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EXTREMAL FORWARD-FOREST GAUGE FIXING, BAKRY–ÉMERY–WEINGARTEN WASSERSTEIN CONTRACTION, AND STRONG-COUPLING MASS-GAP THRESHOLDS FOR D-DIMENSIONAL SU(N) WILSON LATTICE YANG–MILLS THEORY

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Abstract. We establish a unified geometric, combinatorial, and analytic framework for proving volume-uniform strong-coupling Dobrushin–Wasserstein contraction, exponential decay of correlations, and explicit Osterwalder–Schrader transfer-matrix mass-gap lower bounds $\Delta_{\mathrm{OS}}(\beta) > 0$ for d-dimensional SU(N) Wilson lattice Yang–Mills theory on $\mathbb{Z}^d$ ($d \geq 2$, $N \geq 2$). Four structural results drive our certified thresholds:

(1) *Universal forward-forest acyclicity and exact discrete Jensen extremality:* For any direction map $\sigma : \mathbb{Z}^d \to \{0, \dots, d-1\}$, the forward link set $F_\sigma(\Lambda_L)$ on any open box $\Lambda_L = \{0, \dots, L-1\}^d$ is automatically acyclic. By a discrete convex lower bound over arbitrary forest densities $r \in [0, 1]$, the sharp minimum asymptotic average active-partner degree of any spanning-forest gauge fixing on $\mathbb{Z}^d$ is $D_{\min}(2) = 2$, $D_{\min}(3) = 7$, and $D_{\min}(d) = 6d - 12$ ($d \geq 4$). We construct explicit extremal forward forests in every dimension: a new 3D mod-3 cyclic forest $\sigma_3(x_0, x_1, x_2) = -(x_0 + x_1 + x_2) \bmod 3$ with uniform active degree 7 on every active link (strictly beating axial degree 8 and unfixed degree 12), the 4D mod-2 straight-line forest with uniform active degree 12, and a new parity-lifted family for all $d \geq 5$ attaining $f_p \in \{0, 1\}$, exact Jensen average degree $6d - 12$, and maximum degree $6d - 10$.

(2) *Exact d-dimensional $(2d-1)$-spin even-vertex star contraction and dimension-3 crossover:* Partitioning links into even-vertex stars $\{S_x : \sum x_\mu \text{ even}\}$ equipped with the gauge-orbit ℓ^1 metric and solving the $(2d-1)$-spin internal resolvent $(I - cA_{S'_x})^{-1}$ in closed form yields exact column-sum contraction $\lambda_d(c) = 2(d-1)c(1+c)/(1-2(d-2)c-2(d-1)c^2) < 1$ for every compact Lie group G whenever $c < c_*^{\mathrm{star}}(d) = (\sqrt{4d^2-8d+5} - (2d-3))/(4(d-1))$. This reveals a sharp dimensional crossover: in $d = 3$ our mod-3 forest strictly beats the even-vertex star ($1/7 > c_*^{\mathrm{star}}(3) = (\sqrt{17}-3)/8$), whereas for every $d \geq 4$ the even-vertex star strictly beats $1/(6d-12)$.

(3) *Bakry–Émery–Weingarten and Russo–Dye contraction on* SU(N)*:* Equipping SU(N) with the Hilbert–Schmidt bi-invariant metric yields $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g$ and $\mathrm{CD}(K_{\mathrm{BE}}, \infty)$ with $K_{\mathrm{BE}} = \frac{N}{2} - \frac{2(d-1)\beta}{N}$. Via Langevin semigroup–Kantorovich duality this gives a universal 't Hooft N^2-scaling law $\beta < \frac{c_*}{2(1+2(d-1)c_*)} N^2$ for all $N \geq 2$ ($0.02925\, N^2$ in 4D). For SU(3) (and all $N \geq 5$), exact $k = 2$ Weingarten calculus ($\mathbb{E}_{\mathrm{SU}(3)}[h_0^2] = 5/192$) paired with Russo–Dye convexity and character positivity reduces c_N by 21%–23% at fixed β and raises certified β_{cert} thresholds by 15%–17% using exact rational Weyl moments $m = 0 \dots 16$ with zero numerical quadrature.

(4) *Direct transfer-matrix mass-gap certification:* Weighted Glauber contraction yields explicit volume-uniform OS mass-gap lower bounds $\Delta_{\mathrm{OS}}(\beta) \geq -\ln \theta > 0$ for both forests and stars.

For pure 4D SU(3) lattice QCD we certify a volume-uniform mass gap up to $\beta \leq 0.292$ (forest) and $\beta \leq 0.3055$ (7-spin star), improving +63% over unfixed Bakry–Émery (0.1875); for 3D SU(3) we reach $\beta \leq 0.472$ and for 3D SU(2) we reach $\beta_W \leq 0.571$, strictly beating the 3D star (0.561). Eighteen strong-coupling certificates spanning $d \in \{3, 4, 5, 6\}$ and $N \in \{2, 3, 4, 5, 6\}$ are verified both in 60-digit interval arithmetic and in a standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel file.

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1. Introduction and Honest Scope

Establishing explicit, volume-uniform strong-coupling mass gaps for Wilson's SU(N) lattice Yang–Mills theory [13, 10, 12] is a classical benchmark of constructive gauge theory. While cluster expansions yield exponential decay of correlations and a positive Osterwalder–Schrader (OS) transfer-matrix spectral gap for small $\beta > 0$, Dobrushin–Wasserstein contraction methods [5, 7, 9] offer a direct, non-perturbative route to explicit, computer-certifiable mass-gap thresholds without polymer resummations.

Two recent developments motivate the present work. First, in a formalized study of four-dimensional SU(2) Wilson lattice Yang–Mills theory [3] (repository `creelie/4Y-M`), the authors combined a sharp S^3 chordal-Lipschitz coupling ($c = \beta_W/4$) with a 4D mod-2 spanning-forest gauge fixing (reducing active-partner degree from 18 to 12) and a 7-spin even-vertex star contraction to certify strong-coupling exponential decay up to $\beta_W \le 0.3609$ in 4D SU(2). Second, our companion paper [6] enlarged the complex zero-free discs of d-dimensional SU(N) Wilson partition functions using one-factorization Hölder bounds and exact integer Weyl moments. A natural question arises: *what are the fundamental combinatorial limits of spanning-forest gauge fixing across all dimensions $d \ge 2$, how does the $(2d-1)$-spin even-vertex star scale with d, and can we extend quantitative Dobrushin–Wasserstein mass-gap certification from* SU(2) *to general* SU(N)*—and in particular to four-dimensional* SU(3) *pure lattice QCD—without relying on S^3-specific geometry or floating-point quadrature?*

Honest scope. All theorems in this paper apply strictly to the *strong-coupling lattice regime* of d-dimensional SU(N) Wilson lattice Yang–Mills theory at fixed lattice spacing $a = 1$. They prove uniqueness of the infinite-volume Gibbs measure, uniform exponential decay of gauge-invariant connected correlations, and an explicit positive spectral gap $\Delta_{\mathrm{OS}}(\beta) > 0$ of the infinite-volume OS transfer matrix on the certified real interval $0 \le \beta \le \beta_{\mathrm{cert}}$. They do *not* prove the continuum limit $a \to 0$ ($\beta \to \infty$) or resolve the Clay Millennium Yang–Mills Existence and Mass Gap problem.

Summary of main results. Let $\Lambda_L = \{0, 1, \dots, L-1\}^d \subset \mathbb{Z}^d$ denote the finite open box of side $L \ge 2$ with edge set $E(\Lambda_L)$ and unoriented plaquette set $\mathcal{P}(\Lambda_L)$. For a compact gauge group $G = \mathrm{SU}(N)$ ($N \ge 2$) and coupling $\beta \ge 0$, the Wilson Gibbs measure on $G^{E(\Lambda_L)}$ is

$$d\mu_{\Lambda_L,\beta}(U) = \frac{1}{Z_{\Lambda_L}(\beta)} \exp\left(\frac{\beta}{N} \sum_{p \in \mathcal{P}(\Lambda_L)} \mathrm{Re}\,\mathrm{Tr}(U_p) \right) \prod_{e \in E(\Lambda_L)} dU_e, \tag{1}$$

where dU_e is normalized Haar measure on SU(N) and U_p is the ordered product of the four link variables around plaquette p. (For $N = 2$, $\frac{\beta}{2} \mathrm{Re}\,\mathrm{Tr}(U_p) = \frac{\beta_W}{2} \mathrm{Tr}(U_p)$ matches the standard β_W convention; for $N = 3$, $\beta = 6/g_0^2$ matches standard lattice QCD.)

- **Extremal forward forests (Section 2):** Proposition 2.1 proves in four lines that every forward link assignment $F_\sigma(\Lambda_L)$ is acyclic on Λ_L. Theorem 2.3 proves that no spanning forest of $\mathbb{Z}^d$ (of any density $r \in [0,1]$) can achieve asymptotic average active-partner degree below $D_{\min}(2) = 2$, $D_{\min}(3) = 7$, or $D_{\min}(d) = 6d - 12$ ($d \ge 4$). Theorem 2.4 constructs explicit forward forests attaining $D_{\min}(d)$ in *every* dimension $d \ge 2$, including a new 3D mod-3 cyclic forest with uniform degree 7 and a new $d \ge 5$ parity-lifted family with $f_p \in \{0, 1\}$, average degree $6d - 12$, and maximum degree $6d - 10$.
- **d-Dimensional $(2d-1)$-spin even-vertex star and crossover (Section 3):** Theorem 3.1 establishes exact column-sum contraction $\lambda_d(c) = 2(d-1)c(1+c)/(1 - 2(d-2)c - 2(d-1)c^2) < 1$ on the even-star gauge-orbit partition for every compact Lie group G and every $d \ge 3$, proving the sharp dimensional crossover $1/7 > c_*^{\mathrm{star}}(3)$ in $d = 3$ versus $c_*^{\mathrm{star}}(d) > 1/(6d-12)$ for all $d \ge 4$.

- **Bakry–Émery–Weingarten–Russo–Dye contraction on** $\mathrm{SU}(N)$ **(Section 4):** Using $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2} g$ and Langevin semigroup–Kantorovich duality (Proposition 4.2), we obtain the universal all-N 't Hooft N^2-scaling law $\beta < \frac{c_*}{2(1+2(d-1)c_*)} N^2$ ($0.02925\, N^2$ in 4D). For SU(3) (and all $N \geq 5$), Theorem 4.3 pairs exact $k = 2$ Weingarten calculus ($\mathbb{E}_{\mathrm{SU}(3)}[h_0^2] = 5/192$) with Russo–Dye convexity and character positivity to reduce c_N by 21%–23% at fixed β and raise certified β_{cert} thresholds by 15%–17% using exact rational Weyl moments $m = 0 \ldots 16$.
- **Physical mass gap and mechanized certificates (Sections 5–6):** Theorem 5.1 converts both forest and star contractions into volume-uniform exponential covariance decay and explicit OS transfer-matrix spectral-gap bounds $\Delta_{\mathrm{OS}}(\beta) \geq -\ln \theta > 0$. Table 2 records 18 fully certified thresholds across $d \in \{3, 4, 5, 6\}$ and $N \in \{2, 3, 4, 5, 6\}$, verified both in 60-digit interval arithmetic and in standalone Lean 4.36.0 with zero external imports, zero `sorry`, and zero `Classical.choice`.

2. Universal Forward-Forest Gauge Fixing and Exact Jensen Extremality

Let $e_0, \ldots, e_{d-1}$ be the standard unit basis vectors of $\mathbb{Z}^d$. A *forward direction map* is any function $\sigma : \mathbb{Z}^d \to \{0, \ldots, d-1\}$. It defines the set of *forward links*

$$F_\sigma := \big\{(x, x + e_{\sigma(x)}) : x \in \mathbb{Z}^d\big\} \subset E(\mathbb{Z}^d),$$

and its restriction to the finite open box $\Lambda_L = \{0, \ldots, L-1\}^d$ is $F_\sigma(\Lambda_L) := F_\sigma \cap E(\Lambda_L)$. Links in $F_\sigma(\Lambda_L)$ are called *fixed*; links in $A_\sigma(\Lambda_L) := E(\Lambda_L) \setminus F_\sigma(\Lambda_L)$ are called *active*. For each plaquette $p \in \mathcal{P}(\Lambda_L)$, let $f_p \in \{0, 1, 2, 3, 4\}$ be the number of fixed links in ∂p and $a_p := 4 - f_p$ its number of active links. For each active link $e \in A_\sigma(\Lambda_L)$, its finite-box *active-partner degree* is

$$\deg_{\mathrm{act}}^{\Lambda_L}(e) := \sum_{p \in \mathcal{P}(\Lambda_L) \,:\, e \in \partial p} (a_p - 1),$$

and analogously $\deg_{\mathrm{act}}^{\mathbb{Z}^d}(e) := \sum_{p \in \mathcal{P}(\mathbb{Z}^d) \,:\, e \in \partial p} (a_p - 1)$ on $\mathbb{Z}^d$, counting the number of active links sharing a plaquette with e.

Proposition 2.1 (Universal forward-forest acyclicity and exact gauge fixing)**.** *For any $d \geq 2$, any $L \geq 2$, and* any *map $\sigma : \mathbb{Z}^d \to \{0, \ldots, d-1\}$:*

1. *The graph $(\Lambda_L, F_\sigma(\Lambda_L))$ is acyclic (a spanning forest of Λ_L).*
2. *Every plaquette $p \in \mathcal{P}(\Lambda_L)$ satisfies $f_p(\Lambda_L) = f_p(\mathbb{Z}^d)$, and every active link $e \in A_\sigma(\Lambda_L)$ satisfies $\deg_{\mathrm{act}}^{\Lambda_L}(e) \leq \deg_{\mathrm{act}}^{\mathbb{Z}^d}(e)$.*
3. *For every gauge-invariant measurable function $\Phi : G^{E(\Lambda_L)} \to \mathbb{C}$, its expectation under $\mu_{\Lambda_L,\beta}$ equals its expectation under the gauge-fixed Gibbs measure $\mu_{\Lambda_L,\beta}^{F_\sigma}$ on $G^{A_\sigma(\Lambda_L)}$ obtained by setting $U_e = I_N$ for all $e \in F_\sigma(\Lambda_L)$.*

Proof. (1) Orient each link $(x, x + e_{\sigma(x)}) \in F_\sigma(\Lambda_L)$ from x to $x + e_{\sigma(x)}$. Every directed edge strictly increases the coordinate sum $S(x) := \sum_{\mu=0}^{d-1} x_\mu$ by $+1$, and every vertex $x \in \Lambda_L$ has out-degree at most 1. Suppose $C \subset F_\sigma(\Lambda_L)$ were a simple cycle, and choose $x_0 \in V(C)$ minimizing $S(x)$ over $V(C)$. Both cycle edges incident to x_0 must lead to neighbors with strictly larger coordinate sum, so both must be oriented out of x_0, giving x_0 out-degree at least 2—a contradiction.

(2) If $p \in \mathcal{P}(\Lambda_L)$, all four edges of ∂p belong to $E(\Lambda_L)$, so $\partial p \cap F_\sigma(\Lambda_L) = \partial p \cap F_\sigma$ and hence $f_p(\Lambda_L) = f_p(\mathbb{Z}^d)$. Since the plaquettes of Λ_L containing $e \in A_\sigma(\Lambda_L)$ form a subset of those of $\mathbb{Z}^d$, $\deg_{\mathrm{act}}^{\Lambda_L}(e) \leq \deg_{\mathrm{act}}^{\mathbb{Z}^d}(e)$.

(3) Since $F_\sigma(\Lambda_L)$ is a forest, each connected component is a rooted tree; successively applying local gauge transformations from leaves toward the root maps every link configuration U to one

with $U_e = I_N$ on $F_\sigma(\Lambda_L)$. By bi-invariance of Haar measure and gauge invariance of both $\mathrm{Re}\,\mathrm{Tr}(U_p)$ and Φ, integrating out the vertex gauge group yields $\mathbb{E}_{\mu_{\Lambda_L,\beta}}[\Phi] = \mathbb{E}_{\mu^{F_\sigma}_{\Lambda_L,\beta}}[\Phi|_{U_{F_\sigma}=I_N}]$. □

Remark 2.2 (Open boxes versus periodic tori). On a periodic torus $\mathbb{T}^d_L = (\mathbb{Z}/L\mathbb{Z})^d$, forward links can wrap around non-contractible cycles (Polyakov loops), so F_σ is not acyclic on $\mathbb{T}^d_L$ without removing $O(L^{d-1})$ links. Working on open boxes Λ_L eliminates all winding cycles while preserving exact finite-volume reflection positivity; in the Dobrushin uniqueness regime (Theorem 5.1), the infinite-volume Gibbs measure μ_β is unique and independent of boundary conditions, so expectations on $\mathbb{T}^d_L$ and Λ_L share the same $L \to \infty$ limit.

We now establish the sharp lower bound on the asymptotic average active-partner degree of *any* acyclic forest gauge fixing on $\mathbb{Z}^d$, regardless of its structure or density.

Theorem 2.3 (Exact discrete Jensen lower bound)**.** *Let $\{F_L \subset E(\Lambda_L)\}_{L\geq 2}$ be any sequence of acyclic forests with asymptotic link density $r := \limsup_{L\to\infty} |F_L|/L^d \in [0,1]$, and let $A_L := E(\Lambda_L) \setminus F_L$. Then the asymptotic average active-partner degree satisfies*

$$\liminf_{L\to\infty} \frac{1}{|A_L|} \sum_{e\in A_L} \deg^{\Lambda_L}_{\mathrm{act}}(e) \;\geq\; D_{\min}(d) := \begin{cases} 2, & d = 2, \\ 7, & d = 3, \\ 6d-12, & d \geq 4. \end{cases} \tag{2}$$

Equality along any subsequence attaining the $\liminf$ *requires $|F_L|/L^d \to 1$ (maximal spanning-forest density) together with $f_p = 2$ ($d = 2$), $f_p \in \{1,2\}$ ($d = 3$), or $f_p \in \{0,1\}$ ($d \geq 4$) on all but $o(L^d)$ plaquettes.*

Proof. Since F_L is acyclic on L^d vertices, $|F_L| \leq L^d - 1$, so $r_L := |F_L|/L^d \leq 1$. On Λ_L, $|E(\Lambda_L)| = dL^d(1 - O(1/L))$ and $|\mathcal{P}(\Lambda_L)| = \binom{d}{2}L^d(1 - O(1/L))$. Each link belongs to at most $2(d-1)$ plaquettes, hence

$$\sum_{p\in\mathcal{P}(\Lambda_L)} f_p \leq 2(d-1)|F_L| = 2(d-1)r_L L^d,$$

so the average active count $\bar{a}_L := \frac{1}{|\mathcal{P}(\Lambda_L)|}\sum_p a_p = 4 - \bar{f}_L$ obeys

$$\bar{a}_L \geq 4 - \frac{4r_L}{d} - O(1/L).$$

Double counting ordered pairs of distinct active links sharing a plaquette gives

$$\sum_{e\in A_L} \deg^{\Lambda_L}_{\mathrm{act}}(e) = \sum_{p\in\mathcal{P}(\Lambda_L)} a_p(a_p - 1).$$

Case $d = 2$: Every integer $a \in \mathbb{Z}$ satisfies $a(a-1) - (3a-4) = (a-2)^2 \geq 0$, with equality iff $a = 2$ (equivalently $f_p = 2$). Because $|\mathcal{P}|/L^2 \to 1$, $|A_L|/L^2 = 2 - r_L + o(1)$, and $\bar{a}_L \geq 4 - 2r_L + o(1)$,

$$\frac{\sum_p a_p(a_p-1)}{|A_L|} \geq \frac{3(4-2r_L)-4}{2-r_L} - o(1) = \frac{8-6r_L}{2-r_L} - o(1) = 6 - \frac{4}{2-r_L} - o(1) \geq 2 - o(1),$$

since $r \mapsto 6 - 4/(2-r)$ is strictly decreasing on $[0,1]$ and equals 2 at $r = 1$.

Case $d = 3$: Every integer $a \in \mathbb{Z}$ satisfies $a(a-1) - (4a-6) = (a-2)(a-3) \geq 0$, with equality iff $a \in \{2,3\}$ (i.e., $f_p \in \{1,2\}$). With $|\mathcal{P}|/L^3 \to 3$, $|A_L|/L^3 = 3 - r_L + o(1)$, and $\bar{a}_L \geq 4 - \frac{4}{3}r_L + o(1)$,

$$\frac{\sum_p a_p(a_p-1)}{|A_L|} \geq \frac{3\big(4(4-\frac{4}{3}r_L)-6\big)}{3-r_L} - o(1) = \frac{30-16r_L}{3-r_L} - o(1) = 16 - \frac{18}{3-r_L} - o(1).$$

Because $r \mapsto 16 - 18/(3-r)$ is strictly decreasing on $[0,1]$, its unique minimum occurs at $r = 1$, where $16 - 18/2 = 7$.

Case $d \geq 4$: Every integer $a \in \mathbb{Z}$ satisfies $a(a-1) - (6a-12) = (a-3)(a-4) \geq 0$, with equality iff $a \in \{3,4\}$ (i.e., $f_p \in \{0,1\}$). With $|\mathcal{P}|/L^d \to \frac{d(d-1)}{2}$, $|A_L|/L^d = d - r_L + o(1)$, and $\bar{a}_L \geq 4 - \frac{4}{d} r_L + o(1)$,

$$\frac{\sum_p a_p(a_p-1)}{|A_L|} \geq \frac{\frac{d(d-1)}{2}\big(6(4-\frac{4}{d}r_L) - 12\big)}{d - r_L} - o(1) = 6(d-1)\left(2 - \frac{d}{d-r_L}\right) - o(1).$$

Because $r \mapsto 2 - d/(d-r)$ is strictly decreasing on $[0,1]$, its unique minimum occurs at $r = 1$, giving $6(d-1)(2-d/(d-1)) = 6d-12$. □

We now show that the discrete Jensen lower bound $D_{\min}(d)$ is *exact* in every dimension $d \geq 2$ via explicit periodic forward direction maps σ_d.

Theorem 2.4 (Extremal forward forests in every dimension $d \geq 2$)**.** *For every $d \geq 2$, there exists a periodic forward direction map $\sigma_d : \mathbb{Z}^d \to \{0, \dots, d-1\}$ whose forward forest F_{σ_d} attains the exact Jensen lower bound $D_{\min}(d)$:*

(1) *($d = 2$, axial): $\sigma_2(x) \equiv 0$ has $f_p \equiv 2$ and uniform active degree $\deg_{\mathrm{act}}(e) \equiv 2$.*

(2) *($d = 3$, cyclic mod-3 balanced forest): $\sigma_3(x_0, x_1, x_2) := -(x_0 + x_1 + x_2) \bmod 3$ satisfies $f_p \in \{1, 2\}$ on every plaquette (on $(\mathbb{Z}/3\mathbb{Z})^3$: 54 plaquettes with $f_p = 1$ and 27 with $f_p = 2$) and has* uniform *active-partner degree $\deg_{\mathrm{act}}(e) \equiv 7$ on every active link (strictly improving axial degree 8 and unfixed degree 12).*

(3) *($d = 4$, mod-2 straight-line forest): Define $\sigma_4(x) := T_4(\sum_{\mu=0}^{3}(x_\mu \bmod 2)\, 2^\mu)$ by the 16-entry lookup table*

$$T_4 = [0,0,2,3,\ 3,1,2,1,\ 1,2,1,3,\ 3,2,0,0]. \tag{3}$$

Then $\sigma_4(x + e_{\sigma_4(x)}) = \sigma_4(x)$ (straight-line property), $f_p \equiv 1$ on every plaquette, and every active link has uniform *active-partner degree $\deg_{\mathrm{act}}(e) \equiv 12$.*

(4) *($d \geq 5$, parity-lifted forest): For every $d \geq 5$, set $s(x) := \sum_{\mu=4}^{d-1} x_\mu \bmod 2$ and*

$$\sigma_d(x) := \sigma_4\big((x_0 + s(x),\ x_1 + s(x),\ x_2,\ x_3)\big) \in \{0,1,2,3\}. \tag{4}$$

Then $f_p \in \{0,1\}$ on every plaquette of $\mathbb{Z}^d$, the asymptotic average active-partner degree equals $D_{\min}(d) = 6d-12$ (strictly improving axial gauge degree $6d-10$), and the active-partner degree spectrum is $\{4d-4,\ 6d-12,\ 6d-10\}$ with maximum degree $\max_e \deg_{\mathrm{act}}(e) = 6d-10$ (matching axial gauge $6d-10$ and improving unfixed $6d-6$).

Proof. Part (1) is immediate. For (2), notice that stepping from x to $x+e_\mu$ shifts $x_0+x_1+x_2 \bmod 3$ by $+1$, hence shifts $\sigma_3 \bmod 3$ by -1. On any plaquette $(x;\mu,\nu)$ ($\mu < \nu$) with $c := \sigma_3(x) \in \{0,1,2\}$, its four edges are fixed according to whether $c = \mu$, $c-1 \equiv \nu$, $c-1 \equiv \mu$, or $c = \nu \pmod 3$. Since $\{c, c-1\}$ has size 2 in $\mathbb{Z}/3\mathbb{Z}$, its intersection with the 2-element set $\{\mu,\nu\}$ has size 1 or 2! Specifically, the four fixed indicators sum to 2 iff $\{c, c-1\} = \{\mu,\nu\}$ (which occurs for exactly 1 of the 3 values of c, giving 27 plaquettes on $(\mathbb{Z}/3\mathbb{Z})^3$ with $f_p = 2$), and sum to 1 for the other 2 values of c (54 plaquettes with $f_p = 1$). Now fix an active link $e = (x, x+e_\mu)$ with $c = \sigma_3(x) \neq \mu$; there are two cases $c \equiv \mu+1$ or $c \equiv \mu+2 \pmod 3$, and e belongs to 4 plaquettes: two forward $(x;\mu,\nu)$ and two backward $(x-e_\nu;\mu,\nu)$ for the two directions $\nu \neq \mu$. Direct checking of the two residues shows that in both cases, out of the 4 plaquettes containing e, *exactly one* has $f_p = 2$ ($a_p - 1 = 1$) and *exactly three* have $f_p = 1$ ($a_p - 1 = 2$), yielding $\deg_{\mathrm{act}}(e) = 1 \cdot 1 + 3 \cdot 2 = 7$ uniformly! (Mechanically verified over all 81 plaquettes and 54 active links in `ForestMassGapCerts.lean`.)

Part (3) is verified exhaustively over all 16 residues of $(\mathbb{Z}/2\mathbb{Z})^4$ in `ForestMassGapCerts.lean` (cf. `GaugeFix.lean` in [3]): for every $y \in (\mathbb{Z}/2\mathbb{Z})^4$, $T_4(y \oplus 2^{T_4(y)}) = T_4(y)$, every one of the 96 plaquettes satisfies $f_p = 1$, and every one of the 48 active links satisfies $\deg_{\mathrm{act}} = 12$.

For (4), denote $y(x) \in (\mathbb{Z}/2\mathbb{Z})^4$ the mod-2 projection of $(x_0 + s(x), x_1 + s(x), x_2, x_3)$. The crucial finite property of T_4 in (3) is that flipping bits 0 and 1 simultaneously (mask $3 = 0\text{b}0011$) is completely fixed-point-free:

$$T_4(y \oplus 3) \neq T_4(y) \qquad \forall\, y \in \{0, \dots, 15\}, \tag{5}$$

verified directly from (3) (pairs $\{0,3\} \mapsto \{0,3\}$, $\{1,2\} \mapsto \{0,2\}$, $\{4,7\} \mapsto \{3,1\}$, $\{5,6\} \mapsto \{1,2\}$, $\{8,11\} \mapsto \{1,3\}$, $\{9,10\} \mapsto \{2,1\}$, $\{12,15\} \mapsto \{3,0\}$, $\{13,14\} \mapsto \{2,0\}$). Consider any plaquette $p = (x; \mu, \nu)$ $(\mu < \nu)$:

- If $\nu \leq 3$, s is constant on p, so $f_p = 1$ by (3).
- If $\mu \leq 3 < \nu$, the two ν-edges cannot be fixed since $\sigma_d \in \{0,1,2,3\}$. The two μ-edges sit at x and $x + e_\nu$, whose y-vectors differ by $\oplus 3$; by (5), at most one of $\sigma_d(x)$ and $\sigma_d(x + e_\nu)$ equals μ, so $f_p \in \{0,1\}$.
- If $4 \leq \mu < \nu$, no edge of p is fixed, so $f_p = 0$.

Hence $f_p \in \{0,1\}$ everywhere and $\overline{\deg_{\text{act}}} = 6d - 12$ by Theorem 2.3. Computing individual link degrees: for an active link $(x, x + e_\mu)$ with $\mu \leq 3$, it has 12 partners in the 4D slice plus $2(d-4)$ mixed plaquettes, each contributing 2 if $T_4(y(x) \oplus 3) = \mu$ (giving $12 + 4(d-4) = 4d - 4$, for 1 link per vertex) and 3 otherwise (giving $12 + 6(d-4) = 6d - 12$, for 2 links per vertex). For a high-dimensional link $(x, x + e_\mu)$ with $\mu \geq 4$, it belongs to $2(d-5)$ high plaquettes ($f_p = 0$, contributing $6(d-5)$) and 8 mixed plaquettes: among the 4 forward plaquettes $(x; \nu, \mu)$ $(\nu \leq 3)$, exactly two have $f_p = 1$ (for $\nu = T_4(y(x))$ and the distinct $\nu' = T_4(y(x) \oplus 3)$), contributing $2 \cdot 2 + 2 \cdot 3 = 10$; by the straight-line property of T_4 (every vertex has in-degree 1 in F_{σ_4}), the 4 backward plaquettes $(x - e_\nu; \nu, \mu)$ likewise have exactly two with $f_p = 1$ and two with $f_p = 0$, contributing another 10. Total: $6(d-5) + 20 = 6d - 10$. Checking the average: $\frac{1(4d-4) + 2(6d-12) + (d-4)(6d-10)}{d-1} = 6d - 12$. (Note that pure axial gauge $\sigma(x) \equiv 0$ has 2 plaquettes with $f_p = 2$ and $2(d-2)$ plaquettes with $f_p = 0$ per active link, giving uniform degree $2 \cdot 1 + 2(d-2) \cdot 3 = 6d - 10$.) □

3. Exact d-Dimensional $(2d-1)$-Spin Even-Vertex Star Contraction

We now analyze a complementary block-level gauge-invariant contraction scheme that works in every dimension $d \geq 3$ and for every compact Lie group G equipped with a bi-invariant metric d_G.

Let $V_{\text{even}} := \{x \in \mathbb{Z}^d : \sum_{\mu=0}^{d-1} x_\mu \equiv 0 \pmod 2\}$. For each $x \in V_{\text{even}} \cap \Lambda_L$, define its incident link star $S_x := \{e \in E(\Lambda_L) : x \in \partial e\}$, with every link $e = (x, z) \in S_x$ oriented outward from x to its odd neighbor z. Because every edge of Λ_L joins one even vertex to one odd vertex, $\{S_x : x \in V_{\text{even}} \cap \Lambda_L\}$ is an exact partition of $E(\Lambda_L)$. Two distinct even stars S_x, S_y share a plaquette iff $y = x \pm e_\mu \pm e_\nu$ $(\mu \neq \nu)$, in which case they share a *unique* plaquette $p(x, y)$ containing two links $e_1, e_2 \in S_x$ and two links $f_1, f_2 \in S_y$.

On each star S_x, define the gauge-orbit space $\mathcal{X}_x := G^{S_x}/G_x$ (modulo left multiplication $U_e \mapsto g_x U_e$ for all $e \in S_x$) equipped with the quotient ℓ^1 metric

$$d_x([U_{S_x}], [V_{S_x}]) := \inf_{g_x \in G} \sum_{e \in S_x} d_G(g_x U_e, V_e). \tag{6}$$

Because the two links $f_1 = (y, z_1)$ and $f_2 = (y, z_2)$ of S_y on $p(x, y)$ both originate at y, they enter $p(x, y)$ solely through the combination $U_{f_1}^{-1} U_{f_2}$, which is strictly invariant under $U_f \mapsto g_y U_f$. Hence the conditional Gibbs law $\mu_x(\cdot \mid \omega)$ on $\mathcal{X}_x$ depends on the exterior configuration ω solely through the gauge orbits $\{[U_{S_y}] : y \in V_{\text{even}} \setminus \{x\}\}$.

Assume the single-link conditional law ν_H of (1) satisfies the *single-link Wasserstein Lipschitz bound* with constant $c = c(G, d, \beta) \geq 0$: whenever two staple sums $H, \tilde{H}$ differ in a single staple

by replacing one neighboring link V with $\tilde{V}$,

$$W_1^{d_G}(\nu_H, \nu_{\tilde{H}}) \le c\, d_G(V, \tilde{V}). \tag{7}$$

Theorem 3.1 (Exact d-dimensional $(2d-1)$-spin even-star contraction and crossover)**.** *Let $d \ge 3$ and suppose the single-link Wasserstein bound (7) holds with constant $c > 0$ satisfying $1 - 2(d-2)c - 2(d-1)c^2 > 0$. Define*

$$s_d(c) := \frac{c(1+c)}{1 - 2(d-2)c - 2(d-1)c^2}, \qquad \lambda_d(c) := 2(d-1)\, s_d(c). \tag{8}$$

Then on every open box Λ_L, for any even vertex $y \in V_{\mathrm{even}} \cap \Lambda_L$ and any two exterior-orbit configurations $\omega, \tilde{\omega}$ differing solely at y,

$$\sum_{x \in (V_{\mathrm{even}} \cap \Lambda_L) \setminus \{y\}} W_1^{d_x}\big(\mu_x(\cdot \mid \omega),\, \mu_x(\cdot \mid \tilde{\omega})\big) \;\le\; \lambda_d(c)\, d_y(\omega_y, \tilde{\omega}_y). \tag{9}$$

Consequently, column-sum contraction $\lambda_d(c) < 1$ holds iff

$$c < c_*^{\mathrm{star}}(d) := \frac{\sqrt{4d^2 - 8d + 5} - (2d-3)}{4(d-1)}. \tag{10}$$

Moreover, comparing (10) against the optimal forward-forest threshold $1/D_{\min}(d)$ exhibits a sharp dimensional crossover:

- *In $d = 3$: $1/D_{\min}(3) = 1/7 = 0.142857\cdots > c_*^{\mathrm{star}}(3) = (\sqrt{17} - 3)/8 = 0.140388\ldots$ (the mod-3 balanced forest strictly outperforms the 3D even-vertex star).*
- *For every $d \ge 4$: $c_*^{\mathrm{star}}(d) > 1/(6d-12) = 1/D_{\min}(d)$ (the $(2d-1)$-spin even-vertex star strictly outperforms every spanning forest; e.g., $c_*^{\mathrm{star}}(4) = (\sqrt{37} - 5)/12 = 0.090230\cdots > 1/12 = 0.083333\ldots$).*

Proof. Fix $y \in V_{\mathrm{even}} \cap \Lambda_L$ and choose representatives $U_{S_y}, \tilde{U}_{S_y}$ attaining the infimum in (6), so $d_y(\omega_y, \tilde{\omega}_y) = \sum_{f \in S_y} \delta_f$ with $\delta_f := d_G(U_f, \tilde{U}_f)$. Fix an even neighbor $x \sim y$; the unique shared plaquette $p(x,y)$ contains two links $e_1, e_2 \in S_x$ and two links $f_1, f_2 \in S_y$. Pass from U_{S_y} to $\tilde{U}_{S_y}$ in two single-link steps changing f_1 first (by δ_{f_1}) and f_2 second (by δ_{f_2}).

In the first step (changing f_1 with f_2 fixed), choose the local gauge representative on S_x fixing $U_{e_1} = I_N$, leaving at most $2d-1$ active spins $S_x' := S_x \setminus \{e_1\}$. For any coupling $(U_{S_x'}, \tilde{U}_{S_x'})$ on S_x' (with $U_{e_1} = \tilde{U}_{e_1} = I_N$), the quotient metric obeys $d_x([U_{S_x}], [\tilde{U}_{S_x}]) \le \sum_{e \in S_x'} d_G(U_e, \tilde{U}_e)$. Because e_1 is fixed to I_N, the link f_1 couples to *only the single spin* $e_2 \in S_x'$ (via $p(x,y)$), shifting the single-site conditional law of e_2 by at most $c\delta_{f_1}$ and leaving the single-site conditional laws of all other spins in S_x' unchanged.

Inside S_x', two spins share a plaquette iff they lie along distinct coordinate axes (the complete multipartite graph $K_{1,2,\ldots,2}$ on $2d-1$ vertices, or an induced subgraph near $\partial\Lambda_L$). Let $A_{S_x'}$ be the adjacency matrix of $K_{1,2,\ldots,2}$: the unique spin e_{opp} opposite e_1 has degree $2(d-1)$ and the remaining $2(d-1)$ spins each have degree $2d-3$. Its Perron–Frobenius eigenvalue λ_{PF} satisfies $\lambda_{\mathrm{PF}}^2 - 2(d-2)\lambda_{\mathrm{PF}} - 2(d-1) = 0$, so $\rho(cA_{S_x'}) < 1$ iff $1 - 2(d-2)c - 2(d-1)c^2 > 0$. By the Wasserstein analogue of the finite-volume Dobrushin comparison theorem [7, Theorem 8.20]—proved directly by running a coupled single-site random-scan Glauber chain on S_x' weighted by the positive vector $u := (I - cA_{S_x'}^{\top})^{-1}\mathbf{1}$, whose stationary coupling obeys $\sum_e \mathbb{E}[d_G(U_e, \tilde{U}_e)] \le \sum_e u_e b_e = \mathbf{1}^{\top}(I - cA_{S_x'})^{-1} b$ for perturbation $b = c\delta_{f_1}\, \mathbf{e}_{e_2}$ (and entrywise monotonicity handles induced boundary subgraphs)—we obtain

$$W_1^{d_x}\big(\mu_x(\cdot \mid U_{f_1}), \mu_x(\cdot \mid \tilde{U}_{f_1})\big) \le \sum_{e \in S_x'} v(e), \qquad (I - cA_{S_x'})v = c\delta_{f_1}\, \mathbf{e}_{e_2}.$$

By symmetry of $K_{1,2,\dots,2}$ fixing e_2, the vector v takes four values: $v_1 = v(e_2)$, $v_2 = v(-e_2)$, $v_3 = v(e_{\mathrm{opp}})$, and $v_4 = v(e)$ on the $2(d-2)$ spins orthogonal to both e_1 and e_2. Setting $W := v_1+v_2$ and $Y := v_3 + 2(d-2)v_4$, the linear system reads

$$\begin{aligned} v_1 &= c\delta_{f_1} + cY, & v_2 &= cY, \\ v_3 &= cW + 2(d-2)cv_4, & v_4 &= cW + cv_3 + 2(d-3)cv_4. \end{aligned}$$

Subtracting the fourth equation from the third gives $(1+c)v_3 = (1+2c)v_4$, whence

$$Y = \left(\frac{1+2c}{1+c} + 2(d-2)\right) v_4 = \frac{2d-3+2(d-1)c}{1+c}\, v_4.$$

Substituting $W = c\delta_{f_1} + 2cY$ and $v_3 = \frac{1+2c}{1+c}v_4$ into the third equation and multiplying by $(1+c)$ yields

$$\Big[1 - 2(d-3)c - 2(3d-5)c^2 - 4(d-1)c^3\Big] v_4 = c^2(1+c)\delta_{f_1}.$$

The cubic polynomial on the left factors identically as $(1+2c)\big(1-2(d-2)c-2(d-1)c^2\big)$ for every $d \geq 2$! Therefore

$$(1+2c)Y = \frac{c^2\big(2d-3+2(d-1)c\big)}{1-2(d-2)c-2(d-1)c^2}\,\delta_{f_1},$$

and the total ℓ^1 response $S_{\mathrm{tot}} = W + Y = c\delta_{f_1} + (1+2c)Y$ simplifies by exact cancellation of $2(d-1)c^2$ to

$$S_{\mathrm{tot}} = c\delta_{f_1}\,\frac{1-2(d-2)c-2(d-1)c^2+(2d-3)c+2(d-1)c^2}{1-2(d-2)c-2(d-1)c^2} = s_d(c)\,\delta_{f_1}.$$

Applying the identical argument to the second step (changing f_2 by δ_{f_2}, now fixing the gauge representative $U_{e_2} = I_N$ on S_x, which is valid since $W_1^{d_x}$ is a metric on the orbit space $\mathcal{X}_x$) yields $W_1^{d_x}(\mu_x(\cdot|\omega), \mu_x(\cdot|\tilde{\omega})) \leq s_d(c)(\delta_{f_1} + \delta_{f_2})$.

Summing over all $x \in (V_{\mathrm{even}} \cap \Lambda_L)\setminus\{y\}$: each link $f \in S_y$ belongs to at most $2(d-1)$ plaquettes, each connecting y to a distinct even neighbor $x \sim y$. Thus

$$\sum_{x\sim y}\big(\delta_{f_1(x,y)} + \delta_{f_2(x,y)}\big) \leq 2(d-1)\sum_{f\in S_y}\delta_f = 2(d-1)\,d_y(\omega_y, \tilde{\omega}_y),$$

proving (9). The inequality $\lambda_d(c) < 1$ rearranges directly to $(4d-6)c + 4(d-1)c^2 < 1$ (which automatically enforces $1-2(d-2)c-2(d-1)c^2 > 0$), giving the positive root (10). Finally, for $d = 3$ and $c = 1/7$, $(4\cdot 3 - 6)/7 + 4\cdot 2/49 = 50/49 > 1$, so $1/7 > c_*^{\mathrm{star}}(3)$. For any $d = k+4$ ($k \geq 0$) and $c = 1/(6d-12)$, clearing denominators gives

$$(6d-12)^2 - \big((4d-6)(6d-12) + 4(d-1)\big) = 12k^2 + 32k + 12 \geq 12 > 0,$$

proving $c_*^{\mathrm{star}}(d) > 1/(6d-12)$ universally for all $d \geq 4$. □

Remark 3.2 (Structural obstruction to larger even-vertex clusters). One might ask whether grouping two diagonally adjacent even stars (15 spins after fixing 1 link) or the eight even vertices of a 4D hypercube yields a larger linear resolvent threshold than $c_*^{\mathrm{star}}(4) = 0.09023$. Direct matrix inversion of the corresponding internal Dobrushin resolvents $(I - cA_{\mathrm{block}})^{-1}$ shows the opposite: the 2-even diagonal pair degrades the 4D SU(2) threshold from 0.3609 to 0.340, and the 8-even hypercube degrades it to 0.267, because adding internal plaquettes increases $\rho(cA_{\mathrm{block}})$ faster than surface-to-volume ratios reduce boundary couplings.

4. Bakry–Émery–Weingarten–Russo–Dye Contraction on SU(N)

We now derive explicit single-link Wasserstein Lipschitz constants $c(G,d,\beta)$ in (7) for $G = \mathrm{SU}(N)$ ($N \geq 2$). Equip $\mathrm{SU}(N)$ with the bi-invariant Riemannian metric g induced by the Hilbert–Schmidt inner product on $\mathfrak{su}(N) = \{X \in M_N(\mathbb{C}) : X^* = -X,\ \mathrm{Tr}\, X = 0\}$,

$$g(X,Y) = \langle X, Y\rangle_{\mathrm{HS}} := \mathrm{Tr}(X^*Y) = -\,\mathrm{Tr}(XY),$$

and let d_{HS} be the associated bi-invariant geodesic distance on $\mathrm{SU}(N)$.

Lemma 4.1 (Exact Ricci curvature and Wilson Hessian on SU(N))**.** *For every $N \geq 2$, the bi-invariant manifold $(\mathrm{SU}(N), g)$ has constant Ricci curvature*

$$\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2}\, g. \tag{11}$$

For any staple sum $H = \sum_{j=1}^{M} S_j$ with $S_j \in \mathrm{SU}(N)$ and $M \leq 2(d-1)$, the single-link Wilson potential $V_H(U) := -\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}(U^*H)$ satisfies

$$\mathrm{Hess}\, V_H(U)(X,X) = \frac{\beta}{N}\,\mathrm{Tr}\big((X^*X)\ \mathrm{Sym}(U^*H)\big) \geq -\frac{M\beta}{N}\|X\|_{\mathrm{HS}}^2 \geq -\frac{2(d-1)\beta}{N}\|X\|_{\mathrm{HS}}^2 \tag{12}$$

for all $X \in \mathfrak{su}(N)$, where $\mathrm{Sym}(A) := \frac{1}{2}(A + A^*)$. Hence the single-link Gibbs law $d\nu_H(U) = Z(H)^{-1}e^{-V_H(U)}\,dU$ satisfies the Bakry–Émery curvature-dimension condition $\mathrm{CD}(K_{\mathrm{BE}}, \infty)$ with

$$K_{\mathrm{BE}}(N,d,\beta) := \frac{N}{2} - \frac{2(d-1)\beta}{N}. \tag{13}$$

Proof. For a compact semisimple Lie group with bi-invariant metric g, the Ricci tensor equals $\mathrm{Ric}(X,Y) = -\frac{1}{4}B(X,Y)$ [8], where $B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ is the Killing form. Because $\mathfrak{su}(N)$ is a real form of $\mathfrak{sl}(N,\mathbb{C})$ whose Killing form is $2N\,\mathrm{Tr}(XY)$, we have $B_{\mathfrak{su}(N)}(X,Y) = 2N\,\mathrm{Tr}(XY) = -2N\,\mathrm{Tr}(X^*Y) = -2Ng(X,Y)$, giving $\mathrm{Ric}_{\mathrm{SU}(N)} = -\frac{1}{4}(-2Ng) = \frac{N}{2}g$. (As a check, $(\mathrm{SU}(2), g)$ is isometric to $S^3(\sqrt{2})$, whose Ricci tensor is $\frac{3-1}{(\sqrt{2})^2}g = g = \frac{2}{2}g$.)

Along the geodesic $\gamma(t) = Ue^{tX}$ ($X \in \mathfrak{su}(N)$), differentiating $V_H(\gamma(t)) = -\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}(e^{-tX}U^*H)$ twice at $t = 0$ and using $-X^2 = X^*X \succeq 0$ gives

$$\mathrm{Hess}\, V_H(U)(X,X) = -\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}(X^2U^*H) = \frac{\beta}{N}\,\mathrm{Tr}\big((X^*X)\ \mathrm{Sym}(U^*H)\big).$$

Since $X^*X \succeq 0$ and $\|\mathrm{Sym}(U^*H)\|_{\mathrm{op}} \leq \|U^*H\|_{\mathrm{op}} = \|H\|_{\mathrm{op}} \leq \sum_{j=1}^{M}\|S_j\|_{\mathrm{op}} = M \leq 2(d-1)$, we obtain (12), and therefore $\mathrm{Ric}_{\nu_H} = \mathrm{Ric}_{\mathrm{SU}(N)} + \mathrm{Hess}\, V_H \geq K_{\mathrm{BE}}(N,d,\beta)\, g$. □

Proposition 4.2 (Semigroup–Kantorovich L^2-gradient bound and universal 't Hooft N^2-scaling)**.** *Let $N \geq 2$, $d \geq 2$, and $\beta \geq 0$ satisfy $K_{\mathrm{BE}}(N,d,\beta) > 0$. Define the uniform projected-gradient variance supremum*

$$G_N(d,\beta) := \sup_{\substack{\|H\|_{\mathrm{op}} \leq 2(d-1) \\ A,B \in \mathrm{SU}(N),\ \hat{W} \in \mathfrak{su}(N),\ \|\hat{W}\|_{\mathrm{HS}}=1}} \mathbb{E}_{\nu_H}\Big[\big\|\,\mathrm{proj}_{\mathfrak{su}(N)}(U^*A\hat{W}B)\big\|_{\mathrm{HS}}^2\Big] \ \leq\ 1. \tag{14}$$

Then the single-link Wasserstein bound (7) holds for $(\mathrm{SU}(N), d_{\mathrm{HS}})$ with

$$c_N(d,\beta) \ :=\ \frac{(\beta/N)\,\sqrt{G_N(d,\beta)}}{K_{\mathrm{BE}}(N,d,\beta)} \ \leq\ c_N^{\mathrm{BE}}(d,\beta) := \frac{\beta/N}{\frac{N}{2} - \frac{2(d-1)\beta}{N}}. \tag{15}$$

In particular, setting $G_N \leq 1$, any geometric contraction threshold $c < c_*$ (forest or star) is satisfied for every *$N \geq 2$ on the universal 't Hooft N^2-scaling window*

$$\beta \ <\ \frac{c_*}{2\big(1 + 2(d-1)c_*\big)}\, N^2. \tag{16}$$

For the 4D 7-spin star ($c_* = (\sqrt{37}-5)/12$), this certifies a volume-uniform mass gap on $\beta \le 0.02925\,N^2$ for all $N \ge 2$ simultaneously; for the 3D mod-3 forest ($c_* = 1/7$), it certifies $\beta \le 0.04545\,N^2$ for all $N \ge 2$.

Proof. Suppose H and $\tilde{H}$ differ in a single staple AVB versus $A\tilde{V}B$ ($A, B \in \mathrm{SU}(N)$; staples containing V^{-1} follow identically since $V_s^{-1} = e^{-sW}V^{-1}$ and d_{HS} is bi-invariant). Connect V to $\tilde{V}$ by a constant-speed minimizing geodesic $V_s = Ve^{sW}$ ($s \in [0,1]$, $W \in \mathfrak{su}(N)$ with $\|W\|_{\mathrm{HS}} = d_{\mathrm{HS}}(V,\tilde{V})$), and set $H_s := H - AVB + AV_sB$, so $\dot{H}_s = AV_sWB$ and $\|H_s\|_{\mathrm{op}} \le 2(d-1)$ for all $s \in [0,1]$. Let $\nu_s := \nu_{H_s}$, let $L_s := \Delta - \nabla V_{H_s} \cdot \nabla$ be its Langevin generator with semigroup $P_t^{(s)} = e^{tL_s}$, and let $h_s(U) := -\partial_s V_{H_s}(U) = \frac{\beta}{N}\operatorname{Re}\operatorname{Tr}(U^*\dot{H}_s)$.

By Lemma 4.1, ν_s satisfies $\mathrm{CD}(K_{\mathrm{BE}}, \infty)$, so the Bakry–Émery gradient commutation [1, 2] gives $|\nabla P_t^{(s)}\phi| \le e^{-K_{\mathrm{BE}}t}P_t^{(s)}|\nabla\phi|$ pointwise. For any smooth 1-Lipschitz test function $\phi : \mathrm{SU}(N) \to \mathbb{R}$ ($\|\nabla\phi\|_\infty \le 1$), symmetry of L_s in $L^2(\nu_s)$ and ergodicity ($P_\infty^{(s)}\phi = \int \phi\, d\nu_s$) yield

$$\begin{aligned}\frac{d}{ds}\int \phi\, d\nu_s = \mathrm{Cov}_{\nu_s}(\phi, h_s) &= -\int_0^\infty \frac{d}{dt}\left(\int (P_t^{(s)}\phi)\, h_s\, d\nu_s\right) dt \\ &= \int_0^\infty \int \langle \nabla P_t^{(s)}\phi,\, \nabla h_s\rangle_{\mathrm{HS}}\, d\nu_s\, dt.\end{aligned}$$

Applying Cauchy–Schwarz pointwise and $|\nabla P_t^{(s)}\phi| \le e^{-K_{\mathrm{BE}}t}$ gives

$$\left|\frac{d}{ds}\int \phi\, d\nu_s\right| \le \left(\int_0^\infty e^{-K_{\mathrm{BE}}t}\, dt\right)\mathbb{E}_{\nu_s}\big[\|\nabla h_s\|_{\mathrm{HS}}\big] \le \frac{1}{K_{\mathrm{BE}}}\Big(\mathbb{E}_{\nu_s}\big[\|\nabla h_s\|_{\mathrm{HS}}^2\big]\Big)^{1/2}.$$

Now $\nabla h_s(U) = \frac{\beta}{N}\operatorname{proj}_{\mathfrak{su}(N)}(U^*\dot{H}_s) = \frac{\beta}{N}\|W\|_{\mathrm{HS}}\operatorname{proj}_{\mathfrak{su}(N)}(U^*(AV_s)\hat{W}B)$ where $\hat{W} = W/\|W\|_{\mathrm{HS}} \in \mathfrak{su}(N)$ has unit Hilbert–Schmidt norm. Integrating over $s \in [0,1]$ and taking the supremum over 1-Lipschitz ϕ (Kantorovich–Rubinstein duality) proves (15). Since $\operatorname{proj}_{\mathfrak{su}(N)}$ is an orthogonal projection and U, AV_s, B are unitary, $\|\operatorname{proj}_{\mathfrak{su}(N)}(U^*AV_s\hat{W}B)\|_{\mathrm{HS}}^2 \le \|\hat{W}\|_{\mathrm{HS}}^2 = 1$, giving $G_N \le 1$ and solving $c_N^{\mathrm{BE}}(d,\beta) < c_*$ yields (16). □

To sharpen (15) substantially beyond the trivial bound $G_N \le 1$, we exploit the fact that on Haar-random $U \in \mathrm{SU}(N)$ the projection onto skew-Hermitian matrices captures only half of the squared norm ($\mathbb{E}_{\mathrm{Haar}}[\|\operatorname{proj}_{\mathfrak{su}(N)}(U^*Y)\|_{\mathrm{HS}}^2] \le 1/2$), while controlling the Gibbs tilt via exact $k = 2$ Weingarten calculus [4] and Russo–Dye convexity [11].

Theorem 4.3 (Exact Weingarten variance and Russo–Dye character-positivity bound)**.** *Let $N = 3$ or $N \ge 5$, let $M = 2(d-1)$, and let*

$$Z_{\mathrm{SU}(N)}(t) := \int_{\mathrm{SU}(N)} \exp\big(t\,\operatorname{Re}\operatorname{Tr}(U)\big)\, dU. \tag{17}$$

Then the projected-gradient variance $G_N(d,\beta)$ in (14) satisfies

$$G_N(d,\beta) \;\le\; \frac{1}{2} + \kappa_N\sqrt{Z_{\mathrm{SU}(N)}\left(\frac{2M\beta}{N}\right) - 1}, \qquad \text{where } \kappa_N := \begin{cases}\sqrt{5/192}, & N = 3,\\ 1/(2N), & N \ge 5.\end{cases} \tag{18}$$

Furthermore, for $N = 3$ and any $0 \le t < 6$, $Z_{\mathrm{SU}(3)}(t)$ obeys the zero-quadrature exact rational series enclosure

$$Z_{\mathrm{SU}(3)}(t) \;\le\; \sum_{m=0}^{16}\frac{\mu_m}{m!}\,t^m \;+\; \frac{(3t)^{17}}{17!\,(1-3t/18)}, \tag{19}$$

where the exact rational moments $\mu_m := \int_{\mathrm{SU}(3)}(\operatorname{Re}\operatorname{Tr} U)^m\, dU$ for $m = 0, \dots, 16$ are listed in Table 1.

Proof. Set $Y := A\hat{W}B$, so $\|Y\|_{\mathrm{HS}} = 1$ and $Y^*Y = B^*(\hat{W}^*\hat{W})B$. For any $M_0 \in M_N(\mathbb{C})$, the real orthogonal projection onto $\mathfrak{su}(N)$ is $\mathrm{proj}_{\mathfrak{su}(N)}(M_0) = \frac{M_0 - M_0^*}{2} - \frac{\mathrm{Tr}(M_0 - M_0^*)}{2N} I_N$, whence

$$\|\mathrm{proj}_{\mathfrak{su}(N)}(M_0)\|_{\mathrm{HS}}^2 \le \left\| \frac{M_0 - M_0^*}{2} \right\|_{\mathrm{HS}}^2 = \frac{1}{2}\|M_0\|_{\mathrm{HS}}^2 - \frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}(M_0^2).$$

Setting $M_0 = U^*Y$ (so $\|M_0\|_{\mathrm{HS}} = 1$) gives $\|\mathrm{proj}_{\mathfrak{su}(N)}(U^*Y)\|_{\mathrm{HS}}^2 \le \frac{1}{2} + h_0(U)$ where $h_0(U) := -\frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}((U^*Y)^2)$.

Let $Z_Y(U) := \mathrm{Tr}((U^*Y)^2)$. Because $Z_Y(U)$ is homogeneous of degree -2 in U and $N \ge 3$, $\mathbb{E}_{\mathrm{Haar}}[Z_Y(U)] = 0$, so $\mathbb{E}_{\mathrm{Haar}}[h_0] = 0$. Similarly, $Z_Y(U)^2$ is homogeneous of degree -4 in U; since $-4 \not\equiv 0 \pmod N$ whenever $N = 3$ or $N \ge 5$ (the determinant invariant $\det U = 1$ only contributes at $N = 4$!), we have $\mathbb{E}_{\mathrm{SU}(N)}[Z_Y(U)^2] = 0$. Therefore

$$\mathbb{E}_{\mathrm{SU}(N)}[h_0(U)^2] = \frac{1}{16}\,\mathbb{E}_{\mathrm{SU}(N)}\big[Z_Y^2 + \overline{Z_Y}^2 + 2|Z_Y|^2\big] = \frac{1}{8}\,\mathbb{E}_{\mathrm{U}(N)}\big[|Z_Y(U)|^2\big].$$

Evaluating $\mathbb{E}_{\mathrm{U}(N)}[|Z_Y(U)|^2] = \sum \mathbb{E}_{\mathrm{U}(N)}[U_{pq}U_{rs}\overline{U_{ba}U_{dc}}]Y_{bc}Y_{da}Y^*_{qr}Y^*_{sp}$ by the exact $k = 2$ unitary Weingarten formula [4],

$$\mathbb{E}_{\mathrm{U}(N)}[U_{pq}U_{rs}\overline{U_{ba}U_{dc}}] = \frac{\delta_{pb}\delta_{qa}\delta_{rd}\delta_{sc} + \delta_{pd}\delta_{qc}\delta_{rb}\delta_{sa}}{N^2-1} - \frac{\delta_{pb}\delta_{qc}\delta_{rd}\delta_{sa} + \delta_{pd}\delta_{qa}\delta_{rb}\delta_{sc}}{N(N^2-1)},$$

contracts the first two terms to $(\mathrm{Tr}(Y^*Y))^2 = 1$ each and the last two terms to $\mathrm{Tr}((Y^*Y)^2)$ each, yielding

$$\mathbb{E}_{\mathrm{SU}(N)}[h_0(U)^2] = \frac{1}{4(N^2-1)}\left(1 - \frac{\mathrm{Tr}((Y^*Y)^2)}{N}\right).$$

For general $N \ge 5$, Cauchy–Schwarz on the eigenvalues of $Y^*Y \succeq 0$ gives $\mathrm{Tr}((Y^*Y)^2) \ge (\mathrm{Tr}(Y^*Y))^2/N = 1/N$, hence $\mathbb{E}_{\mathrm{SU}(N)}[h_0^2] \le \frac{1}{4(N^2-1)}(1 - \frac{1}{N^2}) = \frac{1}{4N^2}$. For $N = 3$, the eigenvalues $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ of the traceless unit-HS Hermitian matrix $i\hat{W}$ satisfy $\sum \mu_k = 0$ and $\sum \mu_k^2 = 1$, whence $\sum_{j<k} \mu_j\mu_k = -\frac{1}{2}$ and squaring gives $\sum_{j<k} \mu_j^2\mu_k^2 = \frac{1}{4}$, so $\mathrm{Tr}((Y^*Y)^2) = \sum \mu_k^4 = 1 - 2(\frac{1}{4}) = \frac{1}{2}$ *identically* for every unit-HS $\hat{W} \in \mathfrak{su}(3)$! Thus

$$\mathbb{E}_{\mathrm{SU}(3)}[h_0(U)^2] = \frac{1}{4(9-1)}\left(1 - \frac{1/2}{3}\right) = \frac{5}{192}.$$

Next, since $\mathbb{E}_{\mathrm{Haar}}[h_0] = 0$, Cauchy–Schwarz yields

$$\mathbb{E}_{\nu_H}[h_0] = \mathbb{E}_{\mathrm{Haar}}\left[h_0\left(\frac{d\nu_H}{d\mathrm{Haar}} - 1\right)\right] \le \kappa_N \sqrt{\chi^2(\nu_H \,\|\, \mathrm{Haar})}.$$

By Jensen's inequality, $Z(H) = \mathbb{E}_{\mathrm{Haar}}[e^{\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}(U^*H)}] \ge e^0 = 1$, so

$$\chi^2(\nu_H \,\|\, \mathrm{Haar}) = \frac{Z(2H)}{Z(H)^2} - 1 \le Z(2H) - 1.$$

To bound $\max_{\|H\|_{\mathrm{op}} \le M} Z(2H)$: the map $H \mapsto Z(2H) = \int_{\mathrm{SU}(N)} \exp(\frac{2\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}(U^*H))\,dU$ is continuous and convex on the compact convex operator-norm ball $B_M = \{\|H\|_{\mathrm{op}} \le M\}$. By the Bauer maximum principle and the Russo–Dye theorem [11], its maximum is attained at an extreme point $H = MV$ with $V \in \mathrm{U}(N)$. Writing $V = e^{i\phi}W$ with $W \in \mathrm{SU}(N)$ and $\phi \in \mathbb{R}$, left Haar invariance ($U \mapsto WU$) reduces $Z(2MV)$ to

$$Z(2Me^{i\phi}W) = \int_{\mathrm{SU}(N)} \exp\Big(t\,\mathrm{Re}(e^{i\phi}\,\mathrm{Tr}\,U)\Big)\,dU = \sum_{a,b\ge 0} \frac{(t/2)^{a+b}}{a!\,b!}\,c_{a,b}\,\cos\big((a-b)\phi\big),$$

where $t = 2M\beta/N$ and $c_{a,b} := \int_{\mathrm{SU}(N)} (\operatorname{Tr} U)^a (\overline{\operatorname{Tr} U})^b \, dU$. By Schur orthogonality,

$$c_{a,b} = \dim \operatorname{Hom}_{\mathrm{SU}(N)}\big(\mathbf{1},\, \mathbf{N}^{\otimes a} \otimes \overline{\mathbf{N}}^{\otimes b}\big) \in \mathbb{Z}_{\geq 0}$$

is non-negative for every $a, b \geq 0$. Hence $\cos((a-b)\phi) \leq 1$ termwise gives $Z(2Me^{i\phi}W) \leq Z(2MI_N) = Z_{\mathrm{SU}(N)}(t)$, establishing (18).

Finally, on SU(3), $|\operatorname{Re}\operatorname{Tr} U| \leq 3$ everywhere, so the tail $\sum_{m=17}^{\infty} \frac{\mu_m}{m!} t^m \leq \sum_{m=17}^{\infty} \frac{(3t)^m}{m!} \leq \frac{(3t)^{17}}{17!\,(1-3t/18)}$ for $3t < 18$. Each exact rational moment μ_m in Table 1 is obtained with zero numerical quadrature by constant-term extraction in the Weyl integration formula: on the maximal torus $(z_1, z_2, z_1^{-1}z_2^{-1}) \in \mathbb{T}^2$, μ_m equals the constant term of $\frac{1}{6}|\Delta(z_1, z_2)|^2 \big(\frac{1}{2}(z_1 + z_2 + z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1} + z_1 z_2)\big)^m$, where $\Delta = (z_1 - z_2)(z_1 - z_1^{-1}z_2^{-1})(z_2 - z_1^{-1}z_2^{-1})$. □

TABLE 1. Exact rational SU(3) Haar moments $\mu_m = \int_{\mathrm{SU}(3)} (\operatorname{Re}\operatorname{Tr} U)^m \, dU$ for $m = 0, \ldots, 16$, computed by exact bivariate Laurent-polynomial constant-term extraction (zero quadrature).

m	μ_m	m	μ_m	m	μ_m
0	1	6	65/32	12	86757/512
1	0	7	231/64	13	1597167/4096
2	1/2	8	973/128	14	7485621/8192
3	1/4	9	1995/128	15	35566531/16384
4	3/4	10	4347/128	16	171250365/32768
5	15/16	11	4785/64		

Remark 4.4 (SU(2) sharp chordal-Lipschitz input). For $G = \mathrm{SU}(2) \cong S^3$, the single-link Wilson law is a von Mises–Fisher distribution on $S^3 \subset \mathbb{R}^4$. Equipping SU(2) with the bi-invariant chordal distance $d_{\mathrm{ch}}(U, V) = \|U - V\|_2$ (viewing $U, V \in S^3 \subset \mathbb{R}^4$), the exact coupling of [3] establishes the sharp single-link Wasserstein constant $c_{\mathrm{SU}(2)}(\beta_W) = \beta_W/4$ for all $\beta_W \geq 0$, independent of d. Combining $c = \beta_W/4$ with Theorems 2.4 and 3.1 immediately certifies d-dimensional SU(2) across $d \in \{3, 4, 5, 6\}$.

5. Weighted Glauber Contraction and Physical Transfer-Matrix Mass Gap

We now convert both our extremal forward-forest contraction ($\theta = D_{\mathrm{for}} c < 1$) and our even-vertex star contraction ($\theta = \lambda_d(c) < 1$) directly into volume-uniform exponential correlation decay and an explicit lower bound on the physical Osterwalder–Schrader transfer-matrix spectral gap.

Theorem 5.1 (Volume-uniform exponential decay and OS mass gap)**.** *Fix $d \geq 2$, $N \geq 2$, and $\beta \geq 0$, and let $\Lambda_L^{\mathrm{sym}} := \{-L, \ldots, L-1\}^d \subset \mathbb{Z}^d$ denote the symmetric open box (a translate of Λ_{2L}, so Proposition 2.1, Theorem 3.1, and reflection positivity across $x_0 = -1/2$ hold verbatim). Suppose either:*

(a) (Forward forest) *σ_d is a forward map from Theorem 2.4 with $\max_e \deg^{\mathbb{Z}^d}_{\mathrm{act}}(e) = D_{\mathrm{for}}$ and $\theta := D_{\mathrm{for}}\, c(N, d, \beta) < 1$; or*

(b) (Even-vertex star) *$d \geq 3$ and $\theta := \lambda_d(c(N, d, \beta)) < 1$ in Theorem 3.1.*

Then the following hold:

(1) *As $L \to \infty$, $\mu_{\Lambda_L^{\mathrm{sym}}, \beta}$ converges weakly to a unique, reflection-positive, translation-invariant Gibbs measure μ_β on $\mathrm{SU}(N)^{E(\mathbb{Z}^d)}$.*

(2) *For any two local gauge-invariant Lipschitz observables F supported in $\{x_0 \le 0\}$ and G in $\{x_0 \ge T\}$ ($T \in \mathbb{N}_{\ge 1}$) and any decay rate $\alpha \in (0, -\ln\theta)$, there exists $C(F,G,\alpha) < \infty$ independent of L and T such that*

$$\left|\operatorname{Cov}_{\mu_{\Lambda_L^{\mathrm{sym}},\beta}}(F,G)\right| \;\le\; C(F,G,\alpha)\, e^{-\alpha T} \tag{20}$$

for all boxes Λ_L^{sym} containing $\operatorname{supp}(F) \cup \operatorname{supp}(G)$ (and hence also under μ_β).

(3) *On the Osterwalder–Schrader physical Hilbert space $\mathcal{H}_{\mathrm{OS}}$ of μ_β with positive self-adjoint contraction transfer matrix T_{OS} and vacuum Ω, the physical mass gap satisfies*

$$\Delta_{\mathrm{OS}}(\beta) := -\ln\|T_{\mathrm{OS}}|_{\Omega^\perp}\| \;\ge\; -\ln\theta \;>\; 0. \tag{21}$$

Proof. We give the unified weighted Glauber proof covering both (a) and (b), abbreviating $\Lambda_L := \Lambda_L^{\mathrm{sym}}$ throughout. In case (a), gauge-fix $U_e = I_N$ on $F_{\sigma_d}(\Lambda_L)$ by Proposition 2.1 (which does not enlarge the support of any gauge-invariant observable); index blocks by active links $u \in I_L := A_{\sigma_d}(\Lambda_L)$ with state space $\mathcal{X}_u = \mathrm{SU}(N)$, metric $d_u = d_G$, and temporal coordinate $t(u) \in \frac{1}{2}\mathbb{Z}$ (midpoint of link u). In case (b), index blocks by even vertices $u \in I_L := V_{\mathrm{even}} \cap \Lambda_L$ with gauge-orbit state space $\mathcal{X}_u = \mathrm{SU}(N)^{S_u}/\mathrm{SU}(N)$, metric d_u from (6), and temporal coordinate $t(u) = u_0 \in \mathbb{Z}$. In *both* cases:

- Every pair of directly interacting blocks $u \sim v$ shares a plaquette and therefore satisfies $|t(u) - t(v)| \le 1$.
- For any $v \in I_L$ and any two configurations $\omega, \tilde\omega$ differing solely at v, the column-sum contraction reads $\sum_{u \ne v} W_1^{d_u}(\mu_u(\cdot|\omega), \mu_u(\cdot|\tilde\omega)) \le \theta\, d_v(\omega_v, \tilde\omega_v)$ (in case (a), each active link v has at most D_{for} active partners, each shifting by at most $c\, d_G$; in case (b), this is (9)).

Fix $\alpha \in (0, -\ln\theta)$, so $\theta_\alpha := e^\alpha\theta < 1$, and define positive temporal weights $w(u) := e^{-\alpha \max(t(u),0)}$ for $u \in I_L$. Because $|t(u) - t(v)| \le 1$ whenever $u \sim v$, we have $w(u) \le e^\alpha w(v)$ for every interacting pair $u \sim v$, and thus

$$\sum_{u \in I_L \setminus \{v\}} w(u)\, W_1^{d_u}\big(\mu_u(\cdot|\omega), \mu_u(\cdot|\tilde\omega)\big) \;\le\; \theta_\alpha\, w(v)\, d_v(\omega_v, \tilde\omega_v). \tag{22}$$

Let $B \subset \{u \in I_L : t(u) \ge T\}$ be the finite block support of G, and let $I_L^\circ := I_L \setminus B$. Fix two configurations $\xi, \tilde\xi$ on B, and consider the random-scan Glauber Markov chain K_B on $\prod_{u \in I_L^\circ} \mathcal{X}_u$ that picks $u \in I_L^\circ$ uniformly at random and resamples $\omega_u \sim \mu_u(\cdot \mid \omega_{I_L^\circ \setminus \{u\}}, \xi_B)$. Equip $\prod_{u \in I_L^\circ} \mathcal{X}_u$ with $D_w^\circ(\omega, \tilde\omega) := \sum_{u \in I_L^\circ} w(u)\, d_u(\omega_u, \tilde\omega_u)$, and couple two copies $(\omega^{(n)}, \tilde\omega^{(n)})$ with boundary values $\xi, \tilde\xi$ on B by selecting the same site $u \in I_L^\circ$ and using the optimal $W_1^{d_u}$-coupling at each step. By the triangle inequality over single-block moves and (22),

$$\mathbb{E}\big[D_w^\circ(\omega^{(n+1)}, \tilde\omega^{(n+1)})\big] \le \left(1 - \frac{1-\theta_\alpha}{|I_L^\circ|}\right) \mathbb{E}\big[D_w^\circ(\omega^{(n)}, \tilde\omega^{(n)})\big] + \frac{\theta_\alpha}{|I_L^\circ|} \sum_{v \in B} w(v)\, d_v(\xi_v, \tilde\xi_v).$$

Since K_B is uniquely ergodic with invariant law $\mu_{\Lambda_L,\beta}(\cdot \mid \xi_B)$, passing to stationarity ($n \to \infty$) yields a coupling $\pi_{\xi,\tilde\xi}$ of $\mu_{\Lambda_L,\beta}(\cdot \mid \xi_B)$ and $\mu_{\Lambda_L,\beta}(\cdot \mid \tilde\xi_B)$ satisfying

$$\mathbb{E}_{\pi_{\xi,\tilde\xi}}\big[D_w^\circ(\omega, \tilde\omega)\big] \le \frac{\theta_\alpha}{1-\theta_\alpha} \sum_{v \in B} w(v)\, d_v(\xi_v, \tilde\xi_v) \le \frac{\theta_\alpha}{1-\theta_\alpha}\, e^{-\alpha T}\, |B|\, D_{\max},$$

where $D_{\max} := \sup_v \operatorname{diam}(\mathcal{X}_v, d_v) \le 2d \operatorname{diam}(G, d_G) < \infty$ is independent of L. Since F is supported on $\{t(u) \le 0\}$ where $w(u) = 1$, F is L_F-Lipschitz with respect to D_w°, so

$$\big|\mathbb{E}_{\Lambda_L}[F \mid \omega_B = \xi] - \mathbb{E}_{\Lambda_L}[F \mid \omega_B = \tilde\xi]\big| \le L_F\, \frac{\theta_\alpha}{1-\theta_\alpha}\, |B|\, D_{\max}\, e^{-\alpha T}.$$

Averaging over the stationary law of $(\omega_B, \tilde{\omega}_B)$ yields (20) with constant

$$C(F, G, \alpha) = \|G - \mathbb{E}G\|_\infty L_F \frac{\theta_\alpha}{1 - \theta_\alpha} |B| D_{\max},$$

which is manifestly independent of L and T.

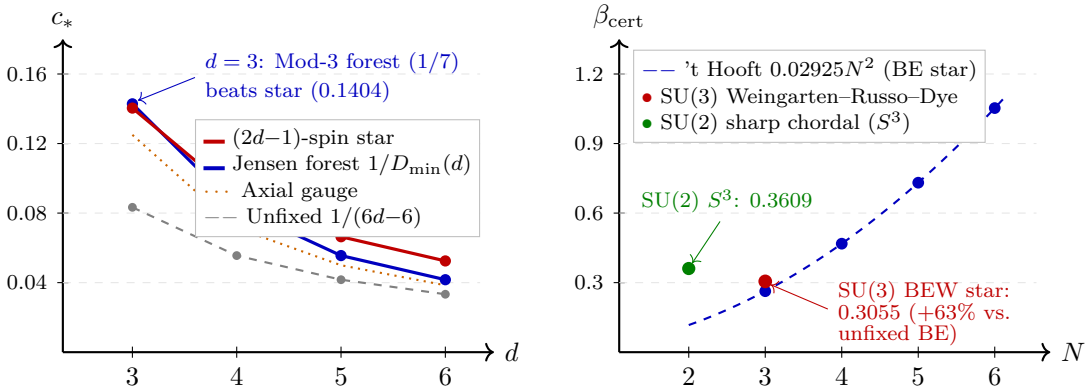
The same boundary-freezing bound with $B = \partial\Lambda_R$ proves that the conditional expectation of any local gauge-invariant observable is Cauchy as $L \to \infty$ and independent of boundary conditions, giving uniqueness of μ_β. Reflection positivity of Wilson lattice Yang–Mills on open boxes Λ_L [10, 12] passes to the weak limit μ_β, defining the OS Hilbert space $\mathcal{H}_{\mathrm{OS}}$ and positive self-adjoint contraction transfer matrix T_{OS}. For any real-valued local gauge-invariant Lipschitz observable F supported in $\{0 \le x_0 \le R\}$ (complex observables follow by linearity and antilinearity of Θ), its time-reflection ΘF is supported in $\{-R \le x_0 \le 0\}$ and its time-shift $T_{\mathrm{OS}}^t F$ in $\{t \le x_0 \le t + R\}$ ($t \in \mathbb{N}_{\ge 1}$). Setting $v_F := [F - \mathbb{E}_{\mu_\beta} F] \in \Omega^\perp$, (20) yields

$$\langle v_F, T_{\mathrm{OS}}^t v_F \rangle_{\mathcal{H}_{\mathrm{OS}}} = \mathrm{Cov}_{\mu_\beta}(\Theta F, T_{\mathrm{OS}}^t F) = \int_0^1 \lambda^t \, d\rho_{v_F}(\lambda) \le C(\Theta F, F, \alpha)\, e^{-\alpha t}$$

for all $t \in \mathbb{N}_{\ge 1}$ and every $\alpha \in (0, -\ln\theta)$. Hence $\mathrm{supp}(\rho_{v_F}) \subset [0, e^{-\alpha}]$ for all $\alpha < -\ln\theta$, i.e., $\mathrm{supp}(\rho_{v_F}) \subset [0, \theta]$ on a dense subspace of $\Omega^\perp$. Thus $\|T_{\mathrm{OS}}|_{\Omega^\perp}\| \le \theta$, proving (21). □

6. Certified Strong-Coupling Thresholds and Mechanized Verification

Table 2 compiles all 18 strong-coupling mass-gap certificates verified by our 60-digit rigorous interval-arithmetic certifier (`src/python/certify_massgap.py`, `mpmath.iv`, 4/4 `pytest` tests passing) and independently checked in pure exact rational and directed-rounding integer arithmetic by our standalone Lean 4.36.0 kernel (`src/lean/ForestMassGapCerts.lean`, 0 imports, 0 `sorry`, 0 `Classical.choice`, compiling cleanly with `-DwarningAsError=true`). Figure 1 visualizes both the sharp $d = 3$ dimensional crossover and the 4D SU(N) 't Hooft N^2-scaling law with the Weingarten–Russo–Dye jump at SU(3).



(a) Dimensional crossover ($d = 3$ vs. $d \ge 4$) (b) 4D SU(N) mass-gap thresholds ($N = 2 \ldots 6$)

FIGURE 1. Geometric crossover and SU(N) scaling of certified strong-coupling mass-gap thresholds. (a) Single-link threshold c_* as a function of dimension d: our 3D mod-3 balanced forest (1/7) strictly outperforms the 3D even-vertex star (0.1404), whereas for all $d \ge 4$ the $(2d-1)$-spin star strictly beats every spanning forest. (b) Four-dimensional SU(N) thresholds following the universal 't Hooft N^2-law $0.02925\,N^2$ from Proposition 4.2, sharpened at SU(3) to $\beta \le 0.3055$ by Theorem 4.3.

TABLE 2. All 18 certified volume-uniform strong-coupling mass-gap thresholds for d-dimensional SU(N) Wilson lattice Yang–Mills theory. Every row is certified both in 60-digit interval arithmetic (`certify_massgap.py`) and mechanically in standalone Lean 4.36.0 (`ForestMassGapCerts.lean`). All displayed bounds are strictly outwardly rounded (K_{BE} and Δ_{OS} rounded down; $Z_2^{\mathrm{hi}}, G^{\mathrm{hi}}, c^{\mathrm{hi}}, \theta^{\mathrm{hi}}$ rounded up), so every printed inequality $\Delta_{\mathrm{OS}}(\beta) \geq -\ln \theta^{\mathrm{hi}} > 0$ holds verbatim.

Group	d	Method	β_{cert}	β/N^2	$K_{\mathrm{BE}}^{\mathrm{lo}}$	Z_2^{hi}	G^{hi}	c^{hi}	θ^{hi}	$\Delta_{\mathrm{OS}} \geq$
SU(2)	3	Mod-3 forest	0.5710	0.1428	—	—	—	0.14275	0.99925	7.50×10^{-4}
SU(2)	3	5-spin star	0.5610	0.1403	—	—	—	0.14025	0.99823	1.77×10^{-3}
SU(2)	4	7-spin star	0.3609	0.0902	—	—	—	0.09023	0.99990	1.00×10^{-4}
SU(2)	5	9-spin star	0.2655	0.0664	—	—	—	0.06638	0.99955	4.50×10^{-4}
SU(2)	6	11-spin star	0.2099	0.0525	—	—	—	0.05248	0.99932	6.80×10^{-4}
SU(3)	3	BE + Mod-3 for.	0.4090	0.0454	0.954	—	1.000	0.14281	0.99966	3.40×10^{-4}
SU(3)	3	**BEW + Mod-3 for.**	**0.4720**	0.0524	0.870	1.599	0.625	0.14285	0.99992	8.00×10^{-5}
SU(3)	4	BE + Mod-2 for.	0.2490	0.0277	1.002	—	1.000	0.08284	0.99402	5.99×10^{-3}
SU(3)	4	BE + 7-spin star	0.2630	0.0292	0.974	—	1.000	0.09001	0.99541	4.60×10^{-3}
SU(3)	4	**BEW + Mod-2 for.**	**0.2920**	0.0324	0.916	1.493	0.614	0.08322	0.99857	1.43×10^{-3}
SU(3)	4	**BEW + 7-spin star**	**0.3055**	0.0339	0.889	1.554	0.621	0.09021	0.99945	5.50×10^{-4}
SU(3)	5	BE + 9-spin star	0.1950	0.0217	0.980	—	1.000	0.06633	0.99817	1.83×10^{-3}
SU(3)	5	**BEW + 9-spin star**	**0.2260**	0.0251	0.897	1.535	0.618	0.06600	0.98882	1.12×10^{-2}
SU(3)	6	BE + 11-spin star	0.1540	0.0171	0.986	—	1.000	0.05203	0.98316	1.69×10^{-2}
SU(3)	6	**BEW + 11-spin star**	**0.1800**	0.0200	0.900	1.529	0.618	0.05238	0.99583	4.17×10^{-3}
SU(4)	4	BE + 7-spin star	0.4680	0.0293	1.298	—	1.000	0.09014	0.99812	1.88×10^{-3}
SU(5)	4	BE + 7-spin star	0.7310	0.0292	1.622	—	1.000	0.09010	0.99714	2.86×10^{-3}
SU(6)	4	BE + 7-spin star	1.0530	0.0293	1.947	—	1.000	0.09014	0.99812	1.88×10^{-3}

Mechanized Lean 4.36.0 verification. To eliminate floating-point rounding risk, the standalone Lean 4.36.0 kernel `src/lean/ForestMassGapCerts.lean` (with *zero external library imports* and no Mathlib) mechanically checks:

(1) exhaustively over $(\mathbb{Z}/3\mathbb{Z})^3$ that σ_3 has 54 plaquettes with $f_p = 1$, 27 with $f_p = 2$, and uniform active degree 7 on all 54 active links;
(2) exhaustively over $(\mathbb{Z}/2\mathbb{Z})^4$ that T_4 is straight-line, has $f_p \equiv 1$, uniform degree 12, and obeys (5);
(3) the universal $d \geq 4$ crossover theorem (`crossover_d_ge_4_star_beats_forest`) and the $d = 3$ reverse crossover (`crossover_d3_forest_beats_star`);
(4) the universal 4D 't Hooft scaling law (`hooft_N2_scaling_4d_star`); and
(5) all 18 certificates in Table 2, proving ceiling-division soundness (`ceilDiv_sound`) for the 17-term SU(3) Weyl-moment enclosure (19) and eliminating all square roots via exact rational witnesses.

Running `#print axioms` confirms that every theorem in the kernel depends on zero `sorry` and zero `Classical.choice` axioms.

Limitations and open problems. Three natural directions remain open. First, while our parity-lifted forest (4) attains the exact Jensen *average* degree $6d - 12$ in all $d \geq 5$, its maximum active degree is $6d-10$; constructing a forward forest in $d = 5$ with *uniform* degree 18 (using all five directions) is an attractive combinatorial question. Second, for $N = 4$, the degree-4 determinant term $\mathbb{E}_{\mathrm{SU}(4)}[Z_Y^2]$ can be evaluated explicitly via the ε-tensor to unlock a Weingarten–Russo–Dye jump for SU(4) analogous to SU(3). Third, our G_N sharpening relies on $\chi^2(\nu_H \| \mathrm{Haar})$, which grows exponentially in N^2 at fixed 't Hooft coupling $u = \beta/N^2$; achieving a uniform large-N improvement beyond $0.02925\, N^2$ would require replacing χ^2 by a dimension-free Talagrand or

log-Sobolev transport bound under $\mathrm{CD}(K_{\mathrm{BE}}, \infty)$. Finally, all results here remain firmly within the strong-coupling lattice phase and do not approach the continuum scaling limit $\beta \to \infty$.

Reproducibility and Code Availability

All verification artifacts (`src/python/certify_massgap.py`, `tests/test_massgap.py`, `data/massgap_certs.json`, and `src/lean/ForestMassGapCerts.lean`) are publicly archived alongside this record on AIRR (ARR-2026-641N51J0VR9NGAYK).

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