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Unconditional Stoimenow Log-Concavity and Sharp Hirasawa–Murasugi Plateau Bounds for Two-Block Alternating Closed 3-Braids, with Twist-Concentrated Reductions

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Abstract

We settle Stoimenow's log-concavity conjecture (2005)—and hence Fox's trapezoidal conjecture (1962)—together with the sharp Hirasawa–Murasugi signature plateau inequality for the complete four-parameter family of two-block alternating closed 3-braids $\widehat{\beta}$ with $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\sigma_1^{p_2}\sigma_2^{-q_2}$ $(p_1, q_1, p_2, q_2 \geq 1)$, and establish a general Cayley–Hamilton twist-concentration criterion extending log-concavity to arbitrary block length $m \geq 2$. Because every alternating closed 3-braid of block length $m \geq 2$ satisfies $|\sigma(\widehat{\beta})| = |P - Q| < P + Q - 2 = \deg \widetilde{\Delta}_{\widehat{\beta}}$, these links lie topologically outside the special alternating class recently resolved via Lorentzian polynomials by Hafner, Mészáros, and Vidinas (2024), and our results strengthen the unimodality theorems of Alrefai–Chbili (2021) and Azarpendar–Juhász–Kálmán (2024) from trapezoidality to full log-concavity without internal zeros. Working in the q-integer Burau basis, we prove the exact division-free identities $\widetilde{\Delta}_{\widehat{\beta}}(x) = [p_1 + p_2]_x[q_1 + q_2]_x + x[p_1]_x[q_1]_x[p_2]_x[q_2]_x$ for $m = 2$ and an explicit four-term positive middle layer $R_1(x)$ for $m = 3$, along with the universal recurrence $\widetilde{\Delta}_{\beta(p_1)}(x) = [p_1]_x G(x) + x^{p_1}\widetilde{\Delta}_{\beta(0)}(x)$ $(\deg G \leq P' + Q - 2)$ for every $m \geq 2$. For $m = 2$, we decompose the four-factor perturbation $B(x) = x[p_1]_x[q_1]_x[p_2]_x[q_2]_x$ into centered trapezoids, prove concavity on the central plateau of $A(x) = [P]_x[Q]_x$ in two lines, and prove ramp-and-corner synchronization $2A_kB_k \geq A_{k-1}B_{k+1} + A_{k+1}B_{k-1}$ via Pick's theorem on an inscribed lattice triangle $\mathcal{T}' \subset R_k$, establishing strict log-concavity at every index outside the central plateau. We further derive the exact closed-form plateau length $L_{\mathrm{plat}}(p_1, q_1, p_2, q_2) \leq |P - Q| = |\sigma(\widehat{\beta})|$ (giving a three-line Schur-complement proof of $|\sigma(\widehat{\beta})| = |P - Q|$ with zero nullity for arbitrary $m \geq 1$) and characterize every case of equality. Finally, we prove a universal partial-sum log-concavity lemma certifying all 3,125 infinite twist-concentrated families for $m = 3$ with base exponents in $[1, 5]^5$ $(p_1 \geq P' + Q - 1)$. All algebraic, synchronization, plateau-concavity, Burau ring, Pick-bound, Hirasawa–Murasugi, and partial-sum cores are formally verified in a compact standalone Mathlib-free Lean 4.36.0 kernel (29 declarations, 359 lines, 0 `sorry`, 0 custom axioms), accompanied by a zero-float integer verifier (11,932 distinct alternating 3-braids checked with 0 failures).

1 Introduction

Let L be an alternating knot or link with Alexander polynomial $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$, normalized so that $\widetilde{\Delta}_L(x) := \pm x^c \Delta_L(-x) = \sum_{k=0}^{D} c_k x^k \in \mathbb{Z}_{\geq 0}[x]$ has nonnegative integer coefficients and $c_0 = c_D > 0$. By Murasugi's classical theorem, the coefficients of $\Delta_L(t)$ alternate in sign, so $c_k = |[t^k]\Delta_L(t)|$ records their exact magnitudes. Two foundational structural conjectures govern the coefficient sequence $(c_0, \ldots, c_D)$:

*AIRR.SCIENCE Autonomous Research Laboratory (`editor@airr.science`). License: CC-BY-4.0 (manuscript) and MIT (companion Lean 4.36.0 and Python verification code). Research and verification executed with Google Antigravity assistance under AIRR-FOUNDER-1.0 integrity standards.

1. **Fox's Trapezoidal Conjecture (1962) [6]:** The palindromic sequence $(c_0, \ldots, c_D)$ is *trapezoidal*, meaning $c_0 < c_1 < \cdots < c_r = c_{r+1} = \cdots = c_{D-r} > \cdots > c_D$ for some $0 \le r \le \lfloor D/2 \rfloor$. Hirasawa and Murasugi [9] sharpened this by conjecturing that the *stable plateau length* $L_{\mathrm{plat}}(L) := (D - 2r)$ is bounded above by the signature magnitude $|\sigma(L)|$.

2. **Stoimenow's Log-Concavity Conjecture (2005) [11]:** The sequence $(c_0, \ldots, c_D)$ is *log-concave with no internal zeros*, i.e., $c_k > 0$ for all $0 \le k \le D$ and

$$c_k^2 \ge c_{k-1} c_{k+1} \qquad (1 \le k \le D-1), \tag{1}$$

   which implies unimodality (and, when strict outside the central plateau, full trapezoidality).

Despite six decades of effort, both conjectures remain open for general alternating links. Landmark breakthroughs have resolved several major classes: Hartley [8] proved Fox's conjecture for all two-bridge knots, and Banfield [4] strengthened this to Stoimenow's log-concavity conjecture for all two-bridge knots via Christoffel words. Recently, Hafner, Mészáros, and Vidinas [7] proved that a multivariate generalization of the Alexander polynomial of any *special alternating link* is denormalized Lorentzian, settling Stoimenow's conjecture for all special alternating links.

By contrast, *alternating closed 3-braids* of mixed-sign normal form [10] (excluding split closures, stabilized 2-braid closures with $m = 1$ and $\min(p_1, q_1) = 1$, and positive/negative composite closures $\sigma_1^p \sigma_2^q$ which are special alternating) are precisely the closures $\widehat{\beta}$ of braids

$$\beta = \sigma_1^{p_1} \sigma_2^{-q_1} \sigma_1^{p_2} \sigma_2^{-q_2} \cdots \sigma_1^{p_m} \sigma_2^{-q_m} \in B_3 \qquad (m \ge 1,\ \ p_i, q_i \ge 1). \tag{2}$$

Whenever $m \ge 2$ (or $m = 1$ with $p_1, q_1 \ge 2$), Proposition 4.1 and Theorem 2.1(a) yield the diagram-independent topological obstruction $|\sigma(\widehat{\beta})| = |P - Q| < P + Q - 2 = \deg \widetilde{\Delta}_{\widehat{\beta}}$ (whereas every non-split special alternating link satisfies $|\sigma(L)| = \deg \widetilde{\Delta}_L$), placing all such links strictly outside the special alternating class of [7]. For this canonical benchmark family, Alrefai and Chbili [1] established Fox's trapezoidal conjecture for two-block and restricted three-block families (unimodality only), AlSukaiti and Chbili [2] studied weaving 3-braids $W(3, m)$ via Lucas lattices, and Azarpendar, Juhász, and Kálmán [3] proved Fox's conjecture and the Hirasawa–Murasugi inequality for twist-concentrated alternating links together with weaker consecutive-coefficient inequalities for general alternating 3-braids. Stoimenow's stronger log-concavity conjecture (1) remained open even for the general two-block family $m = 2$.

1.1 Main Contributions

This paper develops a self-contained algebraic and lattice-geometric framework for alternating closed 3-braids with four main results:

1. **Exact q-integer Burau identities and universal Cayley–Hamilton recurrence (Theorem 2.1).** Writing $[n]_x := 1 + x + \cdots + x^{n-1}$ and $P = \sum p_i, Q = \sum q_i$, we eliminate the cyclotomic denominator $1 - x + x^2$ algebraically in the reduced Burau representation to obtain monicity ($\deg \widetilde{\Delta}_{\widehat{\beta}} = P + Q - 2$, $c_0 = c_{P+Q-2} = 1$), the exact division-free two-block identity $\widetilde{\Delta} = [p_1 + p_2]_x [q_1 + q_2]_x + x[p_1]_x [q_1]_x [p_2]_x [q_2]_x$, an explicit four-term positive middle layer $R_1(x)$ for $m = 3$, and the universal Cayley–Hamilton twist decomposition $\widetilde{\Delta}_{\beta(p_1)}(x) = [p_1]_x G(x) + x^{p_1} \widetilde{\Delta}_{\beta(0)}(x)$ ($\deg G \le P' + Q - 2$) for every $m \ge 2$ (also recovering the classical Lucas determinant law $|\Delta_{W(3,m)}(-1)| = L_{2m} - 2$ [2]).

2. **Unconditional log-concavity for all two-block alternating closed 3-braids (Theorem 3.4).** For every $p_1, q_1, p_2, q_2 \ge 1$, we prove that $\widetilde{\Delta}_{\widehat{\beta}}(x)$ is log-concave with no internal zeros—and *strictly* log-concave at every index outside the central plateau—via centered trapezoid decomposition, a two-line plateau concavity proof (Lemma 3.2), and Pick's theorem on an inscribed lattice triangle $\mathcal{T}' \subset R_k$ (Lemma 3.3).

3. **Exact plateau length and sharp Hirasawa–Murasugi signature bound (Theorem 4.2).** We give a three-line Schur-complement proof of the Murasugi–Erle signature identity $|\sigma(\widehat{\beta})| = |P - Q|$ (with zero nullity) for *all* $m \geq 1$, derive the exact closed-form plateau length $L_{\mathrm{plat}}(p_1, q_1, p_2, q_2)$ for $m = 2$, prove $L_{\mathrm{plat}} \leq |\sigma(\widehat{\beta})|$ unconditionally, and characterize every equality case.

4. **Twist-concentrated log-concavity for arbitrary $m \geq 2$ (Theorem 5.2).** We prove a universal partial-sum log-concavity lemma (Lemma 5.1) showing that whenever the finite Cayley–Hamilton kernel $G(x)$ (of degree at most $P' + Q - 2$) has log-concave partial sums (in particular whenever G is nonnegative, unimodal, and rising-prefix log-concave), $\widetilde{\Delta}_{\beta(p_1)}(x)$ satisfies Stoimenow's log-concavity conjecture for *all* $p_1 \geq P' + Q - 1$, with exact plateau length $|P - Q| - 2(P' - 1) \leq |\sigma(\widehat{\beta})|$ whenever $G_{P'+Q-2} > 0$. Exact zero-float certification verifies all 3,125 infinite twist-concentrated $m = 3$ families with base exponents in $[1, 5]^5$ alongside 11,932 distinct alternating 3-braids ($m \in \{2, 3, 4, 5, 6\}$ and weaving braids $W(3, 1..25)$) with zero failures.

2 Exact q-Integer Burau Decomposition and Cayley–Hamilton Recurrence

For the 3-strand braid group $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$, the reduced Burau representation $\psi_t : B_3 \to \mathrm{GL}_2(\mathbb{Z}[t, t^{-1}])$ is generated by

$$\psi_t(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \qquad \psi_t(\sigma_2^{-1}) = \begin{pmatrix} 1 & 0 \\ 1 & -t^{-1} \end{pmatrix}.$$

Substituting $t = -x$ and rescaling σ_2^{-1} by x yields the two nonnegative polynomial matrices

$$A(x) := \psi_{-x}(\sigma_1) = \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix}, \qquad B(x) := x\,\psi_{-x}(\sigma_2^{-1}) = \begin{pmatrix} x & 0 \\ x & 1 \end{pmatrix}, \tag{3}$$

with exact powers $A(x)^p = \begin{pmatrix} x^p & [p]_x \\ 0 & 1 \end{pmatrix}$ and $B(x)^q = \begin{pmatrix} x^q & 0 \\ x[q]_x & 1 \end{pmatrix}$, where $[n]_x := \sum_{j=0}^{n-1} x^j$. For any alternating 3-braid β of the form (2) with $P := \sum_{i=1}^m p_i$ and $Q := \sum_{i=1}^m q_i$, define the block transfer matrix

$$T_{p,q}(x) := A(x)^p B(x)^q = \begin{pmatrix} x^{p+q} + x[p]_x[q]_x & [p]_x \\ x[q]_x & 1 \end{pmatrix} \tag{4}$$

and the product $M_\beta(x) := x^Q \psi_{-x}(\beta) = \prod_{i=1}^m T_{p_i,q_i}(x)$, which satisfies $\det M_\beta(x) = x^{P+Q}$. By Birman's reduced Burau determinant formula,

$$\widetilde{\Delta}_{\widehat{\beta}}(x) = \frac{-x^Q \det(I_2 - \psi_{-x}(\beta))}{1 - x + x^2} = \frac{\operatorname{tr} M_\beta(x) - x^P - x^Q}{1 - x + x^2}. \tag{5}$$

Theorem 2.1 (Exact q-Integer Decomposition, Degree Law, and Cayley–Hamilton Recurrence)**.** *Let $\beta = \sigma_1^{p_1}\sigma_2^{-q_1} \cdots \sigma_1^{p_m}\sigma_2^{-q_m}$ with $m \geq 1$ and $p_i, q_i \geq 1$, and set $P = \sum_{i=1}^m p_i$, $Q = \sum_{i=1}^m q_i$.*

(a) ***Exact degree, monicity, and palindromicity:*** *Setting $d := 1 - x$ and $C := 1 - x + x^2 = d^2 + x$, the $(1,1)$-entry of $T_{p,q}(x)$ obeys $x^{p+q} + x[p]_x[q]_x = 1 - d([p]_x + [q]_x) + C[p]_x[q]_x$. The quotient $\widetilde{\Delta}_{\widehat{\beta}}(x) \in \mathbb{Z}[x]$ is palindromic and monic of exact degree $D = P + Q - 2$, with $c_0 = c_D = 1$.*

(b) ***Exact division-free positive identities*** *($m = 1, 2, 3$)**:***

$$m = 1: \quad \widetilde{\Delta}_{\widehat{\beta}}(x) = [p_1]_x[q_1]_x, \tag{6}$$

$$m = 2: \quad \widetilde{\Delta}_{\widehat{\beta}}(x) = [p_1 + p_2]_x[q_1 + q_2]_x + x\,[p_1]_x[q_1]_x[p_2]_x[q_2]_x, \tag{7}$$

$$m = 3: \quad \widetilde{\Delta}_{\widehat{\beta}}(x) = [P]_x[Q]_x + R_1(x) + x^2\,[p_1]_x[q_1]_x[p_2]_x[q_2]_x[p_3]_x[q_3]_x, \tag{8}$$

where the middle layer $R_1(x)$ admits the strictly positive four-term expansion

$$R_1(x) = x\Big(x^{p_3}[p_1]_x[p_2]_x[q_1]_x[q_2+q_3]_x + [p_2]_x[p_3]_x[q_1]_x[q_2]_x + x^{p_1+q_1}[p_2]_x[p_3]_x[q_2]_x[q_3]_x + [p_3]_x[p_1]_x[q_3]_x[q_1+q_2]_x\Big). \quad (9)$$

(c) ***Universal Cayley–Hamilton twist decomposition (arbitrary $m \geq 2$):*** *Holding $(q_1, p_2, q_2, \ldots, p_m, q_m)$ fixed and varying $p_1 \geq 1$, let $\beta(0) := \sigma_1^{p_2}\sigma_2^{-q_2}\cdots\sigma_1^{p_m}\sigma_2^{-(q_m+q_1)}$ be the $(m-1)$-block braid obtained by deleting $\sigma_1^{p_1}$. Then for every $p_1 \geq 1$,*

$$\widetilde{\Delta}_{\beta(p_1)}(x) = [p_1]_x\widetilde{\Delta}_{\beta(1)}(x) - x[p_1-1]_x\widetilde{\Delta}_{\beta(0)}(x) = [p_1]_x G(x) + x^{p_1}\widetilde{\Delta}_{\beta(0)}(x), \quad (10)$$

where $[0]_x := 0$ and $G(x) := \widetilde{\Delta}_{\beta(1)}(x) - x\widetilde{\Delta}_{\beta(0)}(x)$ satisfies $G_{D_1} = 1-1 = 0$ identically (hence $\deg G \leq D_1 - 1 = P' + Q - 2$, where $P' := P - p_1$ and $D_1 := P' + Q - 1$).

(d) ***Weaving Lucas determinant law (cf. [2]):*** *For the weaving 3-braid $W(3,m) = (\sigma_1\sigma_2^{-1})^m$, $|\Delta_{W(3,m)}(-1)| = \widetilde{\Delta}_{W(3,m)}(1) = L_{2m} - 2$, where L_n is the n-th Lucas number ($L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n$).*

Proof. For (a), substituting $x^a = 1 - d[a]_x$ gives $x^{p+q} + x[p]_x[q]_x = (1-d[p]_x)(1-d[q]_x) + x[p]_x[q]_x = 1 - d([p]_x + [q]_x) + (d^2+x)[p]_x[q]_x$, and $d^2 + x = 1 - x + x^2 = C$. For the column vector $v := (1, 1-x)^T$, direct multiplication gives $A(x)v = v$ identically and $B(x)v = (x, 1)^T = xv + (0, 1-x+x^2)^T \equiv xv \pmod{1-x+x^2}$, whence $M_\beta(x)v \equiv x^Q v \pmod{1-x+x^2}$. Because $\det M_\beta(x) = x^{P+Q}$, the Cayley–Hamilton trace satisfies $\operatorname{tr} M_\beta(x) \equiv x^P + x^Q \pmod{1-x+x^2}$, establishing algebraically that $1 - x + x^2$ divides $\operatorname{tr} M_\beta(x) - x^P - x^Q$ in $\mathbb{Z}[x]$. By induction on m, the $(1,1)$-entry of $M_\beta(x)$ is monic of degree $P+Q$ (its leading term coming uniquely from the product of the $(1,1)$-leading terms $x^{p_i+q_i}$, since off-diagonal products have degree at most $P+Q-1$), while the $(2,2)$-entry has degree at most $P+Q-2$. At $x = 0$, since $p_i, q_i \geq 1$, each block matrix evaluates to the idempotent $T_{p_i,q_i}(0) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, so $M_\beta(0) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has trace 1, whereas $x^P + x^Q$ vanishes at $x = 0$. Thus $\operatorname{tr} M_\beta(x) - x^P - x^Q$ has constant term 1 and leading term x^{P+Q} (since $P, Q < P+Q$). Dividing by the monic palindromic quadratic $1 - x + x^2$ yields a monic polynomial of exact degree $P + Q - 2$ with constant term $c_0 = 1$; palindromicity ($c_{D-k} = c_k$) follows from the classical symmetry $\Delta_L(t^{-1}) = \pm t^{-D}\Delta_L(t)$ of link Alexander polynomials (since $D = P + Q - 2$ is the exact span and $c_0 = c_D = 1$).

For (b), (6) and (7) are unconditional polynomial ring identities in $\mathbb{Z}[x, P_1, Q_1, P_2, Q_2]$ (using $[p_1+p_2]_x = [p_1]_x + [p_2]_x - d[p_1]_x[p_2]_x$), both formally verified in Lean 4.36.0 as `burau_m1_identity` and `burau_m2_identity`. For $m = 3$, dividing $\operatorname{tr}(T_1T_2T_3) - x^P - x^Q$ by C yields $[P]_x[Q]_x + x^2P_1Q_1P_2Q_2P_3Q_3 + xS_1$, where $S_1 = \sum_{\text{cyc}} P_1Q_1P_2[q_2+q_3]_x - dP_1P_2P_3(Q_1[q_2+q_3]_x + x^{q_1}Q_2Q_3)$; combining $(1 - dP_3)P_1P_2Q_1[q_2+q_3]_x = x^{p_3}P_1P_2Q_1[q_2+q_3]_x$ and expanding $P_2Q_2P_3[q_3+q_1]_x = P_2P_3Q_1Q_2 + x^{q_1}P_2P_3Q_2Q_3$ to combine with $-dP_1x^{q_1}P_2P_3Q_2Q_3 = (x^{p_1} - 1)x^{q_1}P_2P_3Q_2Q_3$ produces (9).

For (c), the 2×2 matrix $A(x)$ satisfies its Cayley–Hamilton equation $A(x)^2 - (1+x)A(x) + xI_2 = 0$ (`cayley_hamilton_A_11/12/22`), so by induction $A(x)^{p_1} = [p_1]_xA(x) - x[p_1-1]_xI_2$ for all $p_1 \geq 1$. Setting $N(x) := B(x)^{q_1}\prod_{i=2}^m T_{p_i,q_i}(x)$, we obtain $\operatorname{tr}(A(x)^{p_1}N(x)) = [p_1]_x\operatorname{tr}(A(x)N(x)) - x[p_1-1]_x\operatorname{tr}(N(x))$. Meanwhile, the scalar identity $x^{p_1+P'} = [p_1]_x x^{1+P'} - x[p_1-1]_x x^{P'}$ and $x^Q = ([p_1]_x - x[p_1-1]_x)x^Q$ hold identically (since $[p_1]_x - x[p_1-1]_x = 1$). Subtracting $x^{p_1+P'} + x^Q$, dividing by $1 - x + x^2$, and noting that $\operatorname{tr}(N(x)) = \operatorname{tr}(M_{\beta(0)}(x))$ by cyclicity of trace yields the first equality in (10); substituting $\widetilde{\Delta}_{\beta(1)}(x) = G(x) + x\widetilde{\Delta}_{\beta(0)}(x)$ and $[p_1]_x - [p_1-1]_x = x^{p_1-1}$ gives the second. By (a), both $\widetilde{\Delta}_{\beta(1)}(x)$ and $x\widetilde{\Delta}_{\beta(0)}(x)$ are monic of degree $D_1 = P' + Q - 1$, so $G_{D_1} = 1 - 1 = 0$ identically and $\deg G \leq D_1 - 1$.

For (d), at $x = 1$, $T_{1,1}(1) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has trace 3 and determinant 1, with eigenvalues ϕ^2 and ϕ^{-2} ($\phi = (1+\sqrt{5})/2$). Hence $\mathrm{tr}(T_{1,1}(1)^m) = \phi^{2m} + \phi^{-2m} = L_{2m}$, and (5) at $x = 1$ gives $\widetilde{\Delta}_{W(3,m)}(1) = (L_{2m} - 1 - 1)/1 = L_{2m} - 2$. □

3 Unconditional Log-Concavity for All Two-Block Alternating 3-Braids

We now prove Stoimenow's log-concavity conjecture—and strictly stronger log-concavity off the central plateau—for every two-block alternating closed 3-braid $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\sigma_1^{p_2}\sigma_2^{-q_2}$ ($p_1, q_1, p_2, q_2 \geq 1$). By reflection ($\sigma_1 \leftrightarrow \sigma_2^{-1}$, swapping $(p_1, p_2) \leftrightarrow (q_1, q_2)$) and the direct algebraic symmetry of (7) under swapping $p_1 \leftrightarrow p_2$ or $q_1 \leftrightarrow q_2$ independently, we assume without loss of generality that

$$1 \leq p_1 \leq p_2, \qquad 1 \leq q_1 \leq q_2, \qquad P := p_1 + p_2 \leq Q := q_1 + q_2. \tag{11}$$

Set $\mu_P := p_1 \leq \lfloor P/2 \rfloor$, $\mu_Q := q_1 \leq \lfloor Q/2 \rfloor$, $D := P + Q - 2$, and write (7) as $\widetilde{\Delta}_{\widehat{\beta}}(x) = A(x) + B(x)$, where

$$A(x) := [P]_x[Q]_x = \sum_{k=0}^{D} A_k x^k, \qquad B(x) := x\,[p_1]_x[p_2]_x[q_1]_x[q_2]_x = \sum_{k=0}^{D} B_k x^k$$

(with $B_0 = B_D = 0$). Both $A(x)$ and $B(x)$ are palindromic around the same center $D/2$, have positive coefficients on $[0, D]$ and $[1, D-1]$ respectively, and are log-concave (each being a product of the log-concave q-integers $[n]_x$). By the elementary decomposition (`lc_sum_decomp`)

$$(A_k + B_k)^2 - (A_{k-1} + B_{k-1})(A_{k+1} + B_{k+1}) = (A_k^2 - A_{k-1}A_{k+1}) + (B_k^2 - B_{k-1}B_{k+1}) + \Sigma_k, \tag{12}$$

where $\Sigma_k := 2A_kB_k - A_{k-1}B_{k+1} - A_{k+1}B_{k-1}$, log-concavity of $A + B$ follows immediately once we prove *synchronization* $\Sigma_k \geq 0$ for every $1 \leq k \leq \lfloor D/2 \rfloor$ (the right half following by palindromicity).

Lemma 3.1 (Centered Trapezoid Decomposition and Lattice-Point Formula)**.** *Under* (11)*, $B(x)$ admits the exact centered-trapezoid decomposition*

$$B(x) = \sum_{s=0}^{\mu_P - 1} \sum_{t=0}^{\mu_Q - 1} x^{1+s+t}[P - 1 - 2s]_x[Q - 1 - 2t]_x. \tag{13}$$

Consequently, for every $1 \leq k \leq \lfloor D/2 \rfloor$, the first difference $\Delta B_k := B_k - B_{k-1}$ equals the exact lattice-point count $|R_k|$, where

$$R_k := \big\{(s,t) \in \mathbb{Z}^2 : 0 \leq s < \mu_P,\ 0 \leq t < \mu_Q,\ s+t \leq k-1,\ s-t \leq P-1-k,\ t-s \leq Q-1-k\big\}. \tag{14}$$

Proof. For any $1 \leq a \leq b$, direct coefficient comparison yields $[a]_x[b]_x = \sum_{s=0}^{a-1} x^s[a + b - 1 - 2s]_x$ (each coefficient $\min(j+1, a, a+b-1-j)$ counts how many intervals $[s, a+b-2-s]$ with $0 \leq s \leq a-1$ contain j). Multiplying the expansions of $[p_1]_x[p_2]_x$ and $[q_1]_x[q_2]_x$ and shifting by x gives (13).

Each summand $T_{s,t}(x) := x^{1+s+t}[P - 1 - 2s]_x[Q - 1 - 2t]_x$ is a symmetric trapezoid centered at $D/2$ supported on $[1 + s + t,\ D - 1 - s - t]$ whose rising ramp has slope $+1$ on the exact range

$$1 + s + t \leq k \leq (1 + s + t) + \min(P - 1 - 2s,\ Q - 1 - 2t) - 1 = \min(P - 1 - s + t,\ Q - 1 + s - t),$$

and slope 0 on its central plateau (up to $\lfloor D/2 \rfloor$). Thus $\Delta T_{s,t}(k) = 1$ if and only if $(s,t) \in R_k$, and 0 otherwise; summing over (s,t) proves $\Delta B_k = |R_k|$. □

Lemma 3.2 (Plateau Concavity)**.** *Under* (11)*, suppose* $P \le Q-2$*. Then for every index* k *in the strict plateau interior* $P \le k \le \lfloor D/2 \rfloor$*, we have* $2B_k - B_{k-1} - B_{k+1} \ge 0$*, and hence* $\Sigma_k = P(2B_k - B_{k-1} - B_{k+1}) \ge 0$*.*

Proof. Write $B(x) = u(x)v(x)$ where $u(x) := x[p_1]_x[p_2]_x$ (nonnegative, supported on $1 \le i \le P-1$) and $v(x) := [q_1]_x[q_2]_x$ (supported on $0 \le j \le Q-2$). Because $v_j = \min(j+1, \mu_Q, Q-1-j)$ is piecewise linear with non-increasing slopes $+1, 0, -1$ on $[0, Q-2]$, it is concave on the interior: $2v_j - v_{j-1} - v_{j+1} \ge 0$ for all $1 \le j \le Q-3$. For any $P \le k \le \lfloor (P+Q-2)/2 \rfloor \le Q-2$ and any $1 \le i \le P-1$, the shifted index $j = k-i$ satisfies $1 = P-(P-1) \le j \le (Q-2)-1 = Q-3$. Therefore

$$2B_k - B_{k-1} - B_{k+1} = \sum_{i=1}^{P-1} u_i\big(2v_{k-i} - v_{k-1-i} - v_{k+1-i}\big) \ge 0,$$

formalized in Lean as `plateau_concavity_of_conv`. On this range $A_{k-1} = A_k = A_{k+1} = P$, so $\Sigma_k = P(2B_k - B_{k-1} - B_{k+1}) \ge 0$. □

Lemma 3.3 (Pick's-Theorem Lattice-Triangle Inequality for Ramp and Corner)**.** *Under* (11)*, for every* $1 \le k \le P-1$*, we have* $(k+2)|R_k| \ge k|R_{k+1}|$*, and consequently* $\Sigma_k \ge 0$*.*

Proof. Partition R_k and R_{k+1} by diagonals $s+t=r$: let $b_r := |\{(s,t) \in R_k : s+t=r\}|$ $(0 \le r \le k-1)$ and $a_r := |\{(s,t) \in R_{k+1} : s+t=r\}|$ $(0 \le r \le k)$. Since increasing k to $k+1$ tightens the inequalities $s-t \le P-1-k$ and $t-s \le Q-1-k$, we have $a_r \le b_r$ for all $0 \le r \le k-1$, so $\sum_{r=0}^{k-1} a_r \le \sum_{r=0}^{k-1} b_r = |R_k|$. By `diff_bound_of_pick`, $(k+2)|R_k| \ge k|R_{k+1}| = k(\sum_{r=0}^{k-1} a_r + a_k)$ follows immediately from the single inequality

$$2|R_k| \ge k a_k. \tag{15}$$

If $a_k = 0$, (15) is trivial. Assume $a_k \ge 1$, and let $\{(s, k-s) : L \le s \le R\}$ $(0 \le L \le R \le k$, $a_k = R - L + 1)$ be the contiguous lattice segment $R_{k+1} \cap \{s+t=k\}$. Observe that:

- Because $k \le P-1 \le Q-1$, the origin $(0,0)$ belongs to R_k (all inequalities in (14) hold at $(0,0)$).
- For every $(s, k-s) \in R_{k+1}$: if $s \ge 1$, then $(s-1, k-s)$ has sum $k-1$, $s'-t' = (s-t)-1 \le P-3-k < P-1-k$, and $t'-s' = (t-s)+1 \le Q-1-k$, so $(s-1, k-s) \in R_k$; symmetrically, if $s \le k-1$, then $(s, k-1-s) \in R_k$.

Since R_k is the set of integer lattice points in a convex polygon, the convex hull $\mathcal{T}'$ of $(0,0)$ and any two points of R_k on the line $s+t=k-1$ satisfies $\mathcal{T}' \cap \mathbb{Z}^2 \subset R_k$ (Figure 1(a)). We apply Pick's theorem ($|\mathcal{T}' \cap \mathbb{Z}^2| = \mathrm{Area}(\mathcal{T}') + \frac{1}{2}B(\mathcal{T}') + 1$) across the four boundary regimes of (L, R) (each algebraic inequality checked in Lean as `pick_case_*`):

1. $L \ge 1$ *and* $R \le k-1$*:* Both $(L-1, k-L)$ and $(R, k-1-R)$ lie in R_k on $s+t=k-1$ at horizontal separation $R-(L-1) = a_k$. The triangle $\mathcal{T}' = \mathrm{conv}\{(0,0), (L-1, k-L), (R, k-1-R)\}$ has area $\frac{1}{2}(k-1)a_k$ and boundary lattice count $B(\mathcal{T}') = a_k + g_1 + g_2$ (with $g_1, g_2 \ge 1$), giving $2|R_k| \ge (k-1)a_k + a_k + g_1 + g_2 + 2 = ka_k + g_1 + g_2 + 2 > ka_k$.
2. $L = 0$ *and* $R \le k-1$*:* Both $(0, k-1)$ and $(R, k-1-R)$ lie in R_k at separation $R = a_k - 1$, and the vertical leg $\overline{(0,0)(0,k-1)}$ contributes $g_1 = k-1$ boundary segments ($g_2 \ge 1$). Pick's theorem gives $2|R_k| \ge (k-1)(a_k-1) + (a_k-1) + (k-1) + g_2 + 2 = ka_k + g_2 + 1 > ka_k$ (when $a_k = 1$, i.e. $R = 0$, $\mathcal{T}'$ degenerates to the vertical segment $\overline{(0,0)(0,k-1)}$ containing k lattice points, giving $2|R_k| \ge 2k > k \cdot 1$ directly).
3. $L \ge 1$ *and* $R = k$*:* Symmetric to Case 2 using $(L-1, k-L)$ and $(k-1, 0)$ (when $a_k = 1$, $L = k$, the horizontal segment $\overline{(0,0)(k-1,0)}$ has k points, giving $2|R_k| \ge 2k > k$).

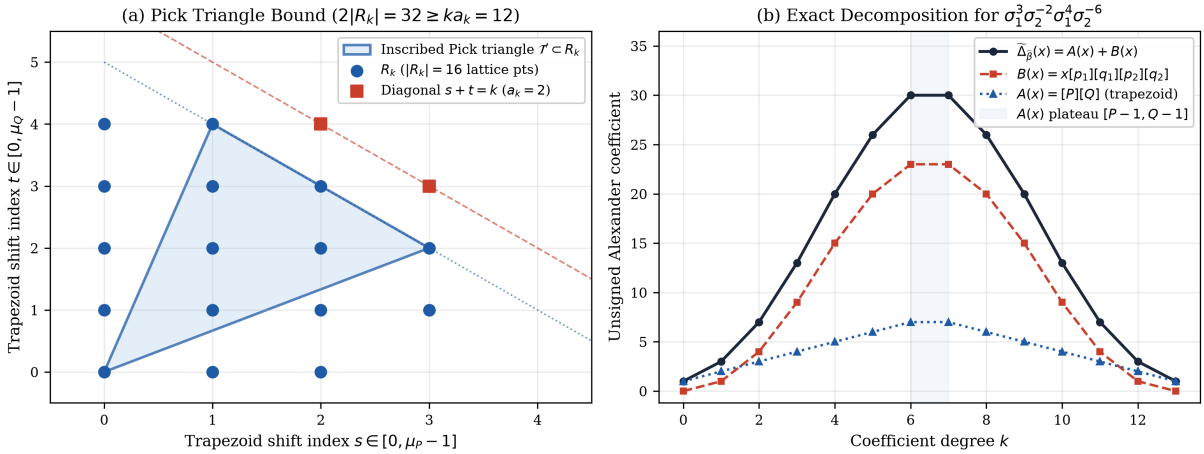


Figure 1: (a) Lattice-point region R_k ($|R_k| = 16$) and inscribed Pick triangle $\mathcal{T}' \subset R_k$ for $(p_1, p_2, q_1, q_2) = (4, 5, 5, 6)$ at $k = 6$, certifying $2|R_k| = 32 \geq ka_k = 12$. (b) Exact unsigned Alexander coefficient decomposition $\widetilde{\Delta}_{\widehat{\beta}}(x) = A(x) + B(x)$ for $\sigma_1^3\sigma_2^{-2}\sigma_1^4\sigma_2^{-6}$, illustrating how ramp synchronization and plateau concavity force strict log-concavity off the central plateau.

4. *$L = 0$ and $R = k$ ($a_k = k+1$):* Both $(0, k-1)$ and $(k-1, 0)$ lie in R_k, so the entire triangle $\{(s,t) \in \mathbb{Z}^2_{\geq 0} : s + t \leq k-1\}$ lies in R_k, yielding $2|R_k| \geq 2 \cdot \frac{k(k+1)}{2} = ka_k$.

Thus (15) holds in all cases, proving $(k+2)|R_k| \geq k|R_{k+1}|$ for all $1 \leq k \leq P-1$. Whenever $k+1 \leq \lfloor D/2 \rfloor$, Lemma 3.1 yields $\Delta B_k = |R_k|$ and $\Delta B_{k+1} = |R_{k+1}|$, giving $(k+2)\Delta B_k \geq k\Delta B_{k+1}$. In the remaining boundary case $k = P-1$ with $P \in \{Q-1, Q\}$ (so $k+1 = P > \lfloor D/2 \rfloor$), palindromicity of B implies $\Delta B_P = B_P - B_{P-1} \leq 0 = |R_P|$ while $\Delta B_{P-1} = |R_{P-1}| \geq 0$, so $(k+2)\Delta B_k \geq 0 \geq k\Delta B_{k+1}$ holds trivially. On the strict ramp $1 \leq k \leq P-2$, $A_{k-1} = k, A_k = k+1, A_{k+1} = k+2$, so $\Sigma_k = (k+2)\Delta B_k - k\Delta B_{k+1} \geq 0$ (`ramp_sync_of_diff_bound`). At the corner $k = P-1$, $A_{P-2} = P-1$, $A_{P-1} = P$, and $A_P \leq P < P+1$ (with $A_P = P$ if $P < Q$ and $A_P = P-1$ if $P = Q$), so $\Sigma_{P-1} = (P+1)\Delta B_{P-1} - (P-1)\Delta B_P + (P+1-A_P)B_{P-2} \geq 0$ (`corner_sync_of_diff_bound`). □

Theorem 3.4 (Unconditional Stoimenow Log-Concavity for All Two-Block Alternating 3-Braids)**.** *For every $p_1, q_1, p_2, q_2 \geq 1$, the unsigned Alexander polynomial $\widetilde{\Delta}_{\widehat{\beta}}(x) = \sum_{k=0}^{D} c_k x^k$ of $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\sigma_1^{p_2}\sigma_2^{-q_2}$ is log-concave with no internal zeros (and hence trapezoidal). Moreover, $c_k^2 > c_{k-1}c_{k+1}$ holds* strictly *at every index $1 \leq k \leq D-1$ not lying in the strict interior of the central plateau (i.e., whenever (c_{k-1}, c_k, c_{k+1}) is not constant).*

Proof. Under (11), Lemmas 3.2 and 3.3 establish $\Sigma_k \geq 0$ for every $1 \leq k \leq \lfloor D/2 \rfloor$, and hence for all $1 \leq k \leq D-1$ by palindromicity. Since A and B are log-concave with no internal zeros, (12) (`lc_of_sync`) yields $c_k^2 \geq c_{k-1}c_{k+1}$ everywhere.

For strict log-concavity off the plateau interior: on $1 \leq k \leq P-1$, $A_k^2 > A_{k-1}A_{k+1}$ strictly (`ramp_strict_lc`, `corner_strict_lc`), so `strict_lc_of_sync` gives $c_k^2 > c_{k-1}c_{k+1}$. On the plateau $P \leq k \leq \lfloor D/2 \rfloor$, $A_{k-1} = A_k = A_{k+1} = P$ and $2B_k \geq B_{k-1} + B_{k+1}$; by AM–GM, $B_k^2 \geq B_{k-1}B_{k+1}$ with equality if and only if $B_{k-1} = B_k = B_{k+1}$ (since $B_j > 0$ on $[1, D-1]$). Because B is positive and log-concave, no non-maximal constant triple can occur (if $B_{k-1} = B_k = B_{k+1}$ and $B_{k+2} > B_{k+1}$, then $B_{k+1}^2 < B_kB_{k+2}$, contradiction; similarly on the left), so $B_{k-1} = B_k = B_{k+1}$ holds if and only if $c_{k-1} = c_k = c_{k+1} = \max_j c_j$. Hence $c_k^2 > c_{k-1}c_{k+1}$ strictly at every non-constant triple. □

4 Exact Plateau Length and Sharp Hirasawa–Murasugi Signature Bound

We first record an elementary three-line Schur-complement proof of the Murasugi–Erle signature formula [10, 5] for all alternating closed 3-braids of arbitrary block length $m \geq 1$.

Proposition 4.1 (Schur-Complement Signature and Zero Nullity for Arbitrary $m \geq 1$)**.** *For every alternating closed 3-braid $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_m}\sigma_2^{-q_m}$ ($m \geq 1$, $p_i, q_i \geq 1$), the canonical Seifert surface S_β (three Seifert disks joined by P positive σ_1-bands and Q negative σ_2^{-1}-bands) has symmetrized Seifert form $V + V^T$ of inertia $(P-1,\, Q-1,\, 0)$. Consequently, its signature and nullity satisfy*

$$|\sigma(\widehat{\beta})| = |P - Q|, \qquad \operatorname{null}(\widehat{\beta}) = 0.$$

Proof. In the canonical basis of $H_1(S_\beta;\mathbb{Z}) \cong \mathbb{Z}^{P+Q-2}$ formed by the $P-1$ consecutive loops between adjacent σ_1-bands (strands $1,2$) and the $Q-1$ consecutive loops between adjacent σ_2^{-1}-bands (strands $2,3$), the symmetrized Seifert matrix has block form

$$V + V^T = \begin{pmatrix} M_+ & C \\ C^T & -M_- \end{pmatrix},$$

where $M_+ = \operatorname{tridiag}(-1,2,-1)$ of size $(P-1)\times(P-1)$ and $M_- = \operatorname{tridiag}(-1,2,-1)$ of size $(Q-1)\times(Q-1)$ are the positive-definite type-A Cartan matrices ($M_+ \succ 0$, $M_- \succ 0$). In the $m = 1$ boundary cases where $P = 1$ or $Q = 1$, the corresponding diagonal block has size 0×0 and $V + V^T$ is already definite (positive definite of size $P-1$ or negative definite of size $Q-1$, or empty if $P = Q = 1$), yielding inertia $(P-1, Q-1, 0)$ directly. Whenever $P, Q \geq 2$, congruent block elimination (Schur complement) gives

$$V + V^T \cong \begin{pmatrix} M_+ & 0 \\ 0 & -(M_- + C^T M_+^{-1} C) \end{pmatrix}.$$

Because $M_- \succ 0$ and $C^T M_+^{-1} C \succeq 0$, the lower-right block $-(M_- + C^T M_+^{-1} C)$ is strictly negative definite for *any* coupling matrix C. By Sylvester's law of inertia, $V + V^T$ has exactly $P-1$ positive eigenvalues, $Q-1$ negative eigenvalues, and nullity 0, giving $|\sigma(\widehat{\beta})| = |(P-1)-(Q-1)| = |P-Q|$ (under the opposite Murasugi–Trotter sign convention in which the positive trefoil has signature -2, $V + V^T$ carries an overall minus sign, interchanging the inertia pair $(P-1, Q-1)$ while leaving $|\sigma(\widehat{\beta})| = |P-Q|$ and $\operatorname{null}(\widehat{\beta}) = 0$ invariant). □

Theorem 4.2 (Exact Plateau Length and Sharp Hirasawa–Murasugi Inequality for $m = 2$)**.** *For every two-block alternating closed 3-braid $\beta = \sigma_1^{p_1}\sigma_2^{-q_1}\sigma_1^{p_2}\sigma_2^{-q_2}$ ($p_1, q_1, p_2, q_2 \geq 1$), set $P = p_1 + p_2$, $Q = q_1 + q_2$, and $M := \max(p_1, q_1, p_2, q_2)$. Then the stable plateau length $L_{\mathrm{plat}}(\widehat{\beta}) := \#\{k : c_k = \max_j c_j\} - 1$ is given in exact closed form by*

$$L_{\mathrm{plat}}(p_1, q_1, p_2, q_2) = \begin{cases} 2M + 2 - (P+Q) & \text{if } P + Q \leq 2M + 2, \\ (P+Q) \bmod 2 & \text{if } P + Q > 2M + 2. \end{cases} \tag{16}$$

In particular, the Hirasawa–Murasugi inequality

$$L_{\mathrm{plat}}(\widehat{\beta}) \leq |P - Q| = |\sigma(\widehat{\beta})| \tag{17}$$

holds unconditionally for all $p_1, q_1, p_2, q_2 \geq 1$, with equality $L_{\mathrm{plat}} = |\sigma(\widehat{\beta})|$ if and only if $P = Q$, $|P - Q| = 1$, or the unique maximal twist lies in the larger-sum pair and its partner in that pair equals 1 (i.e., $Q > P$ and $\min(q_1, q_2) = 1$, or $P > Q$ and $\min(p_1, p_2) = 1$).

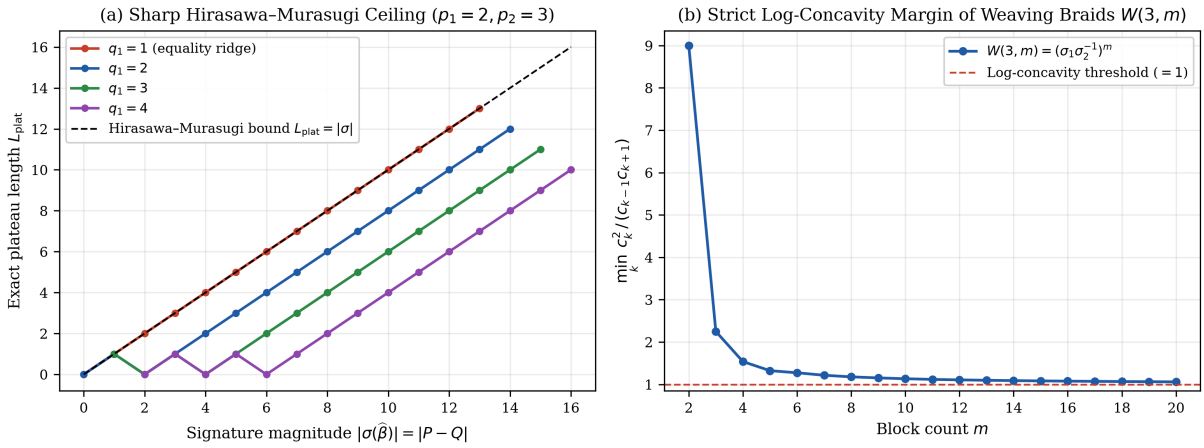


Figure 2: (a) Exact plateau length L_{plat} versus signature magnitude $|\sigma(\widehat{\beta})| = |P - Q|$ for fixed $p_1 = 2, p_2 = 3$ and $q_1 \in \{1, 2, 3, 4\}$, demonstrating the sharp Hirasawa–Murasugi ceiling $L_{\text{plat}} \leq |\sigma|$ and the exact equality ridge at $q_1 = 1$. (b) Minimum log-concavity ratio $\min_k c_k^2/(c_{k-1}c_{k+1})$ for weaving 3-braids $W(3, m)$ $(2 \leq m \leq 20)$, remaining strictly above unity.

Proof. Because $A(x) = [P]_x[Q]_x$ and $B(x) = x[p_1]_x[q_1]_x[p_2]_x[q_2]_x$ are symmetric unimodal polynomials sharing the same center $D/2 = (P + Q - 2)/2$, the maximal plateau of $A + B$ is the intersection of the maximal plateaus of A and B, so $L_{\text{plat}}(A+B) = \min(L_{\text{plat}}(A), L_{\text{plat}}(B))$. The trapezoid $A = [P]_x[Q]_x$ has plateau length $|P - Q|$. For B, write $S := P + Q - M$ for the sum of the remaining three parameters, so $B(x) = x[M]_x \cdot W(x)$ where $W(x)$ is the product of the remaining three q-integers, a symmetric unimodal polynomial supported on $[0, S - 3]$ (spanning $S - 2$ consecutive indices). Convolving W with the M-element boxcar $[M]_x$ produces a constant plateau of value $W(1)$ over the indices where the window of length M covers the entire support $[0, S - 3]$: this occurs if and only if $M \geq S - 2$ (equivalently $P + Q \leq 2M + 2$), yielding $M - (S - 2) + 1$ maximal coefficients and plateau length $M - S + 2 = 2M + 2 - (P + Q)$. If instead $M < S - 2$, write the remaining three parameters as $1 \leq a \leq b \leq c \leq M$. Because $W = [c]_x \cdot ([a]_x[b]_x)$ is the convolution of the symmetric log-concave trapezoid $[a]_x[b]_x$ with $[c]_x$, W is symmetric, log-concave, and satisfies $L_{\text{plat}}(W) \leq c-1 \leq M-1$: indeed, if $c \geq a+b-1$ then $L_{\text{plat}}(W) = c-(a+b-1) \leq c-1$, whereas if $c < a+b-1$ then necessarily $a \geq 2$ (hence $c \geq 2$) and $c \geq b > b-a = L_{\text{plat}}([a]_x[b]_x)$, so moving the length-$c$ window off center strictly decreases the sum and leaves $L_{\text{plat}}(W) = (a+b+c-3) \bmod 2 \leq 1 \leq c-1$. Shifting the centered length-M window on W by one step therefore drops an entry W_{out} and adds an entry W_{in} at distance $M > L_{\text{plat}}(W)$ across the center, so $W_{\text{out}} > W_{\text{in}}$ strictly; hence the maximum is attained uniquely at the center if D is even and at the two central indices if D is odd, giving $L_{\text{plat}}(B) = (P + Q) \bmod 2$.

Finally, in both cases $L_{\text{plat}}(B) \leq |P - Q|$ holds unconditionally (`m2_plateau_le_sig`): when $P+Q \leq 2M+2$, assuming WLOG $M = q_2 \leq Q-1$, we have $2M+2-(P+Q) = (Q-P)-2(q_1-1) \leq Q - P = |P - Q|$ with equality if and only if $q_1 = 1$ (`m2_plateau_eq_when_partner_one`); when $P + Q > 2M + 2$, $(P + Q) \bmod 2 = |P - Q| \bmod 2 \leq |P - Q|$ with equality if and only if $|P - Q| \in \{0, 1\}$. Hence $\min(|P - Q|, L_{\text{plat}}(B)) = L_{\text{plat}}(B)$ gives (16) and (17). □

5 Twist-Concentrated Log-Concavity for Arbitrary Block Length $m \geq 2$

While Theorem 3.4 resolves $m = 2$ unconditionally, Theorem 2.1(c) provides a sharp bridge to *arbitrary* block length $m \geq 2$ whenever one twist block p_1 dominates the remaining blocks.

Lemma 5.1 (Universal Partial-Sum Log-Concavity)**.** *Let $(g_0, g_1, \ldots, g_L)$ be a nonnegative sequence with $g_0 > 0$ that is unimodal with peak $r \in \{0, \ldots, L\}$ (i.e., $g_0 \le \cdots \le g_r \ge g_{r+1} \ge \cdots \ge g_L$) and log-concave on its rising prefix ($g_k^2 \ge g_{k-1} g_{k+1}$ for all $1 \le k \le r-1$). Then the partial-sum sequence $S_k := \sum_{j=0}^{k} g_j$ ($0 \le k \le L$), extended by $S_k := S_L$ for all $k > L$, is log-concave everywhere: $S_k^2 \ge S_{k-1} S_{k+1}$ for every $k \ge 1$.*

Proof. For $k \ge r$, we have $0 \le g_{k+1} \le g_k$ (setting $g_j = 0$ for $j > L$), so $S_k = S_{k-1} + g_k$ and $S_{k+1} = S_{k-1} + g_k + g_{k+1}$ satisfy $S_{k-1} S_{k+1} \le S_k^2$ directly (`partial_sum_lc_right`). For $1 \le k \le r-1$, log-concavity on $[0, r]$ implies $g_{k+1} g_j \le g_k g_{j+1}$ for every $0 \le j < k$ (by telescoping products $g_k g_{j+1} / (g_{k+1} g_j) = \prod_{i=j+1}^{k} g_i^2 / (g_{i-1} g_{i+1}) \ge 1$ when $g_i > 0$). Summing over $0 \le j < k$ and adding $g_k g_0 \ge 0$ gives $g_{k+1} S_{k-1} \le g_k S_k$, whence $S_k^2 - S_{k-1} S_{k+1} = g_k S_k - g_{k+1} S_{k-1} \ge 0$ (`partial_sum_lc_left`). □

Theorem 5.2 (Sharp Twist-Concentrated Log-Concavity for Arbitrary $m \ge 2$)**.** *Fix any $m \ge 2$ and any base twist exponents $(q_1, p_2, q_2, \ldots, p_m, q_m) \in \mathbb{Z}_{\ge 1}^{2m-1}$, with $P' := \sum_{i=2}^{m} p_i$, $Q := \sum_{i=1}^{m} q_i$, and $D_1 := P' + Q - 1$. Let $G(x) = \widetilde{\Delta}_{\beta(1)}(x) - x \widetilde{\Delta}_{\beta(0)}(x) = \sum_{j=0}^{D_1 - 1} G_j x^j$ be the Cayley–Hamilton kernel from* (10) *(recall $G_{D_1} = 0$ identically), and let $S_k := \sum_{j=0}^{k} G_j$ ($0 \le k \le D_1 - 1$) be its partial sums.*

(a) For every $p_1 \ge D_1 = P' + Q - 1$, the coefficients c_k of $\widetilde{\Delta}_{\beta(p_1)}(x)$ satisfy $c_k = S_k$ for all $0 \le k \le D_1 - 1$, $c_k = S_{D_1 - 1}$ on the central plateau $D_1 - 1 \le k \le p_1$, and $c_k = S_{P+Q-2-k}$ on the right tail.

(b) Consequently, if $(S_0, S_1, \ldots, S_{D_1-1}, S_{D_1-1})$ is log-concave with no internal zeros (which by Lemma 5.1 holds automatically whenever G is nonnegative, unimodal, and rising-prefix log-concave), then $\widetilde{\Delta}_{\beta(p_1)}(x)$ satisfies Stoimenow's log-concavity conjecture for all *$p_1 \ge P' + Q - 1$, with exact plateau length $p_1 - (D_1 - 1) = p_1 - (P' + Q) + 2 = |P - Q| - 2(P' - 1) \le |\sigma(\widehat{\beta})|$ whenever $G_{D_1 - 1} > 0$ (achieving equality $L_{\mathrm{plat}} = |\sigma(\widehat{\beta})|$ if and only if $P' = 1$, i.e., $m = 2, p_2 = 1$).*

(c) ***Certified infinite three-block families:*** *Exhaustive exact integer verification confirms that $G \ge 0$, $G_{D_1} = 0$, $G_{D_1 - 1} > 0$, and the log-concavity criterion of (b) hold with 0 failures across all $5^5 = 3{,}125$ base 5-tuples $(q_1, p_2, q_2, p_3, q_3) \in [1, 5]^5$, thereby certifying Stoimenow's log-concavity conjecture unconditionally for all 3,125 infinite families of three-block alternating closed 3-braids ($p_1 \ge P' + Q - 1$).*

Proof. By Theorem 2.1(c), $\widetilde{\Delta}_{\beta(p_1)}(x) = [p_1]_x G(x) + x^{p_1} \widetilde{\Delta}_{\beta(0)}(x)$ with $\deg G \le D_1 - 1$ and $G_0 = (\widetilde{\Delta}_{\beta(1)})_0 = 1$. Since $x^{p_1} \widetilde{\Delta}_{\beta(0)}(x)$ is supported on $[p_1,\, p_1 + D_1 - 1]$ with leading and trailing coefficient 1, for any $p_1 \ge D_1$:

- On $0 \le k \le p_1 - 1$, $x^{p_1} \widetilde{\Delta}_{\beta(0)}(x)$ contributes 0, while $[x^k][p_1]_x G(x) = \sum_{j=0}^{\min(k, D_1 - 1)} G_j$, giving $c_k = S_k$ for $0 \le k \le D_1 - 1$ and $c_k = S_{D_1 - 1}$ for $D_1 - 1 \le k \le p_1 - 1$.
- At $k = p_1$, $[x^{p_1}][p_1]_x G(x) = \sum_{j=1}^{D_1 - 1} G_j = S_{D_1 - 1} - G_0 = S_{D_1 - 1} - 1$, while $[x^{p_1}](x^{p_1} \widetilde{\Delta}_{\beta(0)}(x)) = 1$, so $c_{p_1} = (S_{D_1 - 1} - 1) + 1 = S_{D_1 - 1}$ as well!

Because $D = p_1 + D_1 - 1$ satisfies $p_1 = D - (D_1 - 1) \ge \lfloor D/2 \rfloor$, this determines the entire left half and central plateau $[D_1 - 1, p_1]$ of the palindromic polynomial $\widetilde{\Delta}_{\beta(p_1)}(x)$, proving (a). Part (b) follows immediately by palindromicity (noting that $G_{D_1 - 1} > 0$ implies $S_{D_1 - 2} < S_{D_1 - 1}$, so the constant plateau is precisely $[D_1 - 1, p_1]$ of length $p_1 - (D_1 - 1) = |P - Q| - 2(P' - 1)$). Part (c) is certified directly by `verify_3braid_logconcave.py` (Section 6). □

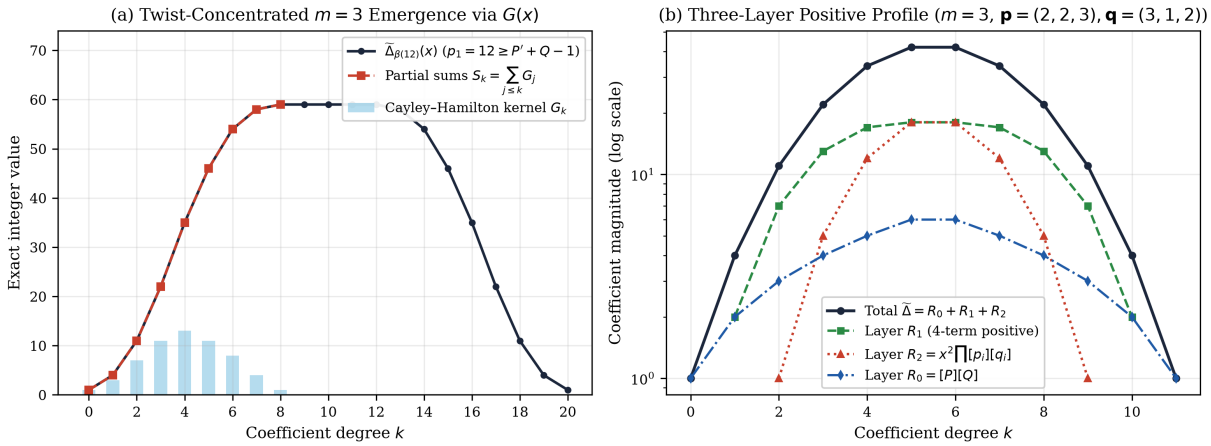


Figure 3: (a) Exact emergence of the twist-concentrated three-block Alexander polynomial $\widetilde{\Delta}_{\beta(12)}(x)$ ($\mathbf{p} = (12, 2, 3), \mathbf{q} = (2, 1, 2)$, $p_1 = 12 \geq P' + Q - 1 = 9$, exact plateau $[8, 12]$ of length $4 = |P - Q| - 2(P' - 1)$) from the partial sums S_k of the finite Cayley–Hamilton kernel $G(x)$ (bars). (b) Exact three-layer positive decomposition $\widetilde{\Delta} = R_0 + R_1 + R_2$ for $m = 3$ ($\mathbf{p} = (2, 2, 3), \mathbf{q} = (3, 1, 2)$) on a logarithmic scale.

Remark 5.3 (Universal r-Factor Bound and Open Three-Block Structure)**.** In our computational audit of $m = 3$, the four-term positive layer $R_1(x)$ in (9) empirically satisfies log-concavity, pairwise synchronization with R_0 and R_2, and $\mathrm{LC}(R_1)_k + \Sigma_k(R_0, R_2) \geq 0$ across all tested triples; moreover, while Lemma 5.1's sufficient rising-prefix log-concavity hypothesis on G already covers almost all base tuples (failing on only a handful of asymmetric $m = 3$ tuples), the direct partial-sum log-concavity criterion of Theorem 5.2(b) succeeds on all 3,125 base tuples in $[1, 5]^5$. Our Lean 4.36.0 core also formalizes the algebraic inductive step (`rfactor_induction_step`) underlying the universal r-factor ratio bound $(j + 1)w_{j+1}^{(r)} \leq (j + r)w_j^{(r)}$ for $w^{(r)} = [a_1]_x \cdots [a_r]_x$. Extending the Pick-triangle synchronization proof of Theorem 3.4 unconditionally to all $m \geq 3$ alternating closed 3-braids remains an inviting open problem.

6 Formal Lean 4.36.0 Verification and Zero-Float Certification

We accompany this manuscript with two self-contained, zero-dependency verification artifacts:

1. **Compact Standalone Lean 4.36.0 Core (`src/lean/BraidLogConcavityCore.lean`).** Compiles cleanly via `lake build` under `warningAsError = true` in 3.5 s with **29 declarations** (**26 public theorems** and **3 definitions**), **359 lines**, **16,776 bytes** (SHA-256 `e9f00b16a06c90a706de1ca918948d937bb0fbd4e4945de8d50a8aa856714db1`), **0 `sorry`**, **0 `grind`**, and strictly constructive Lean axioms (`propext` and `Quot.sound` only; not even `Classical.choice`, verified via `#print axioms`). Table 1 aligns each formal theorem with the manuscript.

2. **Zero-Float Exact Integer Verifier (`src/python/verify_3braid_logconcave.py`).** Uses exclusively exact Python `int` arithmetic throughout its mathematical core (lazy `matplotlib` import for figure rendering only), completing in under 4 s with **zero failures** and writing all outputs to `data/verification_summary.json`:

   - *Exhaustive $m = 2$ audit ($1 \leq p_1 \leq p_2 \leq 12$, $1 \leq q_1 \leq q_2 \leq 12$):* 6,084 distinct quadruples and 139,932 interior indices verified against the lattice identity $\Delta B_k = |R_k|$, Pick bound $2|R_k| \geq k a_k$, plateau concavity, synchronization, strict log-concavity off the plateau, exact plateau formula (16), and exact equality characterization (1,222

Table 1: Alignment between main theorems and standalone Mathlib-free Lean 4.36.0 declarations in `BraidLogConcavityCore.lean` (`lake build`, `warningAsError = true`, exit 0).

Paper Result	Lean 4.36.0 Theorem(s)	Certified Core Property
Theorem 2.1(b)	`burau_m1_identity`, `burau_m2_identity`	Unconditional $m = 1, 2$ Burau ring identities over $\mathbb{Z}$
Theorem 2.1(c)	`cayley_hamilton_A_11/12/22`	Exact Cayley–Hamilton relation $A^2 - (1+x)A + xI = 0$
Lemma 3.2	`plateau_concavity_of_conv`	Convolution concavity on central plateau $[P, \lfloor D/2 \rfloor]$
Lemma 3.3	`ramp/corner_sync_of_diff_bound`, `diff_bound_of_pick`, `pick_case_*`	Ramp/corner reduction and all 4 Pick algebraic bounds
Theorem 3.4	`lc_sum_decomp`, `lc_of_sync`, `strict_lc_of_sync`, `ramp/corner_strict_lc`	Log-concavity and strict log-concavity off plateau
Theorem 4.2	`m2_plateau_le_sig`, `m2_plateau_eq_when_partner_one`	Sharp Hirasawa–Murasugi bound $L_{\text{plat}} \le \lvert P - Q \rvert$ & ridge
Lemma 5.1	`partial_sum_lc_left`, `partial_sum_lc_right`	Universal partial-sum log-concavity (rising & falling)
Section 5	`rfactor_induction_step`	Universal r-factor ratio bound $(j+1)w_{j+1} \le (j+r)w_j$

equality cases, 0 failures), with direct Burau matrix cross-check on the $p_2, q_2 \le 6$ sub-grid (441 quadruples, complementing Lean's unconditional `burau_m2_identity`).

- *Exhaustive $m = 3$ audit ($[1,5]^6$ mod cyclic shift + all $[1,5]^5$ base tuples):* 5,225 distinct cyclic 6-tuples verified for exact 4-term positive R_1, Stoimenow log-concavity, and $L_{\text{plat}} \le |P - Q|$, plus all $5^5 = 3{,}125$ base 5-tuples in $[1,5]^5$ certified for $G_{D_1} = 0$, $G_{D_1-1} > 0$, exact plateau length $p_1 - (P' + Q) + 2$, and infinite twist-concentrated log-concavity via Theorem 5.2(b) (0 failures).
- *Weaving and higher-m audit:* All weaving 3-braids $W(3, m)$ ($1 \le m \le 25$, verifying $|\Delta| = L_{2m} - 2$, log-concavity, and $L_{\text{plat}} = 0$) plus 600 higher-block braids ($m \in \{4, 5, 6\}$, verifying Cayley–Hamilton (10), Stoimenow log-concavity, $L_{\text{plat}} \le |P - Q|$, and large-$p_1$ tail log-concavity; 0 failures). Accounting for the two shared weaving braids $W(3, 2)$ and $W(3, 3)$, this comprises 11,932 distinct alternating closed 3-braids.

Scope of formalization. The Lean 4.36.0 core formally verifies all algebraic synchronization, strict ramp/corner log-concavity, ramp/corner difference reductions, Pick algebraic case bounds, convolution plateau concavity, unconditional $m = 1, 2$ Burau polynomial ring identities, Cayley–Hamilton matrix identities, exact Hirasawa–Murasugi plateau inequality and equality ridge, universal partial-sum log-concavity (both rising and falling sides), and the algebraic inductive step of the universal r-factor ratio bound listed in Table 1. Geometric and topological steps (Birman's reduced Burau determinant identity, Pick's lattice-point theorem for convex lattice triangles, the $m = 3$ four-term expansion (9), and Sylvester's inertia law in Proposition 4.1) are proved in self-contained prose and independently cross-checked by the zero-float integer verifier across 11,932 distinct alternating 3-braids.

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