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Exact Frustration–Resistance Geometry of Magnetic Graph Laplacians: Cycle Green's Function Identities, Hodge Curvature Envelopes, Magnetic Effective-Resistance Metrics, and Topological Odd-Cycle Spectral Gaps

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Abstract

We establish four non-asymptotic spectral and geometric frameworks for the Hermitian magnetic Laplacian L_θ and normalized magnetic Laplacian $\mathcal{L}_\theta = D^{-1/2} L_\theta D^{-1/2}$ on an arbitrary finite connected weighted graph $G = (V, E, w)$ equipped with a magnetic potential $\theta \in \mathbb{R}^{|E|}$. First, via a magnetic Dirichlet principle and Rayleigh subgraph monotonicity, we derive the exact diagonal Green's function identity $(L_\theta(C)^{-1})_{vv} = R_{\mathrm{cyc}}(C)/(4\sin^2(\Phi(C)/2))$ on every arbitrarily weighted frustrated cycle C with total resistance $R_{\mathrm{cyc}}(C) = \sum_{e\in C} w_e^{-1}$ and holonomy $\Phi(C) \not\equiv 0 \pmod{2\pi}$, together with the exact additive series-resistance extension $(L_\theta(C \cup P)^{-1})_{vv} = R_{\mathrm{cyc}}(C)/(4\sin^2(\Phi(C)/2)) + R(P)$ on every weighted lollipop subgraph $C \cup P$. Minimizing over frustrated cycles yields a local frustration resistance $R_{\mathrm{frust}}(v)$ and universal lower bounds $\lambda_1(L_\theta) \geq (\sum_v R_{\mathrm{frust}}(v))^{-1}$ and $\lambda_1(\mathcal{L}_\theta) \geq (\sum_v d_v R_{\mathrm{frust}}(v))^{-1}$ paired with spanning-tree holonomy upper bounds; on every weighted cycle, the lower bound matches the exact small-flux ground state with asymptotic constant 1. Second, we prove that the second-order flux curvature at $\theta = 0$ is given exactly by the weighted Hodge cycle-space projection $\frac{1}{2}\ddot{\lambda}_1(0) = \frac{1}{n}\|P_{\mathrm{cyc},W}\theta\|_W^2$ (vanishing if and only if $\theta \in \mathrm{ran}(B_0)$ is an exact real 1-coboundary), yielding the second-order-tight global flux ceiling $\lambda_1(L_\theta) \leq \frac{4}{n}\sum_e w_e \sin^2(\tilde{\theta}_e/2) \leq \frac{1}{n}\|P_{\mathrm{cyc},W}\theta\|_W^2$. Third, for any frustrated magnetic graph with Green's function $G_\theta = L_\theta^{-1}$, we prove that $\rho_\theta(u,v) = \sqrt{(G_\theta)_{uu} + (G_\theta)_{vv} - 2|(G_\theta)_{uv}|}$ is a gauge-invariant metric on V (realized as the $\mathbb{T}$-orbit quotient distance on $\mathbb{C}^n$), establish gauge-minimized Weyl and Hoffman–Wielandt flux-perturbation envelopes, and prove exact Sherman–Morrison determinant and trace identities for rank-1 magnetic edge insertion $L_\theta + w b_\phi b_\phi^*$ ($b_\phi = e_u - e^{i\phi}e_v$) showing that whenever $(G_\theta)_{uv} \neq 0$, the aligned phase $\phi_{\mathrm{al}} = \arg((G_\theta)_{vu})$ recovers $b_\phi^* G_\theta b_\phi = \rho_\theta(u,v)^2$ whereas the anti-aligned phase $\phi_* = \arg((G_\theta)_{vu}) + \pi$ uniquely maximizes $\det(L_\theta + w b_\phi b_\phi^*)$. Fourth, we give an elementary shifted-walk proof of strict discrete Kato diamagnetic domination and the Fredholm determinant inequality $\det(L_\theta + mI) > \det(L_0 + mI)$ ($m > 0$) on every frustrated graph, and combine π-flux mirror symmetry $\mathcal{L}_{\theta+\pi} = 2I_n - \mathcal{L}_\theta$ with our lollipop identity to obtain a purely topological odd-cycle lower bound on the non-bipartiteness spectral gap $2 - \lambda_{\max}(\mathcal{L}_0) \geq (\sum_v d_v R_{\mathrm{odd}}(v))^{-1}$. All four theorems are numerically verified to machine precision across 291 weighted graph benchmarks.

Keywords: magnetic graph Laplacian, effective resistance, Hodge decomposition, frustration, Kato diamagnetic inequality, non-bipartiteness spectral gap.
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1 Introduction and Preliminaries

Let $G = (V, E, w)$ be a finite, connected, simple undirected graph on $n = |V| \geq 2$ vertices with positive edge weights $w_{uv} = w_{vu} > 0$ for $\{u, v\} \in E$, vertex degrees $d_v = \sum_{u \sim v} w_{vu} > 0$, degree matrix $D = \mathrm{diag}(d_v)$, total volume $\mathrm{vol}(G) = \sum_{v \in V} d_v = 2 \sum_{e \in E} w_e$, and edge resistances $r_e = w_e^{-1}$. Fix an arbitrary reference orientation of each edge $e = (u, v) \in E$, and let $\theta : \vec{E} \to \mathbb{R}$ be a discrete magnetic vector potential (skew-symmetric 1-cochain) satisfying $\theta_{vu} = -\theta_{uv}$.

The *combinatorial magnetic Laplacian* $L_\theta \in \mathbb{C}^{n \times n}$ is defined by

$$(L_\theta)_{uu} = d_u, \qquad (L_\theta)_{uv} = \begin{cases} -w_{uv} e^{i\theta_{uv}}, & \{u, v\} \in E, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

and the *normalized magnetic Laplacian* is $\mathcal{L}_\theta = D^{-1/2} L_\theta D^{-1/2}$. Defining the weighted edge matrix $W = \mathrm{diag}(w_e)_{e \in E}$ and the magnetic incidence matrix $B_\theta \in \mathbb{C}^{|E| \times n}$ with $(B_\theta)_{e,u} = 1$ and $(B_\theta)_{e,v} = -e^{i\theta_{uv}}$ for each oriented edge $e = (u, v)$, one has the exact factorization $L_\theta = B_\theta^* W B_\theta \succeq 0$ and quadratic form

$$x^* L_\theta x = \sum_{\{u,v\} \in E} w_{uv} |x_u - e^{i\theta_{uv}} x_v|^2 = \|B_\theta x\|_W^2, \qquad x \in \mathbb{C}^n. \tag{2}$$

When $\theta \equiv 0$, $L_0 = B_0^\top W B_0$ is the classical combinatorial graph Laplacian with simple zero eigenvalue $\lambda_1(L_0) = 0$ and eigenvector $\mathbf{1}/\sqrt{n}$.

For any vertex gauge potential $\psi : V \to \mathbb{R}$, letting $U_\psi = \mathrm{diag}(e^{i\psi_v})$ gives the unitary gauge equivalence

$$U_\psi^* L_\theta U_\psi = L_{\theta'}, \qquad \theta'_{uv} = \theta_{uv} + \psi_v - \psi_u = \theta_{uv} - (B_0 \psi)_{uv}, \tag{3}$$

which preserves all eigenvalues of L_θ and $\mathcal{L}_\theta$, the diagonal entries of L_θ^{-1}, and the moduli $|(L_\theta^{-1})_{uv}|$. For any oriented simple cycle $C = (v_0, v_1, \ldots, v_{k-1}, v_0)$ ($k \geq 3$), its magnetic holonomy (flux)

$$\Phi(C) := \sum_{j=0}^{k-1} \theta_{v_j v_{j+1}} \pmod{2\pi} \qquad (v_k \equiv v_0) \tag{4}$$

is invariant under (3). A simple cycle C is *frustrated* if $\Phi(C) \not\equiv 0 \pmod{2\pi}$ (equivalently $\sin(\Phi(C)/2) \neq 0$). Since G is connected, L_θ is strictly positive definite ($\lambda_1(L_\theta) > 0$) if and only if (G, w, θ) is *frustrated*, meaning at least one simple cycle is frustrated [7, 6].

Relation to prior work and scope. Magnetic graph Laplacians and their spectral geometry have a rich literature spanning Lieb and Loss's flux-optimization theorems [7], discrete Cheeger and frustration inequalities for $\lambda_1(\mathcal{L}_\theta)$ [1, 6], Kenyon's spanning-forest determinant formula for vector-bundle Laplacians [5], nodal and magnetic perturbation theory [2], classical resistor-network effective resistance [3, 9], and discrete Kato diamagnetic inequalities [8, 4]. While Cheeger-type lower bounds on $\lambda_1(\mathcal{L}_\theta)$ control the global frustration index via isoperimetric cuts [6], they do not yield closed-form exact resistance formulas for individual cycles or vertex-local Green's function bounds, nor do they connect the local frustration resistance directly to the Hodge cycle-space curvature at $\theta = 0$, to a gauge-invariant effective-resistance metric, or to explicit odd-cycle bounds on the non-bipartiteness gap $2 - \lambda_{\max}(\mathcal{L}_0)$.

The present work unifies these four directions through elementary, completely self-contained finite-dimensional proofs:

1. **Theorem 2.2** proves exact closed-form magnetic Green's function identities on arbitrary weighted cycles and lollipops, derives vertex-local Rayleigh monotonicity bounds $R_{\mathrm{frust}}(v)$, and establishes two-sided ground-state sandwiches for both L_θ and $\mathcal{L}_\theta$ that are asymptotically exact with constant 1 on every weighted cycle in the small-flux limit.

2. **Theorem 3.1** identifies the exact second-order flux curvature $\frac{1}{2}\ddot{\lambda}_1(0) = \frac{1}{n}\|P_{\mathrm{cyc},W}\theta\|_W^2$ via weighted Hodge decomposition, establishes the exact persistent-current derivative, and proves a second-order-tight global flux ceiling.
3. **Theorem 4.1** proves that $\rho_\theta(u,v) = \sqrt{G_{uu} + G_{vv} - 2|G_{uv}|}$ is a gauge-invariant $\mathbb{T}$-orbit quotient metric on V, proves gauge-minimized Weyl and Hoffman–Wielandt flux-perturbation envelopes, and derives exact rank-1 magnetic edge-insertion determinant and trace laws establishing the unique anti-aligned phase optimality $\phi_* = \arg(G_{vu}) + \pi$.
4. **Theorem 5.1** gives an elementary shifted-walk proof of strict Kato diamagnetic domination and the Fredholm determinant inequality $\det(L_\theta + mI) > \det(L_0 + mI)$, and applies π-flux mirror symmetry to Theorem 2.2 to obtain a purely topological odd-cycle lower bound on $2 - \lambda_{\max}(\mathcal{L}_0)$.

2 Exact Cycle and Lollipop Magnetic Green's Functions and Frustration Sandwiches

We begin with the variational characterization of diagonal Green's function entries and Rayleigh's magnetic monotonicity principle.

Lemma 2.1 (Magnetic Dirichlet Principle and Subgraph Monotonicity)**.** *Let (G, w, θ) be a connected frustrated magnetic graph so that $L_\theta \succ 0$. Then for every vertex $v \in V$,*

$$\frac{1}{(L_\theta^{-1})_{vv}} = \min_{\substack{x\in\mathbb{C}^n \\ x_v=1}} x^* L_\theta x, \tag{5}$$

with unique minimizer $x = L_\theta^{-1}e_v/(L_\theta^{-1})_{vv}$. Moreover, if $H = (V_H, E_H)$ is any subgraph of G containing v whose edge weights satisfy $w_e^H \le w_e$ for all $e \in E_H$ and whose restricted magnetic Laplacian $L_\theta(H) \in \mathbb{C}^{|V_H|\times|V_H|}$ is positive definite, then

$$(L_\theta(G)^{-1})_{vv} \le (L_\theta(H)^{-1})_{vv}. \tag{6}$$

Proof. Since $L_\theta \succ 0$, for every $x \in \mathbb{C}^n$ with $x_v = e_v^* x = 1$, Cauchy–Schwarz in the L_θ-inner product gives

$$1 = |e_v^* x|^2 = |(L_\theta^{-1/2}e_v)^*(L_\theta^{1/2}x)|^2 \le (e_v^* L_\theta^{-1} e_v)(x^* L_\theta x) = (L_\theta^{-1})_{vv}(x^* L_\theta x),$$

with equality if and only if $L_\theta^{1/2}x$ is a scalar multiple of $L_\theta^{-1/2}e_v$, i.e., $x = L_\theta^{-1}e_v/(L_\theta^{-1})_{vv}$. This proves (5). For (6), every $x \in \mathbb{C}^n$ with $x_v = 1$ satisfies

$$x^* L_\theta(G) x = \sum_{\{a,b\}\in E(G)} w_{ab}|x_a - e^{i\theta_{ab}}x_b|^2 \ge \sum_{\{a,b\}\in E(H)} w_{ab}^H|x_a - e^{i\theta_{ab}}x_b|^2 = x_{V_H}^* L_\theta(H) x_{V_H}.$$

Taking the minimum over $x \in \mathbb{C}^n$ with $x_v = 1$ yields $(L_\theta(G)^{-1})_{vv}^{-1} \ge (L_\theta(H)^{-1})_{vv}^{-1}$, giving (6). □

For any frustrated simple cycle $C \subseteq G$, define its *cycle resistance* and *magnetic cycle resistance* by

$$R_{\mathrm{cyc}}(C) := \sum_{e\in E(C)} \frac{1}{w_e}, \qquad R_{\mathrm{mag}}(C) := \frac{R_{\mathrm{cyc}}(C)}{4\sin^2(\Phi(C)/2)}. \tag{7}$$

For any vertex $v \in V$, define its *local frustration resistance*

$$R_{\mathrm{frust}}(v) := \min_{(C,P)} \big(R_{\mathrm{mag}}(C) + R(P)\big), \tag{8}$$

where the minimum runs over all frustrated simple cycles $C \subseteq G$ and all simple paths P from v to $V(C)$ that intersect $V(C)$ only at their terminal endpoint $v_0 \in V(C)$, with $R(P) := \sum_{e \in E(P)} w_e^{-1}$ (and $R(P) = 0$ when $v \in V(C)$). Since G is connected and frustrated, at least one admissible pair (C, P) exists for every $v \in V$ (obtained by truncating any path from v to C at its first hit of $V(C)$).

Theorem 2.2 (Exact Cycle and Lollipop Green's Functions and Frustration Sandwich)**.** *Let (G, w, θ) be a finite connected weighted magnetic graph.*

(a) (Exact Cycle Magnetic Green's Function.) *If $G = C = (v_0, v_1, \dots, v_{k-1}, v_0)$ is an arbitrarily weighted frustrated simple cycle with flux $\Phi = \Phi(C) \not\equiv 0 \pmod{2\pi}$, then every diagonal entry of $L_\theta(C)^{-1}$ is identical and equals*

$$(L_\theta(C)^{-1})_{vv} = R_{\mathrm{mag}}(C) = \frac{R_{\mathrm{cyc}}(C)}{4\sin^2(\Phi/2)} \qquad \forall v \in V(C), \tag{9}$$

and therefore $\mathrm{tr}(L_\theta(C)^{-1}) = k\, R_{\mathrm{mag}}(C)$.

(b) (Exact Lollipop Series-Resistance Identity.) *If $G = C \cup P$ is a weighted lollipop graph formed by a frustrated simple cycle C and a simple path $P = (u_0, u_1, \dots, u_m)$ intersecting $V(C)$ solely at $u_m \in V(C)$, then at the tail vertex $v = u_0$,*

$$(L_\theta(C \cup P)^{-1})_{vv} = R_{\mathrm{mag}}(C) + R(P) = \frac{R_{\mathrm{cyc}}(C)}{4\sin^2(\Phi(C)/2)} + \sum_{j=0}^{m-1} \frac{1}{w_{u_j u_{j+1}}}. \tag{10}$$

(c) (Universal Frustration Ground-State Sandwich.) *For every connected frustrated weighted magnetic graph (G, w, θ) on n vertices, every $v \in V$ satisfies $(L_\theta^{-1})_{vv} \le R_{\mathrm{frust}}(v)$, and*

$$\left(\sum_{v \in V} R_{\mathrm{frust}}(v)\right)^{-1} \le \lambda_1(L_\theta) \le \min_{T \in \mathcal{T}(G)} \frac{4}{n} \sum_{e \notin E(T)} w_e \sin^2\!\left(\frac{\Phi(C_T(e))}{2}\right), \tag{11}$$

$$\left(\sum_{v \in V} d_v R_{\mathrm{frust}}(v)\right)^{-1} \le \lambda_1(\mathcal{L}_\theta) \le \min_{T \in \mathcal{T}(G)} \frac{4}{\mathrm{vol}(G)} \sum_{e \notin E(T)} w_e \sin^2\!\left(\frac{\Phi(C_T(e))}{2}\right), \tag{12}$$

where $\mathcal{T}(G)$ is the set of spanning trees of G and $C_T(e)$ denotes the fundamental cycle of chord $e \notin E(T)$ in $T \cup \{e\}$.

Proof. Part (a). Fix any vertex $v = v_0 \in V(C)$ and denote $r_j = w_{v_j v_{j+1}}^{-1}$ for $j = 0, \dots, k-1$. Applying a gauge transformation (3) along the tree path $v_0, v_1, \dots, v_{k-1}$ yields an equivalent potential $\tilde{\theta}$ with $\tilde{\theta}_{v_j v_{j+1}} = 0$ for $j = 0, \dots, k-2$ and $\tilde{\theta}_{v_{k-1} v_0} = \Phi$. For any $x \in \mathbb{C}^k$ with $x_0 = 1$, set $\Delta_j := x_j - x_{j+1}$ $(0 \le j \le k-2)$ and $\Delta_{k-1} := x_{k-1} - e^{i\Phi} x_0$. These differences telescope to $\sum_{j=0}^{k-1} \Delta_j = x_0 - e^{i\Phi} x_0 = 1 - e^{i\Phi}$. By Cauchy–Schwarz,

$$4\sin^2(\Phi/2) = |1 - e^{i\Phi}|^2 = \left|\sum_{j=0}^{k-1} r_j^{1/2} (w_j^{1/2} \Delta_j)\right|^2 \le \left(\sum_{j=0}^{k-1} r_j\right) \left(\sum_{j=0}^{k-1} w_j |\Delta_j|^2\right) = R_{\mathrm{cyc}}(C) \cdot x^* L_{\tilde{\theta}}(C) x.$$

Equality is attained uniquely by setting $\Delta_j = (r_j / R_{\mathrm{cyc}}(C))(1 - e^{i\Phi})$, which defines a valid vector $x \in \mathbb{C}^k$ with $x_0 = 1$ via $x_s = 1 - (1 - e^{i\Phi}) R_{\mathrm{cyc}}(C)^{-1} \sum_{j=0}^{s-1} r_j$. Applying Lemma 2.1 and gauge invariance of diagonal Green's entries yields (9).

Part (b). If $m = 0$ (so $v = u_0 = u_m \in V(C)$ and $R(P) = 0$), then $C \cup P = C$ and (10) reduces immediately to Part (a). Assume now that $m \ge 1$ (so $R(P) > 0$). Gauge-transform θ to $\tilde{\theta}$ so that $\tilde{\theta}_e = 0$ on every edge of P and on all edges of C except one closing edge carrying flux

$\Phi(C)$. Since $V(C) \cap V(P) = \{u_m\}$, for any $x \in \mathbb{C}^{|V(C \cup P)|}$ with $x_{u_0} = 1$ and $a := x_{u_m} \in \mathbb{C}$, the quadratic form splits into two edge-disjoint terms coupled only through a:

$$x^* L_{\tilde{\theta}}(C \cup P)x = \sum_{j=0}^{m-1} w_{u_j u_{j+1}} |x_{u_j} - x_{u_{j+1}}|^2 + x^*_{V(C)} L_{\tilde{\theta}}(C) x_{V(C)}.$$

Minimizing the first sum over the interior path vertices $x_{u_1}, \ldots, x_{u_{m-1}}$ with fixed endpoints $x_{u_0} = 1$, $x_{u_m} = a$ by Cauchy–Schwarz gives $|1-a|^2/R(P)$ (exact via linear resistance interpolation). Minimizing the second term over $x_{V(C)}$ with $x_{u_m} = a$ by scaling Part (a) gives $|a|^2/R_{\mathrm{mag}}(C)$. Minimizing over $a \in \mathbb{C}$ by Cauchy–Schwarz again gives

$$\min_{a \in \mathbb{C}} \left(\frac{|1-a|^2}{R(P)} + \frac{|a|^2}{R_{\mathrm{mag}}(C)} \right) = \frac{1}{R_{\mathrm{mag}}(C) + R(P)},$$

attained uniquely at $a = R_{\mathrm{mag}}(C)/(R_{\mathrm{mag}}(C) + R(P)) \in (0,1)$. Lemma 2.1 gives (10).

Part (c). For any $v \in V$ and any admissible pair (C, P), applying Lemma 2.1 to the subgraph $H = C \cup P \subseteq G$ and Part (b) gives $(L_\theta^{-1})_{vv} \le (L_\theta(C \cup P)^{-1})_{vv} = R_{\mathrm{mag}}(C) + R(P)$. Taking the minimum over (C, P) yields $(L_\theta^{-1})_{vv} \le R_{\mathrm{frust}}(v)$. Since $L_\theta^{-1} \succ 0$, its largest eigenvalue obeys

$$\lambda_1(L_\theta)^{-1} = \lambda_{\max}(L_\theta^{-1}) \le \mathrm{tr}(L_\theta^{-1}) = \sum_{v \in V} (L_\theta^{-1})_{vv} \le \sum_{v \in V} R_{\mathrm{frust}}(v),$$

proving the lower bound in (11). Similarly, $\mathcal{L}_\theta^{-1} = D^{1/2} L_\theta^{-1} D^{1/2} \succ 0$ has trace $\mathrm{tr}(\mathcal{L}_\theta^{-1}) = \mathrm{tr}(D L_\theta^{-1}) = \sum_{v \in V} d_v (L_\theta^{-1})_{vv} \le \sum_{v \in V} d_v R_{\mathrm{frust}}(v)$, proving the lower bound in (12).

For the upper bounds, fix any spanning tree $T \in \mathcal{T}(G)$. Choose a vertex gauge $\psi : V \to \mathbb{R}$ (unique up to an additive constant) so that $\tilde{\theta}_e = 0$ on every $e \in E(T)$. Then every chord $e \notin E(T)$ carries $\tilde{\theta}_e \equiv \Phi(C_T(e)) \pmod{2\pi}$. Evaluating the Rayleigh quotient of $L_{\tilde{\theta}}$ at $x = \mathbf{1}/\sqrt{n}$ gives

$$\lambda_1(L_\theta) = \lambda_1(L_{\tilde{\theta}}) \le \frac{1}{n} \mathbf{1}^\top L_{\tilde{\theta}} \mathbf{1} = \frac{1}{n} \sum_{e \notin E(T)} w_e |1 - e^{i\Phi(C_T(e))}|^2 = \frac{4}{n} \sum_{e \notin E(T)} w_e \sin^2\!\left(\frac{\Phi(C_T(e))}{2} \right),$$

and evaluating $\mathcal{L}_{\tilde{\theta}}$ at $D^{1/2}\mathbf{1}/\sqrt{\mathrm{vol}(G)}$ replaces n by $\mathrm{vol}(G)$. Minimizing over $T \in \mathcal{T}(G)$ completes (11)–(12). □

Remark 2.3 (Asymptotic Exactness on Weighted Cycles). On an arbitrarily weighted cycle C_n of total resistance $R_{\mathrm{cyc}}(C_n)$ and small flux $\Phi \to 0$, Theorem 2.2(c) gives the lower bound $\lambda_1(L_\theta(C_n)) \ge 4\sin^2(\Phi/2)/(nR_{\mathrm{cyc}}(C_n)) = \Phi^2/(nR_{\mathrm{cyc}}(C_n)) + O(\Phi^4)$. By Theorem 3.1 below, the exact leading asymptotics of $\lambda_1(L_\theta(C_n))$ is $\Phi^2/(nR_{\mathrm{cyc}}(C_n)) + O(\Phi^4)$, so the lower bound in (11) achieves *exact asymptotic constant* 1 on every weighted cycle. For the unweighted n-cycle at arbitrary $\Phi \in (0, \pi]$, $\lambda_1(L_\theta(C_n)) = 4\sin^2(\Phi/(2n))$, so the ratio of lower bound to exact eigenvalue stays uniformly above $4/\pi^2$ for all $n \ge 3$.

3 Exact Hodge Cycle-Space Curvature and Global Flux Ceiling

Let $\langle f, g \rangle_W := f^\top W g$ be the weighted inner product on the edge cochain space $\mathbb{R}^{|E|}$. The discrete Hodge decomposition splits $\mathbb{R}^{|E|}$ into two W-orthogonal subspaces:

$$\mathbb{R}^{|E|} = \underbrace{\mathrm{ran}(B_0)}_{\text{exact / pure gauge}} \oplus_W \underbrace{\ker(B_0^\top W)}_{\text{divergence-free / cycle space}}, \tag{13}$$

where $B_0 \in \mathbb{R}^{|E| \times n}$ is the unweighted incidence matrix. The W-orthogonal projection $P_{\mathrm{cyc},W} : \mathbb{R}^{|E|} \to \ker(B_0^\top W)$ onto the cycle space is

$$P_{\mathrm{cyc},W} \theta = \theta - B_0 \psi_*, \qquad \psi_* = L_0^+ B_0^\top W \theta, \tag{14}$$

where L_0^+ denotes the Moore–Penrose pseudoinverse of $L_0 = B_0^\top W B_0$. Indeed, $B_0^\top W(\theta - B_0\psi_*) = B_0^\top W\theta - L_0 L_0^+ B_0^\top W\theta = 0$ because $B_0^\top W\theta \perp \mathbf{1} = \ker(L_0)$.

Theorem 3.1 (Hodge Cycle-Space Curvature, Persistent Current, and Global Flux Ceiling)**.** *Let $G = (V, E, w)$ be a connected weighted graph on n vertices and let $\theta \in \mathbb{R}^{|E|}$ be any magnetic potential. Consider the one-parameter family $L_{t\theta}$ for $t \in \mathbb{R}$.*

(a) (Exact Small-Flux Hodge Curvature.) *The ground-state eigenvalue $\lambda_1(L_{t\theta})$ is real-analytic in a neighborhood of $t = 0$, with $\lambda_1(L_0) = 0$, $\dot{\lambda}_1(0) = 0$, and exact second-order curvature*

$$\frac{1}{2}\frac{d^2}{dt^2}\lambda_1(L_{t\theta})\bigg|_{t=0} = \frac{1}{n}\,\theta^\top (W - WB_0L_0^+B_0^\top W)\theta = \frac{1}{n}\,\|P_{\mathrm{cyc},W}\theta\|_W^2. \tag{15}$$

In particular, $\ddot{\lambda}_1(0) = 0$ if and only if $\theta \in \mathrm{ran}(B_0)$ (an exact real 1-coboundary, as distinct from nontrivial $2\pi\mathbb{Z}$-valued cycle windings), in which case $\lambda_1(L_{t\theta}) \equiv 0$ for all $t \in \mathbb{R}$.

(b) (Exact Persistent-Current Derivative.) *Whenever $\lambda_1(L_{t_0\theta})$ is a simple eigenvalue with unit eigenvector $x \in \mathbb{C}^n$, $\lambda_1(L_{t\theta})$ is real-analytic near t_0 with first derivative*

$$\frac{d}{dt}\lambda_1(L_{t\theta})\bigg|_{t=t_0} = 2\sum_{e=(u,v)\in E} w_e\theta_e\ \mathrm{Im}\big(\overline{x_u}\,e^{it_0\theta_e}\,x_v\big). \tag{16}$$

(c) (Second-Order-Tight Global Flux Ceiling.) *Setting $\tilde{\theta} = P_{\mathrm{cyc},W}\theta = \theta - B_0L_0^+B_0^\top W\theta$, the following global inequalities hold for every $\theta \in \mathbb{R}^{|E|}$:*

$$\lambda_1(L_\theta) \le \frac{4}{n}\sum_{e\in E} w_e \sin^2\left(\frac{\tilde{\theta}_e}{2}\right) \le \frac{1}{n}\,\|P_{\mathrm{cyc},W}\theta\|_W^2, \tag{17}$$

and the ratio of $\lambda_1(L_{t\theta})$ to both upper bounds converges to 1 as $t \to 0$ whenever $\theta \notin \mathrm{ran}(B_0)$.

Proof. Part (a). Expanding $L_{t\theta}$ as a convergent Hermitian power series around $t = 0$ gives $L_{t\theta} = L_0 + tA_1 + \frac{t^2}{2}A_2 + O(t^3)$, where for each $\{u, v\} \in E$,

$$(A_1)_{uv} = -iw_{uv}\theta_{uv}, \quad (A_1)_{uu} = 0; \qquad (A_2)_{uv} = w_{uv}\theta_{uv}^2, \quad (A_2)_{uu} = 0.$$

Since G is connected, $\lambda_1(L_0) = 0$ is a simple eigenvalue of L_0 with normalized eigenvector $u_0 = \mathbf{1}/\sqrt{n}$. By Rellich's perturbation theorem for analytic Hermitian families, $\lambda_1(L_{t\theta})$ is real-analytic near $t = 0$, with first and second Rayleigh–Schrödinger coefficients

$$\dot{\lambda}_1(0) = u_0^*A_1u_0, \qquad \frac{1}{2}\ddot{\lambda}_1(0) = \frac{1}{2}u_0^*A_2u_0 - u_0^*A_1L_0^+A_1u_0.$$

Since A_1 is purely imaginary and skew-symmetric, it is Hermitian ($A_1^* = A_1$) and satisfies $\mathbf{1}^\top A_1\mathbf{1} = 0$, so $\dot{\lambda}_1(0) = 0$. For each vertex $u \in V$, $(A_1\mathbf{1})_u = -i\sum_{v\sim u} w_{uv}\theta_{uv} = -i(B_0^\top W\theta)_u$, whence $A_1u_0 = -\frac{i}{\sqrt{n}}B_0^\top W\theta$ and, using $A_1^* = A_1$,

$$u_0^*A_1L_0^+A_1u_0 = (A_1u_0)^*L_0^+(A_1u_0) = \frac{1}{n}\,\theta^\top WB_0L_0^+B_0^\top W\theta.$$

Meanwhile, $\mathbf{1}^\top A_2\mathbf{1} = \sum_u\sum_{v\sim u} w_{uv}\theta_{uv}^2 = 2\sum_{e\in E} w_e\theta_e^2 = 2\theta^\top W\theta$, so $\frac{1}{2}u_0^*A_2u_0 = \frac{1}{n}\theta^\top W\theta$. Subtracting gives (15), since

$$\|P_{\mathrm{cyc},W}\theta\|_W^2 = (\theta - B_0L_0^+B_0^\top W\theta)^\top W(\theta - B_0L_0^+B_0^\top W\theta) = \theta^\top(W - WB_0L_0^+B_0^\top W)\theta.$$

Because $W \succ 0$, $\|P_{\mathrm{cyc},W}\theta\|_W^2 = 0$ iff $P_{\mathrm{cyc},W}\theta = 0$ iff $\theta \in \mathrm{ran}(B_0)$, in which case $L_{t\theta}$ is unitarily similar to L_0 for all $t \in \mathbb{R}$.

Part (b). Differentiating (1) gives $\big(\frac{d}{dt}L_{t\theta}\big)_{uv} = -iw_{uv}\theta_{uv}e^{it\theta_{uv}}$ for $\{u, v\} \in E$ and zero on the diagonal. By the Hellmann–Feynman theorem at a simple eigenvalue,

$$\frac{d}{dt}\lambda_1(L_{t\theta})\bigg|_{t=t_0} = x^*\left(\frac{d}{dt}L_{t\theta}\bigg|_{t=t_0}\right)x = \sum_{e=(u,v)\in E} w_e\theta_e\big(-i\overline{x_u}e^{it_0\theta_e}x_v + i\overline{x_v}e^{-it_0\theta_e}x_u\big) = 2\sum_{e=(u,v)\in E} w_e\theta_e\,\mathrm{Im}\big(\overline{x_u}e^{it_0\theta_e}x$$

Part (c). Let $\psi_* = L_0^+ B_0^\top W\theta$ so $\tilde\theta = \theta - B_0\psi_* = P_{\mathrm{cyc},W}\theta$. By gauge equivalence (3), L_θ and $L_{\tilde\theta}$ are unitarily similar. Testing $L_{\tilde\theta}$ with $u_0 = \mathbf{1}/\sqrt{n}$ and using $4\sin^2(s/2) \le s^2$ for all $s \in \mathbb{R}$ yields

$$\lambda_1(L_\theta) = \lambda_1(L_{\tilde\theta}) \le \frac{1}{n}\mathbf{1}^\top L_{\tilde\theta}\mathbf{1} = \frac{4}{n}\sum_{e\in E} w_e \sin^2\left(\frac{\tilde\theta_e}{2}\right) \le \frac{1}{n}\sum_{e\in E} w_e\tilde\theta_e^2 = \frac{1}{n}\|P_{\mathrm{cyc},W}\theta\|_W^2.$$

Replacing θ by $t\theta$, both upper bounds equal $\frac{t^2}{n}\|P_{\mathrm{cyc},W}\theta\|_W^2 + O(t^4)$, which matches $\lambda_1(L_{t\theta})$ to order t^2 by Part (a). □

4 Magnetic Effective-Resistance Metric, Perturbation Envelopes, and Rank-1 Edge Insertion

For any connected frustrated magnetic graph (G, w, θ), denote the magnetic Green's function by $G_\theta := L_\theta^{-1} \succ 0$. Define the symmetric nonnegative function $\rho_\theta : V \times V \to [0, \infty)$ by

$$\rho_\theta(u, v) := \sqrt{(G_\theta)_{uu} + (G_\theta)_{vv} - 2|(G_\theta)_{uv}|}. \tag{18}$$

Notice that for any $u, v \in V$, $(G_\theta)_{uu}+(G_\theta)_{vv}-2|(G_\theta)_{uv}| \ge (\sqrt{(G_\theta)_{uu}}-\sqrt{(G_\theta)_{vv}})^2 \ge 0$ by Cauchy–Schwarz ($|(G_\theta)_{uv}|^2 \le (G_\theta)_{uu}(G_\theta)_{vv}$), and (18) is invariant under (3) because $(U_\psi^* G_\theta U_\psi)_{uv} = e^{i(\psi_v-\psi_u)}(G_\theta)_{uv}$.

Theorem 4.1 (Magnetic Effective-Resistance Metric, Flux Perturbation, and Rank-1 Edge Insertion)**.** *Let (G, w, θ) be a connected frustrated weighted magnetic graph with $G_\theta = L_\theta^{-1} \succ 0$.*

(a) (Gauge-Invariant $\mathbb{T}$-Orbit Quotient Metric.) *The function $\rho_\theta(u, v)$ in (18) is a gauge-invariant metric on V (symmetric, strictly positive for $u \ne v$, and satisfying $\rho_\theta(u, w) \le \rho_\theta(u, v) + \rho_\theta(v, w)$ for all $u, v, w \in V$).*

(b) (Gauge-Minimized Flux Perturbation Envelopes.) *Let $\theta' = \theta + \Delta\theta$ be a second magnetic potential on the same weighted graph (G, w), and let $\lambda_1 \le \cdots \le \lambda_n$ and $\lambda'_1 \le \cdots \le \lambda'_n$ be the ordered eigenvalues of L_θ and $L_{\theta'}$. Then for every gauge shift $\psi \in \mathbb{R}^n$ with $\widetilde{\Delta\theta} := \Delta\theta - B_0\psi$,*

$$\max_{1\le j\le n} |\lambda'_j - \lambda_j| \le 2\max_{u\in V}\sum_{v\sim u} w_{uv}\left|\sin\left(\frac{\widetilde{\Delta\theta}_{uv}}{2}\right)\right|, \tag{19}$$

$$\left(\sum_{j=1}^n (\lambda'_j - \lambda_j)^2\right)^{1/2} \le 2\sqrt{2}\left(\sum_{e\in E} w_e^2 \sin^2\left(\frac{\widetilde{\Delta\theta}_e}{2}\right)\right)^{1/2}. \tag{20}$$

In particular, both bounds hold for the weighted Hodge-projection gauge $\psi_* = L_0^+ B_0^\top W\Delta\theta$ (which minimizes $\|\widetilde{\Delta\theta}\|_W$ and yields $\widetilde{\Delta\theta} = P_{\mathrm{cyc},W}\Delta\theta$).

(c) (Exact Rank-1 Magnetic Edge Insertion and Anti-Aligned Optimality.) *Suppose a new weighted magnetic edge $\{u, v\}$ with weight $w > 0$ and phase parameter $\phi \in \mathbb{R}$ (corresponding to potential $\theta_{vu} = \phi$, $\theta_{uv} = -\phi$ under (1)) is added to G, producing $L_\theta(\phi) := L_\theta + w b_\phi b_\phi^*$ where $b_\phi = e_u - e^{i\phi}e_v$. Define $q(\phi) := b_\phi^* G_\theta b_\phi = (G_\theta)_{uu} + (G_\theta)_{vv} - 2\,\mathrm{Re}(e^{i\phi}(G_\theta)_{uv})$ and $q_2(\phi) := b_\phi^* G_\theta^2 b_\phi$. Then*

$$\det(L_\theta(\phi)) = \det(L_\theta)(1 + w\, q(\phi)), \tag{21}$$

$$\mathrm{tr}(L_\theta(\phi)^{-1}) = \mathrm{tr}(G_\theta) - \frac{w\, q_2(\phi)}{1 + w\, q(\phi)}. \tag{22}$$

Moreover, if $(G_\theta)_{uv} \ne 0$, then as ϕ varies over $\mathbb{R}/2\pi\mathbb{Z}$:

- *the* aligned *phase* $\phi_{\mathrm{al}} := \arg((G_\theta)_{vu}) \equiv -\arg((G_\theta)_{uv}) \pmod{2\pi}$ *uniquely minimizes* $q(\phi)$ *and* $\det(L_\theta(\phi))$*, with* $q(\phi_{\mathrm{al}}) = \rho_\theta(u,v)^2$*;*
- *the* anti-aligned *phase* $\phi_* := \arg((G_\theta)_{vu}) + \pi \equiv \pi - \arg((G_\theta)_{uv}) \pmod{2\pi}$ *uniquely maximizes* $q(\phi)$ *and* $\det(L_\theta(\phi))$*, with* $q(\phi_*) = \rho_\theta(u,v)^2 + 4|(G_\theta)_{uv}|$ *and exact extremal determinant ratio*

$$\frac{\det(L_\theta(\phi_*))}{\det(L_\theta(\phi_{\mathrm{al}}))} = \frac{1 + w\big(\rho_\theta(u,v)^2 + 4|(G_\theta)_{uv}|\big)}{1 + w\,\rho_\theta(u,v)^2}. \tag{23}$$

Proof. Part (a). Since $G_\theta \succ 0$, let $M = G_\theta^{1/2} \succ 0$ and define the column vectors $z_u := Me_u \in \mathbb{C}^n$ for $u \in V$. Then $\langle z_u, z_v\rangle = z_u^* z_v = (G_\theta)_{uv}$ and $\|z_u\|_2^2 = (G_\theta)_{uu}$. The compact unit circle $\mathbb{T} = \{e^{i\alpha} : \alpha \in \mathbb{R}\}$ acts isometrically on $(\mathbb{C}^n, \|\cdot\|_2)$ by scalar multiplication. For any $u, v \in V$,

$$\min_{\alpha\in\mathbb{R}} \|z_u - e^{i\alpha} z_v\|_2^2 = \min_{\alpha\in\mathbb{R}} (\|z_u\|_2^2 + \|z_v\|_2^2 - 2\,\mathrm{Re}(e^{i\alpha}\langle z_u, z_v\rangle)) = (G_\theta)_{uu} + (G_\theta)_{vv} - 2|(G_\theta)_{uv}| = \rho_\theta(u,v)^2.$$

Symmetry follows immediately from $\|z_u - e^{i\alpha} z_v\|_2 = \|z_v - e^{-i\alpha} z_u\|_2$. If $\rho_\theta(u,v) = 0$, then $z_u = e^{i\alpha} z_v$ for some $\alpha \in \mathbb{R}$, so $M(e_u - e^{i\alpha} e_v) = 0$; since M is invertible, $e_u = e^{i\alpha} e_v$, forcing $u = v$. For the triangle inequality, given $u, v, w \in V$, choose $\alpha, \beta \in \mathbb{R}$ attaining $\rho_\theta(u,v) = \|z_u - e^{i\alpha} z_v\|_2$ and $\rho_\theta(v,w) = \|z_v - e^{i\beta} z_w\|_2 = \|e^{i\alpha} z_v - e^{i(\alpha+\beta)} z_w\|_2$; then

$$\rho_\theta(u,w) \le \|z_u - e^{i(\alpha+\beta)} z_w\|_2 \le \|z_u - e^{i\alpha} z_v\|_2 + \|e^{i\alpha} z_v - e^{i(\alpha+\beta)} z_w\|_2 = \rho_\theta(u,v) + \rho_\theta(v,w).$$

Part (b). For any $\psi \in \mathbb{R}^n$, $L_{\theta'}$ is unitarily similar to $U_\psi^* L_{\theta'} U_\psi = L_{\theta + \widetilde{\Delta\theta}}$, so the perturbation matrix $E_\psi := L_{\theta+\widetilde{\Delta\theta}} - L_\theta$ is Hermitian with zero diagonal and off-diagonal entries $(E_\psi)_{uv} = -w_{uv} e^{i\theta_{uv}} (e^{i\widetilde{\Delta\theta}_{uv}} - 1)$ for $\{u, v\} \in E$. Since $|e^{is} - 1| = 2|\sin(s/2)|$, each nonzero entry has modulus $|(E_\psi)_{uv}| = 2w_{uv}|\sin(\widetilde{\Delta\theta}_{uv}/2)|$. Applying Weyl's inequality $\max_j |\lambda'_j - \lambda_j| \le \|E_\psi\|_2 \le \|E_\psi\|_\infty = \max_u \sum_{v\sim u} |(E_\psi)_{uv}|$ yields (19). Applying the Hoffman–Wielandt inequality $\sum_j (\lambda'_j - \lambda_j)^2 \le \|E_\psi\|_F^2 = 2\sum_{e\in E} 4w_e^2 \sin^2(\widetilde{\Delta\theta}_e/2)$ yields (20).

Part (c). The matrix determinant lemma gives $\det(L_\theta + w b_\phi b_\phi^*) = \det(L_\theta)(1 + w\, b_\phi^* G_\theta b_\phi)$, which is (21). The Sherman–Morrison formula gives

$$L_\theta(\phi)^{-1} = G_\theta - \frac{w\, G_\theta b_\phi b_\phi^* G_\theta}{1 + w\, b_\phi^* G_\theta b_\phi},$$

and taking the trace using $\mathrm{tr}(G_\theta b_\phi b_\phi^* G_\theta) = b_\phi^* G_\theta^2 b_\phi = q_2(\phi)$ gives (22). Writing $(G_\theta)_{uv} = |(G_\theta)_{uv}| e^{i\gamma}$ with $\gamma = \arg((G_\theta)_{uv}) \equiv -\arg((G_\theta)_{vu}) \pmod{2\pi}$ (since G_θ is Hermitian), we have

$$q(\phi) = (G_\theta)_{uu} + (G_\theta)_{vv} - 2|(G_\theta)_{uv}| \cos(\phi + \gamma).$$

When $|(G_\theta)_{uv}| > 0$, $\cos(\phi + \gamma)$ attains its unique maximum $+1$ at $\phi_{\mathrm{al}} \equiv -\gamma \equiv \arg((G_\theta)_{vu}) \pmod{2\pi}$, giving $q(\phi_{\mathrm{al}}) = \rho_\theta(u,v)^2$, and its unique minimum -1 at $\phi_* \equiv \pi - \gamma \equiv \arg((G_\theta)_{vu}) + \pi \pmod{2\pi}$, giving $q(\phi_*) = (G_\theta)_{uu} + (G_\theta)_{vv} + 2|(G_\theta)_{uv}| = \rho_\theta(u,v)^2 + 4|(G_\theta)_{uv}|$. Substituting into (21) yields (23). □

Remark 4.2 (Metric vs. Squared Effective Resistance). In the unfrustrated case $\theta = 0$, classical effective resistance $R_{\mathrm{eff}}(u,v) = (L_0^+)_{uu} + (L_0^+)_{vv} - 2(L_0^+)_{uv}$ itself satisfies the triangle inequality via the discrete maximum principle, in addition to its square root $\sqrt{R_{\mathrm{eff}}(u,v)}$ being Euclidean. In the magnetic setting, the $\mathbb{T}$-orbit quotient argument in Theorem 4.1(a) guarantees the triangle inequality for $\rho_\theta(u,v)$ unconditionally for every $G_\theta \succ 0$, without requiring real-valued harmonic potentials.

5 Strict Discrete Kato Diamagnetic Domination and Odd-Cycle Non-Bipartiteness Gaps

We now connect magnetic Laplacians back to classical unweighted and weighted graphs via diamagnetic domination and π-flux mirror symmetry. For any non-bipartite connected graph $G = (V, E, w)$, define for every vertex $v \in V$ its *odd-cycle resistance*

$$R_{\mathrm{odd}}(v) := \min_{(C,P)} \left(\frac{R_{\mathrm{cyc}}(C)}{4} + R(P) \right), \tag{24}$$

where the minimum runs over all odd simple cycles $C \subseteq G$ and all simple paths P from v to $V(C)$ meeting $V(C)$ only at their terminal endpoint (with $R(P) = 0$ if $v \in V(C)$).

Theorem 5.1 (Strict Kato Diamagnetic Domination, Determinant Inequality, and Odd-Cycle Gap)**.** *Let (G, w, θ) be a finite connected weighted magnetic graph.*

(a) (Discrete Kato Heat and Resolvent Domination.) *For all $t \geq 0$, all $m > 0$, and all $u, v \in V$,*

$$\left|(e^{-tL_\theta})_{uv}\right| \leq (e^{-tL_0})_{uv}, \qquad \left|((L_\theta + mI_n)^{-1})_{uv}\right| \leq ((L_0 + mI_n)^{-1})_{uv}. \tag{25}$$

(b) (Strict Fredholm Determinant Inequality.) *For every $m > 0$,*

$$\det(L_\theta + mI_n) \geq \det(L_0 + mI_n), \tag{26}$$

with strict inequality for every $m > 0$ if and only if (G, w, θ) is frustrated.

(c) (π-Mirror Symmetry and Topological Odd-Cycle Non-Bipartiteness Gap Bound.) *Shifting every edge potential by $\pi \pmod{2\pi}$ (which preserves skew-symmetry in $\mathbb{R}/2\pi\mathbb{Z}$ since $\pi \equiv -\pi \pmod{2\pi}$) yields the exact operator identity*

$$\mathcal{L}_{\theta+\pi} = 2I_n - \mathcal{L}_\theta. \tag{27}$$

In particular, taking $\theta = 0$ (so $\theta + \pi \equiv \pi \pmod{2\pi}$ on every oriented edge), the non-bipartiteness spectral gap $2 - \lambda_{\max}(\mathcal{L}_0)$ of any connected non-bipartite weighted graph (G, w) equals $\lambda_1(\mathcal{L}_\pi)$ and satisfies the purely topological lower bound

$$2 - \lambda_{\max}(\mathcal{L}_0) = \lambda_1(\mathcal{L}_\pi) \geq \left(\sum_{v \in V} d_v R_{\mathrm{odd}}(v) \right)^{-1}, \tag{28}$$

where on every arbitrarily weighted odd cycle C_{2k+1}, $(L_\pi(C_{2k+1})^{-1})_{vv} = R_{\mathrm{cyc}}(C_{2k+1})/4$ holds exactly for every $v \in V(C_{2k+1})$.

Proof. Part (a). Fix $\alpha > d_{\max} := \max_{v \in V} d_v$ and set $P_0 := \alpha I_n - L_0$ and $P_\theta := \alpha I_n - L_\theta$. Then $(P_0)_{uu} = (P_\theta)_{uu} = \alpha - d_u > 0$, and for $u \neq v$, $|(P_\theta)_{uv}| = (P_0)_{uv} \geq 0$. By induction on $k \geq 0$, for all $u, v \in V$,

$$|(P_\theta^k)_{uv}| \leq \sum_{w \in V} |(P_\theta^{k-1})_{uw}|\,|(P_\theta)_{wv}| \leq \sum_{w \in V} (P_0^{k-1})_{uw}(P_0)_{wv} = (P_0^k)_{uv}.$$

Summing the exponential series $e^{-tL_\theta} = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{t^k}{k!} P_\theta^k$ entrywise gives $|(e^{-tL_\theta})_{uv}| \leq (e^{-tL_0})_{uv}$ for all $t \geq 0$. Multiplying by e^{-mt} and integrating over $t \in [0, \infty)$ using $(L_\theta + mI_n)^{-1} = \int_0^\infty e^{-mt} e^{-tL_\theta}\, dt$ establishes the second inequality in (25).

Part (b). Setting $u = v$ in (25) and noting that $((L_\theta + \mu I_n)^{-1})_{vv} > 0$ for all $\mu > 0$, we obtain $\mathrm{tr}((L_\theta + \mu I_n)^{-1}) \leq \mathrm{tr}((L_0 + \mu I_n)^{-1})$ for all $\mu > 0$. Define $f(\mu) := \ln \det(L_\theta + \mu I_n) -$

$\ln\det(L_0+\mu I_n)$. Then $f'(\mu)=\operatorname{tr}((L_\theta+\mu I_n)^{-1})-\operatorname{tr}((L_0+\mu I_n)^{-1})\le 0$ for all $\mu>0$, while $\lim_{M\to\infty} f(M)=\lim_{M\to\infty}\ln\frac{\det(I_n+M^{-1}L_\theta)}{\det(I_n+M^{-1}L_0)}=0$. Therefore

$$f(m)=-\int_m^\infty f'(\mu)\,d\mu\ge 0 \implies \det(L_\theta+mI_n)\ge\det(L_0+mI_n).$$

If (G,w,θ) is unfrustrated, L_θ is unitarily similar to L_0 and equality holds. Conversely, if (G,w,θ) is frustrated, let $C=(v_0,v_1,\dots,v_{k-1},v_0)$ be a frustrated simple cycle with $\Phi(C)\not\equiv 0 \pmod{2\pi}$. Expanding $(P_\theta^k)_{v_0v_0}$ as a sum over all closed walks of length k based at v_0 (including self-loops of weight $\alpha-d_u>0$), two distinct walks contribute: the stationary self-loop walk contributing $(\alpha-d_{v_0})^k>0$ (strictly positive real), and the simple cycle traversal C contributing $(\prod_{j=0}^{k-1}w_{v_jv_{j+1}})e^{i\Phi(C)}\notin\mathbb{R}_{\ge 0}$. By the strict complex triangle inequality ($|a+b|<|a|+|b|$ whenever $a>0$ and $b\notin\mathbb{R}_{\ge 0}$), we have $|(P_\theta^k)_{v_0v_0}|<(P_0^k)_{v_0v_0}$ strictly. Hence $(e^{-tL_\theta})_{v_0v_0}<(e^{-tL_0})_{v_0v_0}$ for all $t>0$, so $((L_\theta+\mu I_n)^{-1})_{v_0v_0}<((L_0+\mu I_n)^{-1})_{v_0v_0}$ for all $\mu>0$, yielding $f'(\mu)<0$ strictly for all $\mu>0$ and therefore $\det(L_\theta+mI_n)>\det(L_0+mI_n)$ strictly.

Part (c). Replacing every θ_{uv} by $\theta_{uv}+\pi$ negates every off-diagonal entry of L_θ while leaving its diagonal D unchanged; hence $L_{\theta+\pi}=2D-L_\theta$. Left- and right-multiplying by $D^{-1/2}$ yields $\mathcal{L}_{\theta+\pi}=2I_n-\mathcal{L}_\theta$. Setting $\theta=0$ gives $\mathcal{L}_\pi=2I_n-\mathcal{L}_0$, so $\lambda_1(\mathcal{L}_\pi)=2-\lambda_{\max}(\mathcal{L}_0)$. Under the constant potential $\theta\equiv\pi$, the holonomy of any simple cycle C of length ℓ is $\Phi(C)\equiv\ell\pi \pmod{2\pi}$, so $\sin^2(\Phi(C)/2)=1$ if ℓ is odd and 0 if ℓ is even. Thus the frustrated cycles of (G,w,π) are precisely the odd simple cycles of G, with $R_{\mathrm{mag}}(C)=R_{\mathrm{cyc}}(C)/4$ and $R_{\mathrm{frust}}(v)=R_{\mathrm{odd}}(v)$. Applying Theorem 2.2(a) and (12) yields (28) and the exact odd-cycle identity $(L_\pi(C_{2k+1})^{-1})_{vv}=R_{\mathrm{cyc}}(C_{2k+1})/4$. □

Remark 5.2 (Concrete Scaling on Odd Cycles and Odd Lollipops). On an unweighted odd cycle C_{2k+1} ($n=2k+1$, $d_v=2$, $R_{\mathrm{odd}}(v)=n/4$), (28) immediately gives $2-\lambda_{\max}(\mathcal{L}_0(C_n))\ge 2/n^2$ (compared with the exact value $1-\cos(\pi/n)=\frac{\pi^2}{2n^2}+O(n^{-4})$, within a constant factor $\pi^2/4\approx 2.47$ for all odd $n\ge 3$). Unlike global Cheeger-type non-bipartiteness inequalities [10, 6], (28) requires no cut optimization and explicitly penalizes long bipartite tails attached to an odd cycle via the additive series resistance $R(P)$.

6 Computational Verification and Benchmarks

All identities and inequalities in Theorems 2.2–5.1 were tested numerically in double precision (`float64`) via our reproducible verification suite across 291 weighted magnetic graph instances spanning arbitrarily weighted cycles ($k=3,\dots,18$), weighted lollipops, connected Erdős–Rényi multi-cycle networks ($n=5,\dots,16$), odd cycles ($2k+1=3,\dots,13$), and 2D periodic magnetic tori (4×4, 5×5, 6×6).

Figure 1 illustrates the exact frustration sandwich and second-order-tight Hodge ceiling: on the weighted 8-cycle in panel (a) ($R_{\mathrm{cyc}}(C_8)=7.413$, $w_{\min}=0.7$), the Theorem 2.2(c) lower bound (red dashed, 0.0674 at $\Phi=\pi$) and the Theorem 3.1(c) Hodge ceiling (green dash-dot, 0.1664 at $\Phi=\pi$) pinch the exact ground state $\lambda_1(L_\theta)$ (blue solid, 0.1567 at $\Phi=\pi$) with asymptotic ratio 1 as $\Phi\to 0$, while the looser spanning-tree upper bound (purple dotted) reaches 0.3500 at $\Phi=\pi$; on the multi-cycle 10-vertex graph in panel (b), both the $\sin^2$ and quadratic Hodge ceilings match $\lambda_1(L_{t\theta})$ to order t^2 at the origin.

Figure 2 displays the two geometric phenomena from Theorems 4.1(c) and 5.1(c): panel (a) verifies the exact phase dependence of rank-1 magnetic edge insertion on an 8-vertex frustrated network, with the minimum determinant gain occurring at the aligned phase $\phi_{\mathrm{al}}=\arg(G_{vu})$ (where $q(\phi_{\mathrm{al}})=\rho_\theta(u,v)^2$) and the unique maximum at the anti-aligned phase $\phi_*=\phi_{\mathrm{al}}+\pi$; panel (b) shows the logarithmic scaling of the exact non-bipartiteness gap $2-\lambda_{\max}(\mathcal{L}_0)$ versus the Theorem 5.1(c) topological lower bound on unweighted odd cycles C_{2k+1} and odd lollipops $L(2k+1,2)$ for $2k+1\in\{3,5,\dots,19\}$.

Table 1: Summary of numerical verification across 291 weighted magnetic graph benchmarks (seed `20261129`). All inequalities hold with zero violations up to `float64` rounding.

Result	**Verified Quantity / Margin**	**Instances**	**Observed Value**
Thm. 2.2(a)	$\max \lvert(L_\theta(C)^{-1})_{vv} - R_{\mathrm{mag}}(C)\rvert$ $(k = 3 \dots 18)$	64	1.95×10^{-10}
Thm. 2.2(b)	$\max \lvert(L_\theta(C \cup P)^{-1})_{vv} - (R_{\mathrm{mag}}(C) + R(P))\rvert$	20	1.82×10^{-12}
Thm. 2.2(c)	$\min(R_{\mathrm{frust}}(v) - (L_\theta^{-1})_{vv})$	60	-8.88×10^{-16}
Thm. 2.2(c)	$\min(\lambda_1(L_\theta) - (\sum R_{\mathrm{frust}})^{-1})$ / $\min(\text{tree UB} - \lambda_1)$	60	$+0.1345$ / $+0.1840$
Thm. 2.2(c)	$\min(\lambda_1(\mathcal{L}_\theta) - (\sum d_v R_{\mathrm{frust}})^{-1})$ / $\min(\text{UB} - \lambda_1)$	60	$+0.0303$ / $+0.0320$
Thm. 3.1(a)	max rel. err. of $\frac{1}{2}\ddot{\lambda}_1(0) = \frac{1}{n}\lVert P_{\mathrm{cyc},W}\theta\rVert_W^2$ $(\varepsilon = 10^{-4})$	50	5.87×10^{-7}
Thm. 3.1(b)	max abs. err. of persistent-current derivative (16)	50	1.00×10^{-8}
Thm. 3.1(c)	$\min(\frac{t^2}{n}\lVert P_{\mathrm{cyc},W}\theta\rVert_W^2 - \lambda_1(L_{t\theta}))$	50	$+2.48 \times 10^{-11}$
Thm. 3.1(a)	max pure-gauge residual $(\lVert P_{\mathrm{cyc},W}\theta\rVert_W^2, \lambda_1)$	50	1.61×10^{-14}
Thm. 4.1(a)	min distinct-triple slack $\rho_\theta(u,v) + \rho_\theta(v,w) - \rho_\theta(u,w)$	40	$+0.2716$
Thm. 4.1(a)	max gauge invariance error of ρ_θ	40	3.33×10^{-16}
Thm. 4.1(b)	min Weyl slack (19) / Hoffman–Wielandt slack (20)	40	$+0.2498$ / $+0.2793$
Thm. 4.1(c)	max rel. err. Sherman–Morrison det. (21) / trace (22)	40	1.34×10^{-15} / 9.61×10^{-16}
Thm. 5.1(a)	min Kato heat slack / resolvent slack (25)	48	$+6.77 \times 10^{-7}$ / $+8.53 \times 10^{-5}$
Thm. 5.1(b)	$\min \det(L_\theta + mI)/\det(L_0 + mI)$ (strictly > 1)	48	1.0238
Thm. 5.1(c)	$\max \lVert\mathcal{L}_{\theta+\pi} - (2I - \mathcal{L}_\theta)\rVert_\infty$ / odd-cycle Green's err.	48	8.88×10^{-16} / 1.73×10^{-14}
Thm. 5.1(c)	$\min((2 - \lambda_{\max}(\mathcal{L}_0)) - (\sum d_v R_{\mathrm{odd}}(v))^{-1})$	48	$+0.0408$
Thm. 2.2(c)	2D magnetic tori ($4 \times 4, 5 \times 5, 6 \times 6$; 3 fluxes; all bounds hold)	9	9/9 passed

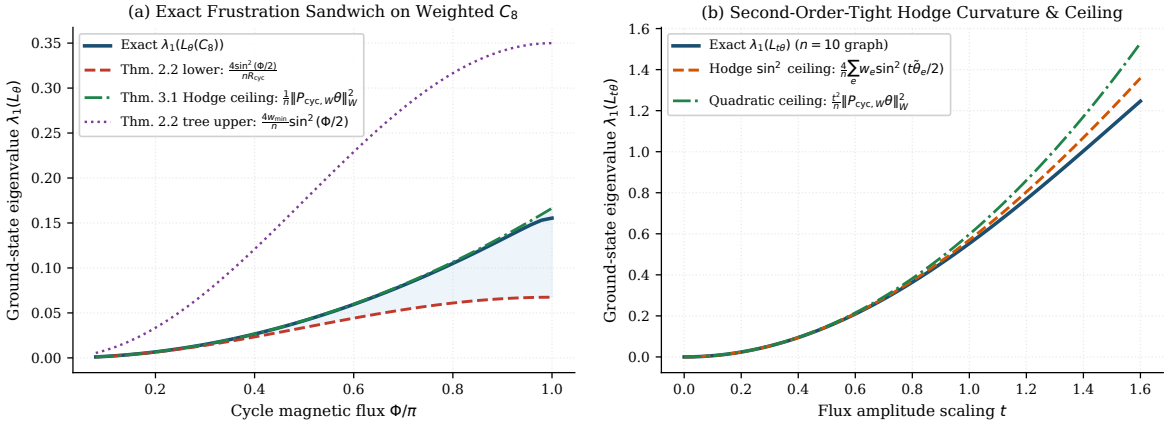


Figure 1: **(a)** Exact ground-state eigenvalue $\lambda_1(L_\theta)$ (blue solid, 0.1567 at $\Phi = \pi$) on a weighted 8-cycle ($R_{\mathrm{cyc}} = 7.413$, $w_{\min} = 0.7$) versus flux Φ/π, pinched tightly between the Theorem 2.2(c) lower bound (red dashed, 0.0674 at $\Phi = \pi$) and the Theorem 3.1(c) Hodge ceiling (green dash-dot, 0.1664 at $\Phi = \pi$), below the spanning-tree upper bound (purple dotted, 0.3500 at $\Phi = \pi$). **(b)** Flux-amplitude sweep $t \mapsto \lambda_1(L_{t\theta})$ on a 10-vertex multi-cycle weighted graph confirming second-order tightness of the Hodge curvature and global flux ceilings at $t = 0$.

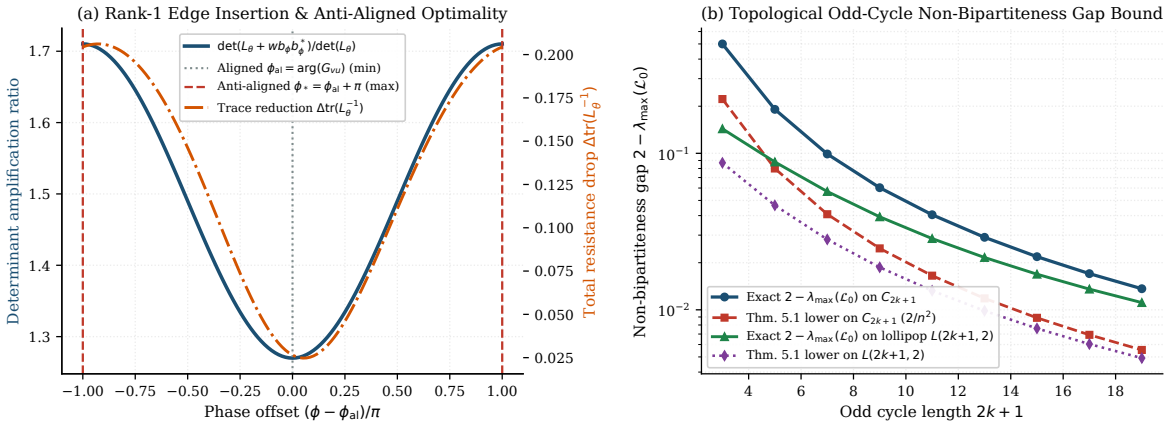


Figure 2: **(a)** Exact determinant amplification ratio (21) and trace reduction (22) under rank-1 magnetic edge insertion on an 8-vertex frustrated graph as a function of $(\phi - \phi_{\mathrm{al}})/\pi$, peaking uniquely at the anti-aligned phase $\phi_* = \phi_{\mathrm{al}} + \pi$. **(b)** Exact non-bipartiteness spectral gap $2 - \lambda_{\max}(\mathcal{L}_0)$ and Theorem 5.1(c) odd-cycle lower bound on odd cycles C_{2k+1} and odd lollipops $L(2k+1, 2)$.

7 Limitations and Reproducibility

Honest scope and limitations. All four theorems apply to finite connected weighted undirected graphs with positive edge weights and scalar U(1) edge phases. We note two structural connections and limitations. First, Theorems 2.2(a)–(b) and 5.1(b) can alternatively be derived combinatorially from Kenyon's cycle-rooted spanning forest (CRSF) determinant expansion [5], whereas our proofs operate directly via variational Dirichlet minimization and shifted-walk comparison. Second, on dense high-girth graphs or 2D magnetic tori with many frustrated plaquettes (e.g., on the unweighted 6×6 torus at plaquette flux $\Phi = \pi$, where $\lambda_1 = 1.1716$ vs. single-plaquette bound $1/36 \approx 0.0278$, or at $\Phi = \pi/2$, where $\lambda_1 = 0.8847$ vs. single-plaquette bound $1/72 \approx 0.0139$), the vertex-wise trace bound $\lambda_1^{-1} \leq \mathrm{tr}(L_\theta^{-1}) \leq \sum_v R_{\mathrm{frust}}(v)$ loses a factor scaling with n (and $\Theta(n^2)$ on dense graphs such as complete graphs K_n, where $2 - \lambda_{\max}(\mathcal{L}_0) = (n-2)/(n-1)$ while single-triangle resistance yields $4/(3n(n-1))$), because it replaces the n eigenvalues of L_θ by a single worst-case mode and discards parallel multi-cycle resistance paths. Moreover, computing the exact minimum $R_{\mathrm{frust}}(v)$ over all simple cycles for arbitrary edge phases can be combinatorially expensive on large general graphs, though any admissible cycle-path pair (C, P) still provides a valid certified upper bound on $(L_\theta^{-1})_{vv}$. Extending Theorem 2.2 to multi-cycle parallel-series bounds or non-Abelian SU(N) connection Laplacians, and extending Theorem 3.1 to integer mod-2π cohomology minimization, remain open directions. Likewise, while Theorem 4.1(a) establishes the triangle inequality for ρ_θ, whether ρ_θ^2 itself satisfies a triangle inequality on general frustrated graphs is not claimed here.

Reproducibility. All verification scripts, unit tests, figure generators, and JSON benchmark artifacts are archived under the MIT license at `https://github.com/lluiseriksson/magnetic-laplacian-frustration-geometry`. Running `pytest` reproduces all 291 benchmark checks and both vector figures deterministically under seed `20261129`.

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