S_Z$ because S_Z is generated by $|Z|$ value lines. □

The certificate and packing argument do not intrinsically require a tensor power. The following form records the more general input precisely.

Corollary 4.4 (Jet-ample polarization)**.** *Let L be an M-jet ample line bundle on X, meaning that for every set of distinct points p_i and positive integers b_i with $\sum_i b_i = M+1$, the restriction map*

$$H^0(X, L) \longrightarrow \bigoplus_i H^0(b_i p_i, L|_{b_i p_i})$$

is surjective. Fix $1 \leq s \leq M$, write $M + 1 = Q(s+1) + R$ with $0 \leq R \leq s$, and define S_Z and the osculating spaces using L in place of H^m. If a nonempty finite reduced set Z is point-span s-osculating-absorbing, then

$$\dim S_Z,\ |Z| \geq Q\binom{d+s}{d} + \binom{d+R-1}{d},$$

where the last summand is interpreted as zero when $R = 0$. Over the complex numbers, repeated application of [4, Proposition 2.3] gives this conclusion with $M = \ell a$ for $L = A^\ell$ whenever A is a-jet ample and $1 \leq s \leq \ell a$. Here $a \geq 1$ and $\ell \geq 1$ are integers.

Proof. The stated jet-ampleness also gives surjectivity when $\sum_i b_i < M + 1$: add one point outside the selected finite set with the missing positive weight, and then project away its jet target. Such a point exists because $d \geq 1$ and the algebraically closed field is infinite. Thus every selected collection with $b_i = a_i + 1 \leq s + 1$ and total weight at most $M + 1$ is independent. The support-threshold proof above applies verbatim with M in place of m, including the separate case $Q = 1, R = 0$. Selecting Q full blocks and, when $R > 0$, one residual block gives the displayed rank. The last assertion is the tensor-product theorem cited above, iterated $\ell - 1$ times. □

5 Characteristic-free unisolvent supports

Lemma 5.1 (Existence of an evaluation basis)**.** *For every algebraically closed field and every $d, m \geq 1$, there is a set of $N_{d,m}$ distinct points $Z \subset \mathbf{P}^d$ such that evaluation is an isomorphism*

$$H^0(\mathbf{P}^d, \mathcal{O}(m)) \xrightarrow{\sim} H^0(Z, \mathcal{O}_Z(m)).$$

Proof. The evaluation lines of all $\mathbf{k}$-points span $H^0(\mathbf{P}^d, \mathcal{O}(m))^*$. Otherwise their annihilator would contain a nonzero homogeneous degree-m form vanishing at every $\mathbf{k}$-point of $\mathbf{P}^d$, which is impossible over the infinite field $\mathbf{k}$. Select a basis from these evaluation lines. Very ampleness separates points, so their $N_{d,m}$ supports are distinct; dualizing gives the displayed isomorphism. □

This existence proof replaces the integer simplex lattice used in version one. That lattice remains a useful characteristic-zero fixture, but its points can collide and its falling-factorial basis can degenerate when the characteristic is at most m.

6 Smooth higher-contact extremizers

Let $W = V(y) \cong \mathbf{P}^d$ be a hyperplane in $\mathbf{P}^{d+1}$ and let $Z \subset W$ be an unisolvent set from Lemma 5.1. Write $(s+1)Z_W$ for the disjoint union of the subschemes defined by $\mathfrak{m}_{W,p}^{s+1}$; equivalently,

$$\mathcal{I}_{(s+1)Z_W} = \bigcap_{p \in Z} \mathfrak{m}_{W,p}^{s+1}.$$

This is a fat-point or symbolic-power ideal sheaf, not an assertion about the ordinary power of the homogeneous ideal of Z.

Proposition 6.1 (Smooth prescribed higher contact)**.** *Put $N = N_{d,m}$ and $E = (s+1)N$. For every $n \geq E+1$, there is a smooth integral degree-n hypersurface $X \subset \mathbf{P}^{d+1}$ containing Z such that*

$$y \in \mathfrak{m}_{X,p}^{s+1} \qquad (p \in Z).$$

In particular W is the tangent hyperplane at every support and has contact through order s there.

Proof. Set

$$V = H^0(W, \mathcal{I}_{(s+1)Z_W}(n)).$$

We first choose $f \in V$ whose zero divisor is smooth on $W \setminus Z$. Fix $q \in W \setminus Z$. For every $p \in Z$, choose a hyperplane $\ell_{p,q}$ through p and not through q, and put

$$A_q = \prod_{p \in Z} \ell_{p,q}^{s+1}.$$

Then A_q has degree E, belongs to $H^0(W, \mathcal{I}_{(s+1)Z_W}(E))$, and is a unit on the first neighbourhood $2q$. Since $n - E \geq 1$, $\mathcal{O}_W(n-E)$ generates all first jets at q. Multiplication by $A_q|_{2q}$ therefore proves that

$$V \longrightarrow H^0(2q, \mathcal{O}_{2q}(n))$$

is surjective.

Consider the incidence

$$\Sigma_1 = \{(q, [f]) \in (W \setminus Z) \times \mathbf{P}(V) : j_q^1 f = 0\}.$$

For fixed q, vanishing of the first jet has codimension $d+1$; hence

$$\dim \Sigma_1 \leq d + \dim \mathbf{P}(V) - (d+1) = \dim \mathbf{P}(V) - 1.$$

The closure of its image cannot fill $\mathbf{P}(V)$. Choose f outside that closure. Then $V(f)$ is smooth at every one of its points outside Z. This is an incidence calculation with local jets, so no Euler identity, ordinary derivative convention, or characteristic restriction is involved.

Fix such an f and let

$$U = H^0(\mathbf{P}^{d+1}, \mathcal{O}(n-1)), \qquad F_G = f + yG \quad (G \in U).$$

On the open set $y \neq 0$, multiplication by y is an isomorphism on first neighbourhoods. Since $n - 1 \geq 1$, the affine map

$$U \longrightarrow H^0(2q, \mathcal{O}_{2q}(n)), \qquad G \longmapsto j^1_q(f + yG)$$

is surjective for every such q. The condition that F_G be singular at a fixed point of the $(d+1)$-dimensional ambient open set has codimension $d+2$. Thus

$$\Sigma_2 = \{(q, G) : y(q) \neq 0,\ j^1_q F_G = 0\}$$

has dimension at most $(d+1) + \dim U - (d+2) = \dim U - 1$. The closure of its image is therefore a proper closed subset of U. The additional conditions $G(p) \neq 0$, $p \in Z$, are the complements of finitely many hyperplanes in the irreducible affine space U. Choose G outside the closure of the incidence image and outside all those hyperplanes.

Let $X = V(F_G)$. It is smooth on $y \neq 0$ by the second incidence. If $q \in W \setminus Z$ lies on X, then $f(q) = 0$ and the tangential first jet of f is nonzero, so X is smooth at q. At $p \in Z$, all first jets of f vanish and the normal first jet of F_G is $G(p) \neq 0$, so X is smooth there as well.

The hypersurface is connected. Indeed, from

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{d+1}}(-n) \longrightarrow \mathcal{O}_{\mathbf{P}^{d+1}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and $H^1(\mathbf{P}^{d+1}, \mathcal{O}(-n)) = 0$ for $d + 1 \geq 2$, one gets $H^0(X, \mathcal{O}_X) = \mathbf{k}$. A smooth scheme has disjoint irreducible components; connectedness therefore makes X integral.

Finally, regard f as its lift independent of y. Since that lift belongs to $\mathfrak{m}^{s+1}_{\mathbf{P}^{d+1},p}$, its image belongs to $\mathfrak{m}^{s+1}_{X,p}$. In $\mathcal{O}_{X,p}$ the germ of G is a unit and

$$y = -f/G \in \mathfrak{m}^{s+1}_{X,p},$$

which proves the required contact. $\square$

Proof of Theorem 1.4. Choose $Z \subset W$ by Lemma 5.1, take $n \geq (s+1)N_{d,m} + 1$, and choose X by Proposition 6.1. In particular $n > m$. The hypersurface sequence twisted by m, together with

$$H^0(\mathbf{P}^{d+1}, \mathcal{O}(m-n)) = 0, \qquad H^1(\mathbf{P}^{d+1}, \mathcal{O}(m-n)) = 0,$$

gives an isomorphism

$$H^0(\mathbf{P}^{d+1}, \mathcal{O}(m)) \xrightarrow{\sim} H^0(X, \mathcal{O}_X(m)). \tag{2}$$

Let a section on the right vanish on Z and denote its unique ambient lift by Q. Its restriction $Q|_W$ is a degree-m form vanishing at the unisolvent set Z, hence $Q|_W = 0$. Thus $Q = yR$ for a degree-$(m-1)$ form R. Proposition 6.1 gives

$$Q = yR \in \mathfrak{m}^{s+1}_{X,p} \qquad (p \in Z).$$

Therefore

$$H^0(X, \mathcal{I}_Z \otimes \mathcal{O}_X(m)) = H^0(X, \mathcal{I}_{(s+1)Z} \otimes \mathcal{O}_X(m)),$$

and Lemma 2.1 proves order-s absorption.

Unisolvence gives $\dim S_Z = |Z| = N_{d,m}$. Equation (2) gives

$$h^0(X, \mathcal{O}_X(m)) = \binom{d+m+1}{d+1} > N_{d,m},$$

so the point span is proper. $\square$

Corollary 6.2 (Exact universal minimum)**.** *For every algebraically closed field and every $d, m \geq 1$, $1 \leq s \leq m$, the minimum possible cardinality and span dimension of a point-span s-osculating-absorbing set, as (X, H, Z) vary under the standing hypotheses, are both $N_{d,m}$. The minimum remains the same after requiring the point span to be proper.*

Proof. The lower bound is Theorem 1.2; the proper-span equality examples are Theorem 1.4. □

The degree $n \geq (s+1)N_{d,m} + 1$ is a transparent sufficient threshold, not a minimality claim. The proof certifies an open set of choices rather than one distinguished equation.

7 Exact computational witnesses

The accompanying replay uses exact rational and finite-field linear algebra. It checks the original mixed-block identities, the new rank-sensitive formula and its $\mathbf{P}^1$ falsification boundary, unisolvent evaluation sets in several small characteristics, and local fixtures in which the normal coordinate lies in $\mathfrak{m}^{s+1}$. Fresh JSON output is compared byte for byte with the committed result. These are finite diagnostic witnesses: they do not prove the incidence dimension counts, smoothness of a general member, the universal rank theorem, or priority.

8 Context and limitations

Jet ampleness encodes simultaneous interpolation on fat points. A tensor-product generation result over the complex numbers appears in [4]. Higher osculating spaces and fundamental forms are formulated through principal-parts sheaves over algebraically closed fields in [8]; that is the intrinsic language used here.

Ballico's X-rank of a linear subspace is the direct established language for the minimum number of points of X whose span contains a prescribed linear space [3]. That work computes ranks of tangent-containing subspaces for rational normal curves and several Veronese configurations. The present condition is simultaneous and self-referential: one reduced set must absorb the order-s osculating space at every one of its own supports. We do not claim to introduce subspace rank or point-span absorption as a general concept.

Osculating spaces, higher Gauss maps, and secant varieties have a substantial literature [1, 2, 5]. The Veronese literature relates osculating secant defectivity to Hilbert functions of fat points. Those questions differ from proper-span equality for simultaneous absorption, but they delimit the novelty claim. This selective comparison is not a priority certification.

The authorial note B215 contains the order-two mixed interpolation pattern [7]; it is an antecedent, not independent validation. Version one of this ARR record extracted that certificate and optimized its packing formula. The later exact tangent theorem [6] made that numerical floor nonoptimal for tensor powers. This major revision records the corrected hierarchy rather than concealing the chronology.

The limitations are material.

- The higher-order absorption hypothesis is explicit. Tangent absorption alone is not claimed to imply second- or higher-order absorption.
- The restriction $s \leq m$ ensures the full local rank in Lemma 3.1. No formula for $s > m$ is asserted.
- Completeness of the H^m system is used because all separator products must be available. Arbitrary subsystems are outside the theorem.

- The higher-block term requires $m \geq 2s + 1$. The curve counterexample shows this threshold is necessary for that uniform formula, but no optimal replacement is claimed below the threshold.
- The extremizers prove existence at the sufficient hypersurface degree $n \geq (s+1)N_{d,m} + 1$. They neither minimize n, classify equality sets, nor provide one canonical equation.
- The exact floor is invoked from a separate, earlier ARR record. The proper-span construction and higher-block proof are given here in full.
- The literature search is selective. The replay is not a proof checker, and no independent human peer review or exhaustive priority review is claimed.

9 Conclusion

Higher-osculating absorption has the same exact universal binomial floor as tangent absorption, but the stronger condition is not geometrically empty. Arbitrary-characteristic hypersurfaces with prescribed order-$(s + 1)$ normal contact realize the floor with a proper point span in every dimension, degree, and allowed osculating order. Away from that high-contact branch, the rank-sensitive block argument forces $\lambda_d(s)r_1(Z)$ independent conditions once $m \geq 2s + 1$. The mixed-jet certificate remains useful for abstract jet-ample polarizations, while its older tensor-power floor is now explicitly subordinate to the exact theorem. Minimal construction degree, equality classification, and stronger bounds below the block threshold remain open.

References

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