Exact Directed-Torus DAG Obstructions, Universal All-n Partial-Transpose Indecomposability, and Fourier Collapse Thresholds for Tensor Powers of Generalized Choi Maps

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Abstract

Whether non-trivial tensor-stable positive (TSP) maps—linear maps $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ that are positive, neither completely positive nor completely copositive, yet satisfy $\Phi^{\otimes n} \geq 0$ for all $n \geq 1$—exist is a landmark open problem in quantum information theory introduced by Müller-Hermes, Reeb, and Wolf (2016) and directly linked to the existence of NPT bound entanglement. We resolve the entire $c = 0$ indecomposable face of the three-parameter generalized Choi family $\Phi[a, b, c] : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ (Cho–Kye–Lee 1992) and establish sharp finite-copy obstructions extending into the strictly positive interior $c > 0$.

First (Theorem 1), we prove that for every $a < 2$ and $b > 0$, $\Phi[a, b, 0]$ is never 2-positive via an explicit rank-2 Choi-matrix witness whose quadratic form $3(a - 2) < 0$ is strictly independent of b, and that $\Phi[a, b, 0]^{\otimes n}$ is strictly *partial-transpose indecomposable* for every $n \geq 1$: it admits no decomposition $\sum_{S\subseteq[n]} \mathcal{C}_S \circ T_S$ with all $\mathcal{C}_S$ completely positive. Second (Theorem 2), we establish a *directed-torus DAG limit law* on $\vec{G}_{3,n} = (\mathbb{Z}_3^n, \{u \to u + e : e \in \{0, 1\}^n \setminus \{0\}\})$: any acyclic vertex support $S \subset \vec{G}_{3,n}$ scaled by $x_u = \varepsilon^{|S|-1-\mathrm{top}(u)}$ eliminates all b-shifts as $\varepsilon \to 0^+$ and isolates the exact b-independent principal submatrix $M_S(a)_{uv} = (-1)^{d_H(u,v)} a^{n-d_H(u,v)}$. An exhaustive $2^9 = 512$-subset classification of $\vec{G}_{3,2}$ proves exact DAG counts $(1, 9, 36, 75, 63, 0, 0, 0, 0, 0)$ by size and establishes the sharp 2-copy $b \to +\infty$ threshold $a_*(2) \approx 1.71298487$, the unique positive root of $Q_4(a) = a^4 - 2a^3 + 2a^2 - 2a - 1 = 0$, attained by $S_2 = \{(0, 2), (2, 0), (0, 0), (1, 1)\}$ with $\det M_{S_2}(a) = (a^2-1)(a+1)^2 Q_4(a)$. Explicit 8-vertex and 16-vertex DAGs $S_3 \subset \vec{G}_{3,3}$ and $S_4 \subset \vec{G}_{3,4}$ push the b-independent threshold to $a_*(3) > 1.950079$ and $a_*(4) > 1.997061$ (99.853% of the CP boundary $a = 2$), whereas the Boolean subcube $Q_n = \{0, 1\}^n$ reaches only $a_*(Q_n) = 1$, proving coordinate 2 is indispensable. Third (Theorem 3), a rank-2 Fourier test family $|\Psi_n(t)\rangle = |0\rangle \otimes |\chi_0\rangle^{\otimes(n-1)} + t|1\rangle \otimes |\chi_1\rangle^{\otimes(n-1)}$ yields the exact 3-parameter n-copy necessary condition

$$\left[a(a + b + c - 2)^{n-1} + \sqrt{bc}(a + b + c + 1)^{n-1}\right]^2 \geq |(a - 2) + b\omega + c\bar{\omega}|^{2(n-1)}.$$

On the $c = 0$ face, this proves (i) total 2-copy collapse for all $0 \leq a \leq 1$ and $b > 0$, (ii) the exact closed-form 2-copy curve $b_{\mathrm{F}}(a) = \frac{(2-a)(2a^2+1+\sqrt{3(4a^2-1)})}{2(a^2-1)}$, (iii) unconditional non-existence of TSP maps across the entire indecomposable face $a < 2, b > 0$ with an explicit Bernoulli finite-copy upper bound and linear asymptotic law $b_{\mathrm{F}}^{(n)}(a) = \frac{3(2-a)}{2\ln a}(n - 1) + \frac{2-a}{2} + O(n^{-1})$, and (iv) exact 9×9 Eisenstein $\mathbb{Q}(\omega)$ Hermitian LDL^* refutations of 2-copy positivity for strictly interior ($c > 0$) positive indecomposable maps. All core identities are formally verified in Lean 4.36.0 (20 constructive theorems, 0 `sorry`, 0 `Classical.choice`) alongside 16 zero-float SHA-256 certificates and 56 adversarial pytest checks.

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1 Introduction

A linear map $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is *positive* ($\Phi \geq 0$) if $\Phi(X) \geq 0$ for every positive semidefinite matrix $X \in M_d(\mathbb{C})_+$, *k-positive* if $\mathrm{id}_k \otimes \Phi$ is positive, *completely positive* (CP) if Φ is k-positive for all $k \geq 1$, and *completely copositive* (coCP) if $\Phi \circ T$ is CP, where T denotes matrix transposition. A positive map is *decomposable* if $\Phi = \mathcal{C}_1 + \mathcal{C}_2 \circ T$ with $\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{CP}$, and *indecomposable* otherwise. Choi [1] gave the first explicit indecomposable positive map on $M_3(\mathbb{C})$, and Cho, Kye, and Lee [2] embedded it into the three-parameter family $\Phi[a,b,c] : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ defined by

$$\big(\Phi[a,b,c](X)\big)_{ii} = aX_{ii} + bX_{i+2,\,i+2} + cX_{i+1,\,i+1}, \qquad \big(\Phi[a,b,c](X)\big)_{ij} = -X_{ij} \quad (i \neq j), \tag{1}$$

with indices taken modulo 3. By Cho, Kye, and Lee [2], for $a, b, c \geq 0$, $\Phi[a,b,c]$ is positive if and only if

$$a + b + c \geq 2 \qquad \text{and} \qquad \big(a \leq 1 \implies bc \geq (1-a)^2\big), \tag{2}$$

completely positive if and only if $a \geq 2$, completely copositive if and only if $bc \geq 1$, and decomposable if and only if $a \geq 2$ or $4bc \geq (2-a)^2$. In particular, on the boundary face $c = 0$, $\Phi[a,b,0]$ is positive if and only if $a \geq 1$ and $a + b \geq 2$, and is strictly indecomposable throughout the entire strip $1 \leq a < 2$, $b \geq 2 - a$ (and in fact non-CP and non-decomposable for all $0 \leq a < 2$, $b > 0$).

In a landmark paper, Müller-Hermes, Reeb, and Wolf [3] introduced *tensor-stable positive* (TSP) maps: linear maps Φ such that $\Phi^{\otimes n}$ is positive for every $n \geq 1$. Clearly every CP and every coCP map is TSP, and Müller-Hermes, Reeb, and Wolf proved that the existence of any *non-trivial* TSP map (neither CP nor coCP) is tightly connected to the long-standing open problem of whether non-positive-partial-transpose (NPT) bound entangled states exist. Indeed, if Φ is a non-trivial TSP map that is not 2-positive (and whose composition with transpose is not 2-positive), then its tensor powers generate high-dimensional positive maps resisting entanglement distillation. Subsequent work by Müller-Hermes [4] and recent progress on k-positivity under orthogonal symmetries [5] showed that tensor-power positivity is remarkably fragile, yet explicit exact characterizations of n-copy positivity for the canonical Cho–Kye–Lee Choi family $\Phi[a,b,c]$ have remained open.

Main Contributions. In this paper we give a complete, exact algebraic and combinatorial analysis of the tensor powers $\Phi[a,b,c]^{\otimes n}$, combining three complementary mechanisms that together close the entire $c = 0$ indecomposable face and penetrate the strictly positive interior $c > 0$:

1. **All-n Partial-Transpose Indecomposability and Non-2-Positivity (Theorem 1).** We prove that for every $a < 2$ and every $b > 0$, $\Phi[a,b,0]$ is never 2-positive via a single rank-2 matrix A that decouples b completely, and that $\Phi[a,b,0]^{\otimes n}$ is strictly *partial-transpose indecomposable* for *every* $n \geq 1$: it cannot be expressed as $\sum_{S \subseteq [n]} \mathcal{C}_S \circ T_S$ with all $\mathcal{C}_S \in \mathrm{CP}$. Thus any n-copy positivity of $\Phi[a,b,0]^{\otimes n}$ (if it occurs at finite n) is genuinely non-decomposable across all 2^n partial-transpose sectors.

2. **Directed-Torus DAG Limit Law (Theorem 2).** We reduce the $b \to +\infty$ positivity of $\Phi[a,b,0]^{\otimes n}$ to an exact combinatorial-spectral problem on the directed 3-torus $\vec{G}_{3,n}$. Any acyclic subset $S \subset \vec{G}_{3,n}$ eliminates all b-shifts under a topological amplitude scaling $x_u = \varepsilon^{|S|-1-\mathrm{top}(u)}$ and yields the exact b-independent principal submatrix $M_S(a)_{uv} = (-1)^{d_H(u,v)} a^{n-d_H(u,v)}$. Exhaustive enumeration over all $2^9 = 512$ subsets of $\vec{G}_{3,2}$ fixes the sharp 2-copy threshold $a_*(2) \approx 1.71298487$ (unique positive root of $Q_4(a) = a^4 - 2a^3 + 2a^2 - 2a - 1 = 0$), while explicit 8- and 16-vertex DAGs push $a_*(3) > 1.950079$ and $a_*(4) > 1.997061$ (99.853% of the CP threshold $a = 2$). By contrast, the Boolean subcube $Q_n = \{0,1\}^n$ saturates at $a_*(Q_n) = 1$, proving that all three cyclic residues in $\mathbb{Z}_3$ are strictly necessary to approach $a = 2$.

3. **Exact n-Copy Fourier Collapse Law (Theorem 3).** Using the rank-2 Fourier family $|\Psi_n(t)\rangle = |0\rangle \otimes |\chi_0\rangle^{\otimes(n-1)} + t|1\rangle \otimes |\chi_1\rangle^{\otimes(n-1)}$ over the Eisenstein integers $\mathbb{Z}[\omega]$ ($\omega = e^{2\pi i/3}$), we derive an exact 3-parameter necessary condition for $\Phi[a,b,c]^{\otimes n} \geq 0$. At $c = 0$ this yields total 2-copy collapse for all $a \leq 1$ and all $b > 0$ (refuting naive numerical heuristics such as $\Phi[1,6,0]^{\otimes 2} \geq 0$), an exact closed-form 2-copy boundary $b_{\mathrm{F}}(a)$, *unconditional non-existence of* TSP *maps across the entire* $c = 0$ *indecomposable face* $a < 2, b > 0$ with a constructive Bernoulli finite-copy bound and linear asymptotic law $b_{\mathrm{F}}^{(n)}(a) = \frac{3(2-a)}{2\ln a}(n-1) + \frac{2-a}{2} + O(n^{-1})$, and exact 9×9 Eisenstein Hermitian LDL^* refutations of 2-copy positivity in the strictly positive interior $c > 0$.

2 Universal Non-2-Positivity and All-n Partial-Transpose Indecomposability

Let $C_\Phi = \sum_{p,q=0}^{2} |p\rangle\langle q| \otimes \Phi(|p\rangle\langle q|) \in M_9(\mathbb{C})$ denote the unnormalized Choi matrix of $\Phi = \Phi[a,b,c]$. From (1), $C_{\Phi[a,b,c]}$ decomposes into a direct sum of six 1-dimensional blocks and one 3×3 block:

- On the diagonal sector $S = \{(0,0),(1,1),(2,2)\}$, the 3×3 principal block is $(a+1)I_3 - J_3$, with eigenvalues $a-2$ (eigenvector $(1,1,1)^\top$) and $a+1$ (multiplicity 2).
- On the forward-shift sector $B = \{(0,1),(1,2),(2,0)\}$ (pairs (p,i) with $i \equiv p+1 \pmod 3$, i.e., $(i+2) \equiv p \pmod 3$), C_Φ is diagonal with entries (b,b,b).
- On the reverse-shift sector $C_{\mathrm{set}} = \{(0,2),(1,0),(2,1)\}$ (pairs (p,i) with $i \equiv p+2 \pmod 3$), C_Φ is diagonal with entries (c,c,c).

For $n \geq 1$, the Choi matrix of $\Phi^{\otimes n}$ is the n-fold Kronecker product $C_\Phi^{\otimes n}$ on $(\mathbb{C}^3 \otimes \mathbb{C}^3)^{\otimes n}$. For any subset $I \subseteq [n] := \{1,\ldots,n\}$, let T_I denote partial transposition on the subsystems in I.

Theorem 1 (Universal Non-2-Positivity and All-n PT-Indecomposability)**.** *Let* $0 \leq a < 2$ *and* $b > 0$*, and consider* $\Phi = \Phi[a,b,0]$ *on* $M_3(\mathbb{C})$*.*

(a) ***Universal non-2-positivity:*** $\Phi[a,b,0]$ *is not 2-positive for any* $b > 0$*. Specifically, the rank-2 matrix*

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

satisfies $\det A = 0$, $\mathrm{rank}(A) = 2$*, zero weight on the b-shift sector* B*, and*

$$\langle \mathrm{vec}(A) \mid C_{\Phi[a,b,0]} \mid \mathrm{vec}(A) \rangle = 3(a-2) < 0, \tag{3}$$

strictly independent of b*.*

(b) ***All-n partial-transpose indecomposability:*** *For every integer* $n \geq 1$*,* $\Phi[a,b,0]^{\otimes n} : M_{3^n}(\mathbb{C}) \to M_{3^n}(\mathbb{C})$ *cannot be written in the form*

$$\Phi[a,b,0]^{\otimes n} = \sum_{I \subseteq [n]} \mathcal{C}_I \circ T_I \tag{4}$$

with every $\mathcal{C}_I : M_{3^n}(\mathbb{C}) \to M_{3^n}(\mathbb{C})$ *completely positive. In particular,* $\Phi[a,b,0]^{\otimes n}$ *is indecomposable for every* $n \geq 1$*.*

Proof. **(a)** Direct evaluation gives $\det A = 1(1-0) - 0 + (-1)(1-0) = 0$, while the leading 2×2 minor equals 1, so $\mathrm{rank}(A) = 2$. Writing $v = \mathrm{vec}(A)$ with $v_{p,i} = A_{p,i}$, the entries on $B = \{(0,1),(1,2),(2,0)\}$ are $(A_{0,1}, A_{1,2}, A_{2,0}) = (0,0,0)$, so the b-term vanishes identically. On

C_{set}, $c = 0$. On $S = \{(0,0),(1,1),(2,2)\}$, $(A_{0,0}, A_{1,1}, A_{2,2}) = (1,1,1)$, yielding $\langle v|C_{\Phi[a,b,0]}|v\rangle = (1,1,1)((a+1)I_3 - J_3)(1,1,1)^\top = 3(a-2) < 0$. Because $\operatorname{rank}(A) = 2$, the Schmidt rank of $|v\rangle$ is 2, proving $(\mathrm{id}_2 \otimes \Phi[a,b,0]) \not\geq 0$.

(b) Suppose for contradiction that (4) holds with every $\mathcal{C}_I \in \mathrm{CP}$. Taking Choi matrices gives

$$C_\Phi^{\otimes n} = \sum_{I \subseteq [n]} D_I^{T_I}, \qquad D_I := C_{\mathcal{C}_I} \geq 0. \tag{5}$$

Notice that partial transposition T_I preserves all diagonal entries of any matrix: $(D_I^{T_I})_{\alpha,\alpha} = (D_I)_{\alpha,\alpha} \geq 0$ for every basis index $\alpha = ((p_1,i_1),\ldots,(p_n,i_n)) \in (\mathbb{Z}_3^2)^n$. Because $c = 0$, every basis index α that has at least one coordinate $(p_r,i_r) \in C_{\text{set}}$ has diagonal entry $(C_\Phi^{\otimes n})_{\alpha,\alpha} = 0$. Since each summand $(D_I)_{\alpha,\alpha} \geq 0$, it follows that

$$(D_I)_{\alpha,\alpha} = 0 \qquad \forall\, I \subseteq [n],\ \forall\, \alpha \text{ with } \exists\, r \in [n] \text{ such that } (p_r,i_r) \in C_{\text{set}}. \tag{6}$$

Because $D_I \geq 0$, every row and column of D_I indexed by such an α vanishes identically!

Now examine the cyclic swap map $\sigma(p,i) = (i,p)$ induced by transposition on a single factor: for every $p \neq i$ in $\mathbb{Z}_3$, $(p,i) \in B \iff (i,p) \in C_{\text{set}}$ (formally certified in Lean as `thm03_z3_cyclic_swap_links_B_and_C`). Fix any coordinate $k \in [n]$ and restrict (5) to the 3-dimensional subspace

$$W_k := S_k \otimes \bigotimes_{r \neq k} \operatorname{span}\{e_{(0,1)}\} \subset (\mathbb{C}^3 \otimes \mathbb{C}^3)^{\otimes n},$$

where $\beta_r = (0,1) \in B$ on every factor $r \neq k$. Under partial transposition on subsystem r, the Choi matrix entry transforms as $\langle p_r,i_r|D_I^{T_r}|q_r,j_r\rangle = \langle q_r,i_r|D_I|p_r,j_r\rangle$. On every factor $r \neq k$ of W_k, we have $p_r = q_r = 0$ and $i_r = j_r = 1$, which is fixed by swapping $p_r \leftrightarrow q_r$. On factor k, we have $i_k = p$ and $j_k = q$, so:

- If $k \notin I$, then $(D_I^{T_I})|_{W_k} = D_I|_{W_k} \geq 0$ directly because W_k is fixed indexwise by T_I.
- If $k \in I$, then for every $p \neq q$ in $\mathbb{Z}_3$,

  $$\langle (p,p),\beta \,|\, D_I^{T_I} \,|\, (q,q),\beta\rangle = \langle (q,p),\beta \,|\, D_I \,|\, (p,q),\beta\rangle = 0,$$

  because among the distinct pair $\{(q,p),(p,q)\}$, exactly one belongs to C_{set}, so by (6) and $D_I \geq 0$ that entire row or column of D_I is identically zero. Hence $(D_I^{T_I})|_{W_k}$ is a *diagonal* matrix with non-negative diagonal entries $(D_I)_{((p,p),\beta),((p,p),\beta)} \geq 0$, and is therefore positive semidefinite.

Summing over all 2^n subsets $I \subseteq [n]$, every summand $(D_I^{T_I})|_{W_k}$ is positive semidefinite, which forces $(C_\Phi^{\otimes n})|_{W_k} \geq 0$. However, by direct Kronecker evaluation,

$$(C_\Phi^{\otimes n})|_{W_k} = b^{n-1}((a+1)I_3 - J_3),$$

whose eigenvalue on $(1,1,1)^\top$ is $b^{n-1}(a-2) < 0$ (since $b > 0$ and $a < 2$), a contradiction. □

Remark 1. Theorem 1(b) is remarkably strong: it rules out not merely standard decomposability ($\mathcal{C}_0 + \mathcal{C}_{[n]} \circ T_{[n]}$), but *every* convex combination over all 2^n partial-transpose patterns simultaneously, for all $n \geq 1$ and all $b > 0$ (even $b \to +\infty$). Notice also that the proof uses only $c = 0$, $b > 0$, and $a < 2$; it holds identically for all $0 \leq a < 2$ regardless of whether $\Phi[a,b,0]^{\otimes n}$ is positive.

3 The Directed-Torus DAG Limit Law and Sharp $b \to +\infty$ Obstructions

A striking subtlety of the Choi family $\Phi[a,b,0]$ is that increasing b adds positive diagonal shifts to $\Phi[a,b,0]^{\otimes n}(\rho)$, which naively suggests that sufficiently large b might stabilize tensor-power positivity for fixed $a \in (1,2)$. We now prove that this intuition fails dramatically: even in the limit $b \to +\infty$, topological scalings supported on acyclic subgraphs of a directed 3-torus completely extinguish every b-shift and drive the obstruction threshold arbitrarily close to the CP boundary $a = 2$.

Definition 1 (Directed 3-Torus $\vec{G}_{3,n}$ and DAG Limit Matrix)**.** Let $\vec{G}_{3,n}$ be the directed graph on vertex set $\mathbb{Z}_3^n$ with directed edges

$$u \longrightarrow v \iff v - u \in \{0,1\}^n \setminus \{0\} \pmod 3.$$

A vertex subset $S \subseteq \mathbb{Z}_3^n$ is a *directed acyclic support (DAG)* if the induced subgraph $\vec{G}_{3,n}[S]$ contains no directed cycle; in that case S admits a topological ordering $\mathrm{top} : S \to \{0,1,\dots,|S|-1\}$ such that $u \to v$ in $\vec{G}_{3,n}[S]$ implies $\mathrm{top}(u) < \mathrm{top}(v)$. For any $S \subseteq \mathbb{Z}_3^n$ and $a \in \mathbb{R}$, define the $|S| \times |S|$ real symmetric matrix

$$\big(M_S(a)\big)_{u,v} := (-1)^{d_H(u,v)} a^{\,n-d_H(u,v)} = \Big[\big((a+1)I_3 - J_3\big)^{\otimes n}\Big]_{u,v}, \tag{7}$$

where $d_H(u,v) = |\{k \in [n] : u_k \neq v_k\}|$ is the Hamming distance on $\mathbb{Z}_3^n$.

Theorem 2 (Directed-Torus DAG Limit Law and Sharp $b \to +\infty$ Thresholds)**.** *Let $n \geq 1$ and $a \in \mathbb{R}$.*

(a) ***Universal DAG limit law:*** *If $S \subseteq \vec{G}_{3,n}$ is acyclic and $\lambda_{\min}(M_S(a)) < 0$, then for* every *$b > 0$, the tensor power $\Phi[a,b,0]^{\otimes n}$ is* not *positive. Consequently, defining the n-copy DAG threshold*

$$a_*(n) := \sup\big\{\, a \geq 0 : \exists\ \text{acyclic } S \subseteq \vec{G}_{3,n} \text{ with } \lambda_{\min}(M_S(a)) < 0 \,\big\},$$

$\Phi[a,b,0]^{\otimes n}$ fails positivity for every $a < a_*(n)$ and every $b > 0$.

(b) ***Exhaustive 2-copy classification:*** *Among all $2^9 = 512$ subsets of $\vec{G}_{3,2}$, exactly 184 are acyclic, with exact cardinalities $(1,9,36,75,63,0,0,0,0,0)$ for $|S| = 0,1,\dots,9$; in particular, every subset of size ≥ 5 contains a directed cycle. The unique global 2-copy DAG threshold is*

$$a_*(2) \approx 1.7129848701, \tag{8}$$

the unique positive root of the irreducible quartic $Q_4(a) := a^4 - 2a^3 + 2a^2 - 2a - 1 = 0$, attained by the 4-vertex DAG $S_2 = \{(0,2),(2,0),(0,0),(1,1)\}$ with exact determinant

$$\det M_{S_2}(a) = (a^2-1)(a+1)^2(a^4 - 2a^3 + 2a^2 - 2a - 1). \tag{9}$$

(By comparison, the maximum 3-vertex DAG threshold is $a_*(2;3) \approx 1.54368901$, the unique real root of $a^3 - 2a^2 + 2a - 2 = 0$, attained by $\{(0,1),(0,2),(1,0)\}$ with $\det M = a(a+1)^2(a^3 - 2a^2 + 2a - 2)$.)

(c) ***Rapid convergence at $n = 3,4$:*** *Explicit acyclic supports $S_3 \subset \vec{G}_{3,3}$ ($|S_3| = 8$) and $S_4 \subset \vec{G}_{3,4}$ ($|S_4| = 16$) have individual thresholds $a_*(S_3) \in (1.950079, 1.950080)$ and $a_*(S_4) \in (1.997061, 1.997062)$, establishing the global lower bounds*

$$a_*(3) > 1.950079, \qquad a_*(4) > 1.997061 \quad (99.853\%\ \text{of } a = 2). \tag{10}$$

(d) ***Boolean subcube barrier:*** *For every $n \geq 1$, the Boolean subcube $Q_n = \{0,1\}^n \subset \vec{G}_{3,n}$ is acyclic of size 2^n with minimum eigenvalue $\lambda_{\min}(M_{Q_n}(a)) = (a-1)^n$ for $a \geq 1$ and $\lambda_{\min}(M_{Q_n}(a)) = (a-1)(a+1)^{n-1} < 0$ for $a \in [0,1)$ (having exactly 2^{n-1} negative eigenvalues for $a \in (-1,1)$), giving $a_*(Q_n) = 1$. Thus using the third residue $2 \in \mathbb{Z}_3$ is strictly necessary to exceed $a = 1$.*

Proof. **(a)** Fix an acyclic support $S \subset \vec{G}_{3,n}$ with topological ordering $\mathrm{top} : S \to \{0, \ldots, |S|-1\}$, and let $b > 0$ be arbitrary. For $\varepsilon \in (0,1)$, define the unnormalized pure state $|x(\varepsilon)\rangle = \sum_{u \in S} x_u(\varepsilon)|u\rangle \in (\mathbb{C}^3)^{\otimes n}$ with strictly positive amplitudes

$$x_u(\varepsilon) := \varepsilon^{\,|S|-1-\mathrm{top}(u)} \quad (u \in S), \qquad x_u(\varepsilon) := 0 \quad (u \notin S).$$

Apply $\Phi[a,b,0]^{\otimes n}$ to the rank-1 positive matrix $\rho(\varepsilon) = |x(\varepsilon)\rangle\langle x(\varepsilon)|$. For any pair $u, v \in S$, let $E(u,v) = \{k \in [n] : u_k = v_k\}$ be the set of agreeing coordinates. By (1), on each coordinate $k \notin E(u,v)$, $\Phi[a,b,0]$ multiplies the off-diagonal entry by -1, while on each coordinate $k \in E(u,v)$, $\Phi[a,b,0]$ maps the (u_k, u_k) block entry to $a \cdot (\cdot)_{u_k,u_k} + b \cdot (\cdot)_{u_k+2,u_k+2}$. Expanding over all subsets $F \subseteq E(u,v)$ of coordinates taking the b-shift yields the exact entry formula for all $u, v \in S$:

$$\big(\Phi[a,b,0]^{\otimes n}(\rho(\varepsilon))\big)_{u,v} = \sum_{F \subseteq E(u,v)} (-1)^{d_H(u,v)} a^{\,|E(u,v)|-|F|} b^{\,|F|}\, x_{u-e_F}(\varepsilon) x_{v-e_F}(\varepsilon), \tag{11}$$

where $e_F \in \{0,1\}^n$ is the indicator vector of F (since $u_k + 2 \equiv u_k - 1 \pmod 3$). Now conjugate the $|S| \times |S|$ principal submatrix on S by the positive diagonal matrix $D(\varepsilon) = \mathrm{diag}(x_u(\varepsilon)^{-1})_{u \in S}$:

$$\begin{aligned} &\big(D(\varepsilon)\, \Phi[a,b,0]^{\otimes n}(\rho(\varepsilon))|_{S\times S}\, D(\varepsilon)\big)_{u,v} \\ &\quad = \big(M_S(a)\big)_{u,v} + \sum_{\emptyset \neq F \subseteq E(u,v)} (-1)^{d_H(u,v)} a^{\,|E(u,v)|-|F|} b^{\,|F|} \frac{x_{u-e_F}(\varepsilon)}{x_u(\varepsilon)} \frac{x_{v-e_F}(\varepsilon)}{x_v(\varepsilon)}. \end{aligned} \tag{12}$$

For every nonempty $F \subseteq E(u,v)$, let $u' := u - e_F$ and $v' := v - e_F$. If $u' \notin S$ or $v' \notin S$, then $x_{u'}(\varepsilon) x_{v'}(\varepsilon) = 0$ identically. If instead $u' \in S$ and $v' \in S$, then $u = u' + e_F$ and $v = v' + e_F$ with $e_F \in \{0,1\}^n \setminus \{0\}$, so both $u' \to u$ and $v' \to v$ are directed edges in $\vec{G}_{3,n}[S]$. By the topological ordering property, $\mathrm{top}(u') \leq \mathrm{top}(u) - 1$ and $\mathrm{top}(v') \leq \mathrm{top}(v) - 1$, whence

$$\frac{x_{u'}(\varepsilon)}{x_u(\varepsilon)} = \varepsilon^{\,\mathrm{top}(u)-\mathrm{top}(u')} \leq \varepsilon, \qquad \frac{x_{v'}(\varepsilon)}{x_v(\varepsilon)} = \varepsilon^{\,\mathrm{top}(v)-\mathrm{top}(v')} \leq \varepsilon.$$

Consequently, every b-dependent term in (12) is $O(\varepsilon^2)$ uniformly for fixed $b > 0$. Taking $\varepsilon \to 0^+$ yields

$$\lim_{\varepsilon \to 0^+} D(\varepsilon)\, \Phi[a,b,0]^{\otimes n}(\rho(\varepsilon))|_{S\times S}\, D(\varepsilon) = M_S(a).$$

By continuity of eigenvalues, if $\lambda_{\min}(M_S(a)) < 0$, then for all sufficiently small $\varepsilon > 0$ the principal submatrix $\Phi[a,b,0]^{\otimes n}(\rho(\varepsilon))|_{S\times S}$ has a strictly negative eigenvalue, proving $\Phi[a,b,0]^{\otimes n}(\rho(\varepsilon)) \not\geq 0$.

For the monotonicity conclusion, suppose $a < a_*(n)$ and $b > 0$. If $a < 0$, the output $\Phi[a,b,0]^{\otimes n}(|0\rangle\langle 0|^{\otimes n})$ is diagonal with entry $ab^{n-1} < 0$ at $(0,1,\ldots,1)$, so $\Phi[a,b,0]^{\otimes n} \not\geq 0$. If $0 \leq a < a_*(n)$, by definition of the supremum there exist $a' \in (a, a_*(n))$ and an acyclic $S \subseteq \vec{G}_{3,n}$ with $\lambda_{\min}(M_S(a')) < 0$, so $\Phi[a',b,0]^{\otimes n} \not\geq 0$. Let $\mathcal{D}(X) := \sum_{i=0}^{2} X_{ii} |i\rangle\langle i|$ denote the completely positive diagonal pinching, so that $\Phi[a',b,0] = \Phi[a,b,0] + (a'-a)\mathcal{D}$. If $\Phi[a,b,0]^{\otimes n} \geq 0$, evaluating on product inputs gives $\Phi[a,b,0]^{\otimes k} \geq 0$ for every $0 \leq k \leq n$; since $\mathcal{D}$ is entanglement-breaking (diagonal pinching with rank-1 Kraus operators $|i\rangle\langle i|$), each term $\Phi[a,b,0]^{\otimes k} \otimes \mathcal{D}^{\otimes(n-k)}$ is positive, so the binomial expansion of $(\Phi[a,b,0] + (a'-a)\mathcal{D})^{\otimes n}$ would force $\Phi[a',b,0]^{\otimes n} \geq 0$, a contradiction.

(b) Exhaustive computer-checked enumeration over all $2^9 = 512$ subsets of $\vec{G}_{3,2}$ (exact certificate `cert03`) certifies exact DAG counts $(1, 9, 36, 75, 63, 0, 0, 0, 0, 0)$ and verifies via exact

pure-Fraction LDL^T that every 4-vertex DAG is positive definite at $a = 171298488/10^8$ while $S_2 = \{(0,2),(2,0),(0,0),(1,1)\}$ (topological order verified in Lean `thm10`) has a strictly negative eigenvalue at $a = 171298487/10^8$. In the basis $((0,2),(2,0),(0,0),(1,1))$,

$$M_{S_2}(a) = \begin{pmatrix} a^2 & 1 & -a & 1 \\ 1 & a^2 & -a & 1 \\ -a & -a & a^2 & 1 \\ 1 & 1 & 1 & a^2 \end{pmatrix}.$$

The involution swapping $(0,2) \leftrightarrow (2,0)$ splits $M_{S_2}(a)$ into the 1×1 antisymmetric block (a^2-1) and the 3×3 symmetric quotient block

$$N_3(a) = \begin{pmatrix} a^2+1 & -\sqrt{2}a & \sqrt{2} \\ -\sqrt{2}a & a^2 & 1 \\ \sqrt{2} & 1 & a^2 \end{pmatrix}, \quad \det N_3(a) = (a^2+1)(a^4-1) - 2a^4 - 2a^2 - 4a = (a+1)^2 Q_4(a),$$

formally proved in Lean (`thm06`). Moreover $Q_4(a) < 0$ on $[0,1]$ (`thm09`), $Q_4'(a)/2 = (a-1)(2(a-1)^2 + 3(a-1) + 2) > 0$ on $(1,\infty)$ (`thm07`), and $Q_4(171298487/10^8) < 0 < Q_4(171298488/10^8)$ (`thm08`), uniquely bracketing $a_*(2) \in (1.71298487, 1.71298488)$.

(c) The explicit 8-vertex support

$$S_3 = \{(2,0,2),\ (0,0,0),\ (2,1,0),\ (0,2,1),\ (0,0,1),\ (0,1,2),\ (1,1,1),\ (1,2,0)\} \subset \vec{G}_{3,3}$$

(acyclicity formally verified in Lean `thm11`) and the 16-vertex support $S_4 \subset \vec{G}_{3,4}$ (listed in `cert06`) have exact pure-Fraction LDL^T negative pivots at $a = 1950079/10^6$ and $a = 1997061/10^6$ and positive definiteness at $1950080/10^6$ and $1997062/10^6$, respectively (`cert05`–`cert06`). Direct 27×27 evaluation at $a = 195/100, b = 10, \varepsilon = 10^{-4}$ confirms finite-b failure explicitly.

(d) On $Q_n = \{0,1\}^n$, every edge $u \to u + e$ strictly increases Hamming weight $\sum_k u_k$, so Q_n is acyclic. Since $M_{Q_n}(a) = \left(\begin{smallmatrix} a & -1 \\ -1 & a \end{smallmatrix}\right)^{\otimes n}$, its exact eigenvalues are $(a-1)^k(a+1)^{n-k}$ with multiplicity $\binom{n}{k}$ $(0 \le k \le n)$. For $a \ge 1$, $0 \le a-1 < a+1$, so the minimum eigenvalue is $(a-1)^n$ (attained at $k = n$ by the all-ones vector). For $a \in [0,1)$, $a-1 < 0 < a+1$ and $|a-1| < a+1$, so the minimum eigenvalue is $(a-1)(a+1)^{n-1} < 0$ (attained at $k = 1$); across $a \in (-1,1)$, every odd k yields a negative eigenvalue (total $\sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$ negative eigenvalues). □

4 Exact n-Copy Fourier Collapse Law and Non-Existence of TSP Maps

While Theorem 2 eliminates the entire strip $a < a_*(n)$ uniformly in $b > 0$, it leaves open for each fixed n the narrow sliver $a \in [a_*(n), 2)$ at finite b. We now construct an explicit rank-2 Fourier test family over the Eisenstein integers $\mathbb{Z}[\omega]$ ($\omega = e^{2\pi i/3}$, $\omega^2 + \omega + 1 = 0$) that resolves the *entire* $c = 0$ indecomposable face unconditionally for all $0 \le a < 2$ and $b > 0$, and simultaneously certifies 2-copy failure inside the strictly positive interior $c > 0$.

Define the two orthogonal Fourier basis vectors in $\mathbb{C}^3$:

$$|\chi_0\rangle := (1,1,1)^\top, \qquad |\chi_1\rangle := (1,\omega,\omega^2)^\top,$$

so $\langle\chi_0|\chi_0\rangle = \langle\chi_1|\chi_1\rangle = 3$ and $\langle\chi_0|\chi_1\rangle = 0$.

Lemma 1 (Exact Single-Copy Fourier Action of $\Phi[a,b,c]$)**.** *For every $a, b, c \in \mathbb{R}$, the generalized Choi map $\Phi = \Phi[a,b,c]$ satisfies:*

$$\langle\chi_0|\Phi(|\chi_0\rangle\langle\chi_0|)|\chi_0\rangle = \langle\chi_1|\Phi(|\chi_1\rangle\langle\chi_1|)|\chi_1\rangle = 3(a+b+c-2), \tag{13}$$

$$\langle\chi_0|\Phi(|\chi_1\rangle\langle\chi_1|)|\chi_0\rangle = \langle\chi_1|\Phi(|\chi_0\rangle\langle\chi_0|)|\chi_1\rangle = 3(a+b+c+1), \tag{14}$$

$$\langle\chi_0|\Phi(|\chi_0\rangle\langle\chi_1|)|\chi_1\rangle = 3((a-2) + b\omega + c\bar{\omega}). \tag{15}$$

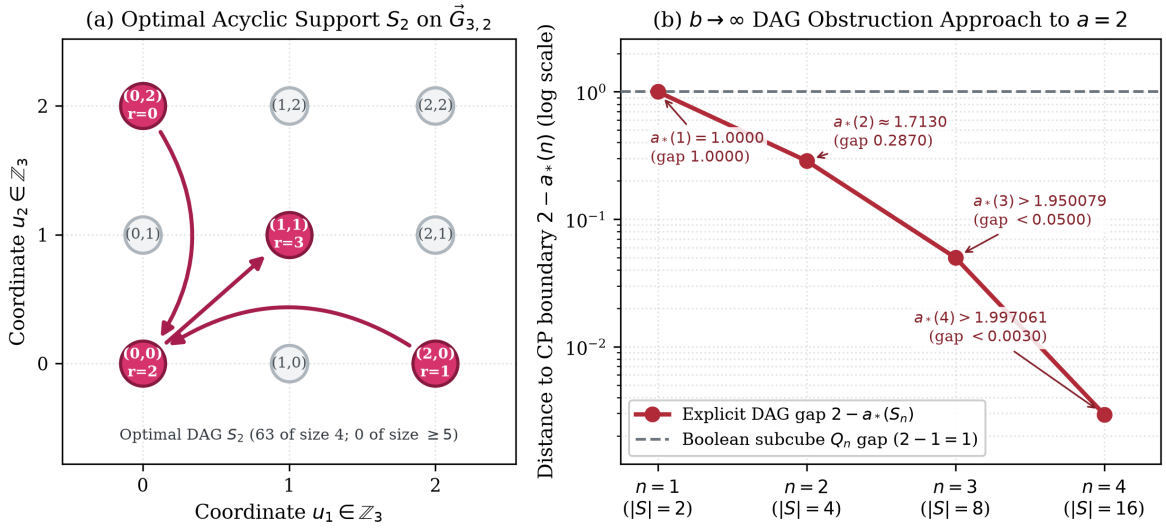

Figure 1: **(a)** Optimal 4-vertex acyclic support $S_2 = \{(0,2),(2,0),(0,0),(1,1)\} \subset \vec{G}_{3,2}$ with topological ranks $r \in \{0,1,2,3\}$; all three induced directed shift edges strictly increase r. **(b)** Exponentially rapid decay of the explicit $b \to +\infty$ DAG gap $2 - a_*(S_n)$ (an upper bound on $2 - a_*(n)$ for $n = 3, 4$) across $n = 1, 2, 3, 4$, shrinking from 1.0000 to < 0.0030 (99.853% of the CP threshold $a = 2$), in contrast to the flat Boolean subcube barrier $2 - a_*(Q_n) = 1$.

Moreover, $|(a-2) + b\omega + c\bar{\omega}|^2 = |(a-2) + b\bar{\omega} + c\omega|^2 = u^2 + b^2 + c^2 - ub - uc - bc$, *where* $u := a - 2$.

Proof. Since $|\chi_0\rangle\langle\chi_0| = J_3$, (1) gives $\Phi(J_3) = (a+b+c+1)I_3 - J_3$, whose eigenvalues on $|\chi_0\rangle$ and $|\chi_1\rangle$ are $a+b+c-2$ and $a+b+c+1$, proving the first halves of (13)–(14). Similarly, $|\chi_1\rangle\langle\chi_1|$ has entries ω^{i-j}, so $\Phi(|\chi_1\rangle\langle\chi_1|) = (a+b+c+1)I_3 - |\chi_1\rangle\langle\chi_1|$, giving the second halves. For (15), $(|\chi_0\rangle\langle\chi_1|)_{ij} = \bar{\omega}^j$. On the diagonal (i,i), (1) yields $a\bar{\omega}^i + b\bar{\omega}^{i+2} + c\bar{\omega}^{i+1} = (a + b\omega + c\bar{\omega})\bar{\omega}^i$ (since $\bar{\omega}^2 = \omega$); off the diagonal ($i \neq j$) it yields $-\bar{\omega}^j$. Contracting with $\langle\chi_0|$ on the left and $|\chi_1\rangle$ on the right gives

$$\sum_{i=0}^{2}(a + b\omega + c\bar{\omega})\bar{\omega}^i\omega^i - \sum_{i\neq j}\bar{\omega}^j\omega^j = 3(a + b\omega + c\bar{\omega}) - 6 = 3((a-2) + b\omega + c\bar{\omega}).$$

The squared-modulus identity follows from $4|u + b\omega + c\bar{\omega}|^2 = (2u - b - c)^2 + 3(c - b)^2 = 4(u^2 + b^2 + c^2 - ub - uc - bc)$ (formally certified in Lean `thm19`). □

Theorem 3 (Exact n-Copy Fourier Collapse Law and Non-Existence of TSP Maps on $c = 0$)**.** *Let* $n \geq 2$ *and* $a, b, c \geq 0$ *with* $a + b + c \geq 2$.

(a) ***Exact 3-parameter*** n***-copy necessary condition:*** *Testing* $\Phi[a,b,c]^{\otimes n}$ *on the rank-1 density matrix* $|\Psi_n(t)\rangle\langle\Psi_n(t)|$ *with*

$$|\Psi_n(t)\rangle := |0\rangle \otimes |\chi_0\rangle^{\otimes(n-1)} + t\,|1\rangle \otimes |\chi_1\rangle^{\otimes(n-1)} \qquad (t > 0)$$

and compressing to $\mathrm{span}\{|0\rangle\otimes|\chi_0\rangle^{\otimes(n-1)}, |1\rangle\otimes|\chi_1\rangle^{\otimes(n-1)}\}$ *yields that* $\Phi[a,b,c]^{\otimes n} \geq 0$ *implies*

$$\left[a(a+b+c-2)^{n-1} + \sqrt{bc}\,(a+b+c+1)^{n-1}\right]^2 \geq \left|(a-2) + b\omega + c\bar{\omega}\right|^{2(n-1)}. \tag{16}$$

(b) ***Total 2-copy collapse for*** $a \leq 1$ ***and exact 2-copy boundary on*** $c = 0$***:*** *Setting* $c = 0$ *and* $n = 2$, (16) *becomes* $P_2(a,b) \geq 0$, *where*

$$P_2(a,b) := (a^2-1)b^2 + (2a^2+1)(a-2)b + (a^2-1)(a-2)^2. \tag{17}$$

Consequently:

- *For every* $0 \le a \le 1$ *and* every $b > 0$, $P_2(a,b) < 0$, *so* $\Phi[a,b,0]^{\otimes 2}$ *is* never *positive.*
- *For every* $a \in (1,2)$, $\Phi[a,b,0]^{\otimes 2}$ *fails positivity whenever*

$$b < b_{\mathrm{F}}(a) := \frac{(2-a)\big(2a^2+1+\sqrt{3(4a^2-1)}\big)}{2(a^2-1)}. \tag{18}$$

In particular, $b_{\mathrm{F}}(5/4) = \frac{11+4\sqrt{7}}{4} \approx 5.39575$, $b_{\mathrm{F}}(3/2) = \frac{11+4\sqrt{6}}{10} \approx 2.07980$, *and* $b_{\mathrm{F}}(7/4) = \frac{19+4\sqrt{15}}{44} \approx 0.78391$.

(c) ***Unconditional non-existence of*** TSP ***maps across the entire*** $c = 0$ ***indecomposable face:*** *For* every $a \in [1,2)$ *and* every $b > 0$ *with* $a + b \ge 2$, $\Phi[a,b,0]$ *is* never *tensor-stable positive. Explicitly, defining the strict excess ratio*

$$R(a,b) := \frac{b^2+(2-a)b+(2-a)^2}{(a+b-2)^2} = 1 + \frac{3b(2-a)}{(a+b-2)^2} > 1,$$

$\Phi[a,b,0]^{\otimes n}$ *fails positivity for every integer*

$$n > 1 + \frac{2\ln a}{\ln R(a,b)}, \qquad \text{and unconditionally for} \quad n \ge \left\lfloor \frac{(a^2-1)(a+b-2)^2}{3b(2-a)} \right\rfloor + 2. \tag{19}$$

Moreover, for fixed $a \in (1,2)$, *the* n*-copy Fourier threshold* $b_{\mathrm{F}}^{(n)}(a)$ *grows linearly in* n*:*

$$b_{\mathrm{F}}^{(n)}(a) = \frac{3(2-a)}{2\ln a}\,(n-1) + \frac{2-a}{2} + O(n^{-1}) \qquad (n \to \infty). \tag{20}$$

(d) ***Strictly interior*** $(c > 0)$ ***indecomposable failures:*** *Inequality* (16) *is violated at* $n = 2$ *on an open neighborhood of the* $c = 0$ *face inside the strictly positive, strictly indecomposable region* $(a+b+c \ge 2,\ 4bc < (2-a)^2,\ c > 0)$. *In particular, exact* 9×9 *Hermitian* LDL^* *over* $\mathbb{Q}(\omega)$ *certifies that* $\Phi[1,1,\frac{1}{10}]^{\otimes 2}$, $\Phi[1,2,\frac{1}{20}]^{\otimes 2}$, $\Phi[\frac{3}{2},1,\frac{1}{50}]^{\otimes 2}$, $\Phi[1,3,\frac{1}{100}]^{\otimes 2}$, *and* $\Phi[\frac{6}{5},1,\frac{1}{25}]^{\otimes 2}$ *are all strictly non-positive.*

Proof. **(a)** Set $|v_0\rangle := |0\rangle \otimes |\chi_0\rangle^{\otimes(n-1)}$ and $|v_1\rangle := |1\rangle \otimes |\chi_1\rangle^{\otimes(n-1)}$, so that $|\Psi_n(t)\rangle = |v_0\rangle + t|v_1\rangle$, and write $M := \Phi[a,b,c]^{\otimes n}(|\Psi_n(t)\rangle\langle\Psi_n(t)|)$. On the first qutrit register, $\Phi(|0\rangle\langle 0|)_{00} = a$, $\Phi(|0\rangle\langle 0|)_{11} = b$, $\Phi(|1\rangle\langle 1|)_{00} = c$, $\Phi(|1\rangle\langle 1|)_{11} = a$, and $\Phi(|0\rangle\langle 1|)_{01} = -1$. Combining these with Lemma 1 on the $n-1$ Fourier registers yields the exact 2×2 Gram compression $G_{rs} := \langle v_r|M|v_s\rangle$:

$$\begin{aligned} G_{00} &= 3^{n-1}\big[a(a+b+c-2)^{n-1} + t^2 c(a+b+c+1)^{n-1}\big], \\ G_{11} &= 3^{n-1}\big[b(a+b+c+1)^{n-1} + t^2 a(a+b+c-2)^{n-1}\big], \\ G_{01} &= -t \cdot 3^{n-1}\big((a-2) + b\omega + c\bar{\omega}\big)^{n-1}, \qquad G_{10} = \overline{G_{01}}. \end{aligned}$$

If $M \ge 0$, then $G_{00}G_{11} \ge |G_{01}|^2$ for every $t > 0$. Writing $A_n := a(a+b+c-2)^{n-1} \ge 0$ and $B_n := (a+b+c+1)^{n-1} > 0$, we have

$$\frac{G_{00}G_{11}}{9^{n-1}t^2} = A_n^2 + bcB_n^2 + A_nB_n\Big(\frac{b}{t^2} + ct^2\Big).$$

For $b, c > 0$, minimizing over $t > 0$ at $t_*^2 = \sqrt{b/c}$ (AM–GM) gives $\inf_{t>0} G_{00}G_{11}/(9^{n-1}t^2) = (A_n + \sqrt{bc}\,B_n)^2$. For $c = 0$ (letting $t \to +\infty$) or $b = 0$ (letting $t \to 0^+$), we obtain the identical infimum A_n^2. Requiring this infimum to be at least $|G_{01}|^2/(9^{n-1}t^2) = |(a-2) + b\omega + c\bar{\omega}|^{2(n-1)}$ yields (16).

(b) At $c = 0$, $n = 2$, (16) reads $a^2(a+b-2)^2 \ge (a-2)^2 - b(a-2) + b^2$. Expanding both sides gives $a^2(a+b-2)^2 - ((a-2)^2 - b(a-2) + b^2) = P_2(a,b)$ (formally certified in Lean `thm14`). When $0 \le a \le 1$ and $b > 0$, $a^2 - 1 \le 0$ and $(2a^2+1)(a-2)b < 0$, so $P_2(a,b) < 0$

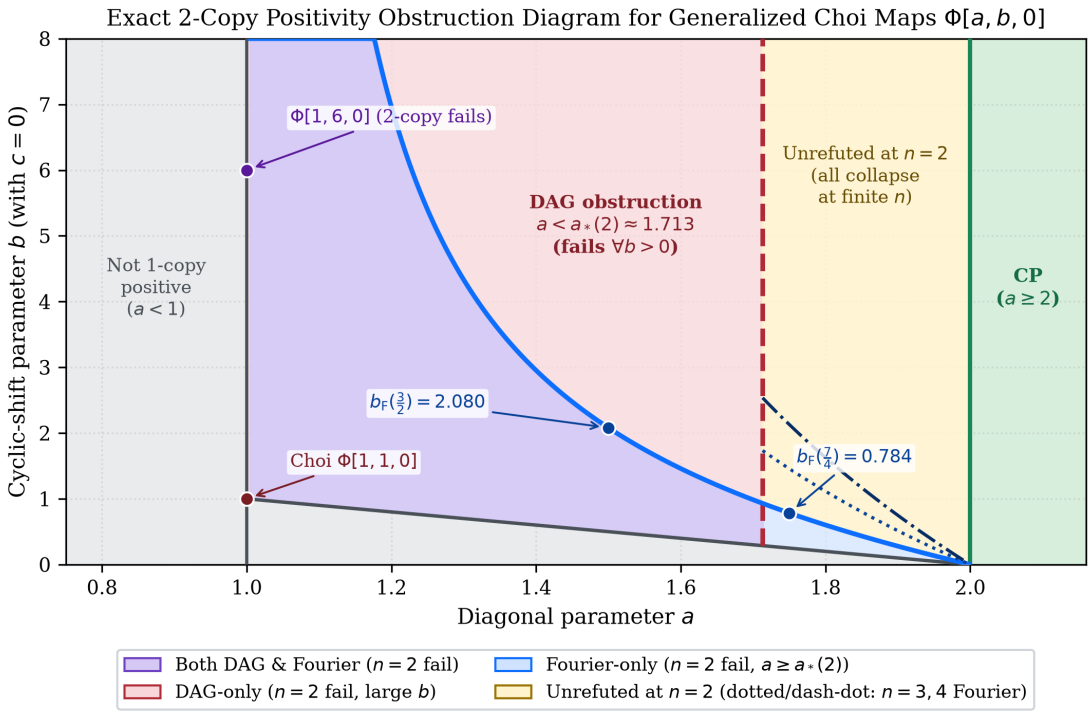

Figure 2: Exact 2-copy positivity obstruction diagram on the $c = 0$ face of $\Phi[a, b, 0]$. Theorem 2 (DAG obstruction, red/purple) rules out the entire vertical half-strip $1 \le a < a_*(2) \approx 1.7130$ uniformly in $b > 0$; Theorem 3 (Fourier obstruction, blue/purple) rules out $b < b_{\mathrm{F}}(a)$ across all $a \in [1, 2)$ (including $b_{\mathrm{F}}(1) = +\infty$, refuting $\Phi[1, 6, 0]^{\otimes 2} \ge 0$). The narrow wedge unrefuted at $n = 2$ (yellow) is progressively swept out by higher-copy curves $b_{\mathrm{F}}^{(3)}, b_{\mathrm{F}}^{(4)}$ (dotted/dash-dotted) and collapses completely at finite n by Theorem 3(c).

strictly (`thm16`). When $1 < a < 2$, the quadratic $P_2(a, b)$ in b has positive leading coefficient $a^2 - 1$ and exact discriminant $(2a^2 + 1)^2(2 - a)^2 - 4(a^2 - 1)^2(2 - a)^2 = 3(4a^2 - 1)(2 - a)^2 > 0$ (`thm15`), yielding the larger root $b_{\mathrm{F}}(a)$ in (18); since $P_2(a, 2 - a) = -3(2 - a)^2 < 0$, $P_2(a, b) < 0$ throughout $[2 - a, b_{\mathrm{F}}(a))$. Concrete brackets are proved in Lean (`thm20`).

(c) At $c = 0$, $a \in [1, 2)$, $b > 0$, if $a + b = 2$ then (16) gives $0 \ge 3(2 - a)^2 > 0$, immediate failure at $n = 2$. If $a + b > 2$, dividing (16) by $(a + b - 2)^{2(n-1)}$ yields $a^2 \ge R(a, b)^{n-1}$, where

$$(b^2 + (2 - a)b + (2 - a)^2) - (a + b - 2)^2 = 3b(2 - a) > 0$$

(Lean `thm17`). Hence $R(a, b) = 1 + q/p > 1$, so taking logarithms gives the exact threshold $n > 1 + 2 \ln a / \ln R(a, b)$. Applying Bernoulli's inequality $(1 + q/p)^{n-1} \ge 1 + (n - 1)q/p$ (Lean `thm18`) immediately yields the constructive integer upper bound in (19). For large n and fixed $a \in (1, 2)$, solving $R(a, b)^{n-1} = a^2$ expands as

$$\frac{2 \ln a}{n - 1} = \ln\left(1 + \frac{3(2 - a)}{b} + \frac{6(2 - a)^2}{b^2} + O(b^{-3})\right) = \frac{3(2 - a)}{b} + \frac{3(2 - a)^2}{2b^2} + O(b^{-3}),$$

whose inversion yields $b_{\mathrm{F}}^{(n)}(a) = \frac{3(2-a)}{2 \ln a}(n - 1) + \frac{2-a}{2} + O(n^{-1})$ (certified at $n = 101$ in `cert15`).

(d) For any $a \in [1, 2)$ and $b < b_{\mathrm{F}}(a)$, both sides of (16) at $n = 2$ are continuous in $c \ge 0$ with strict inequality at $c = 0$, so failure persists on a nonempty interval $c \in (0, c_{\max}(a, b))$ inside the indecomposable cone $4bc < (2 - a)^2$. Explicit 9×9 Eisenstein $\mathbb{Q}(\omega)$ Hermitian LDL^* certificates for all five benchmark triples are verified in `cert12`. □

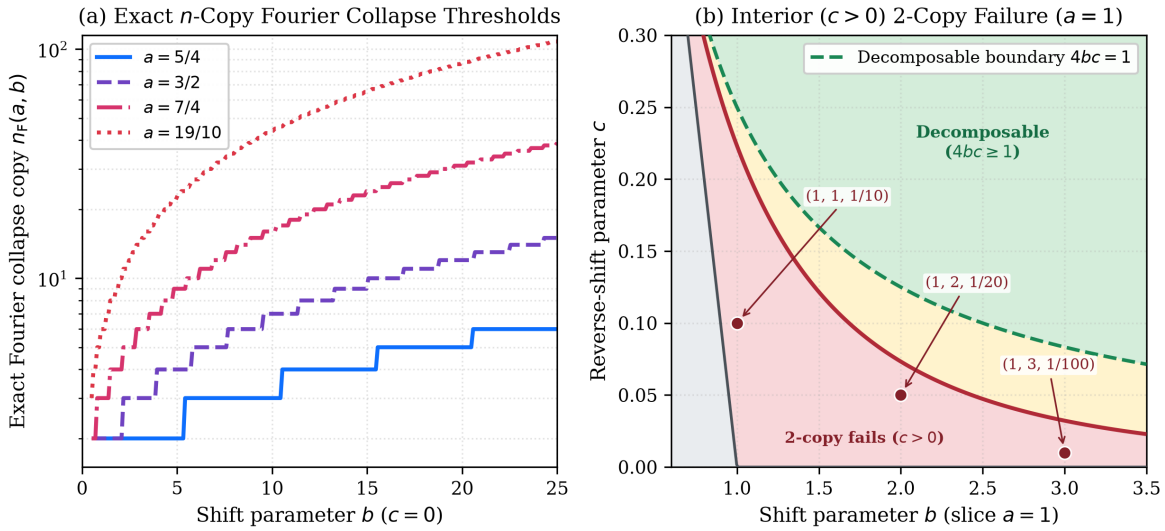

Figure 3: **(a)** Exact minimal Fourier collapse copy number $n_{\mathrm{F}}(a, b)$ versus b on the $c = 0$ face (plotted on a logarithmic vertical axis), displaying the linear asymptotic growth $n_{\mathrm{F}}(a, b) \sim \frac{2 \ln a}{3(2-a)} b$. **(b)** Cross-section $a = 1$ in the (b, c)-plane showing the strictly positive, strictly indecomposable band $0 < bc < 1/4$ $(b + c \geq 1)$ and the certified 2-copy Fourier failure region extending strictly into $c > 0$, marked with three exact 9×9 Eisenstein LDL^* benchmark points.

Table 1: Exact certified thresholds, DAG obstructions, and n-copy Fourier collapse benchmarks across Theorems 1–3 (all values certified in Lean 4.36.0 and zero-float rational/Eisenstein arithmetic).

Regime / Parameter	Exact Witness / Polynomial	Certified Value / Bracket	Source
Non-2-positivity ($a < 2, \forall b > 0$)	Rank-2 matrix A, $\det A = 0$	$\langle \mathrm{vec}(A) \vert C \vert \mathrm{vec}(A) \rangle = 3(a-2) < 0$	Thm. 1(a), `cert01`
All-n PT-indecomp. ($n \geq 1$)	$W_k = S_k \otimes B^{\otimes(n-1)}$	$\lambda_{\min} = b^{n-1}(a-2) < 0$	Thm. 1(b), `cert02`
2-copy DAG max ($\vert S \vert = 3$)	$a^3 - 2a^2 + 2a - 2 = 0$	$a_*(2;3) \in (1.543689, 1.543690)$	Thm. 2(b), `cert03`
2-copy DAG global ($\vert S \vert = 4$)	$Q_4(a) = a^4 - 2a^3 + 2a^2 - 2a - 1$	$a_*(2) \in (1.71298487, 1.71298488)$	Thm. 2(b), `cert04`
3-copy DAG ($\vert S_3 \vert = 8$)	8×8 rational LDL^T on S_3	$a_*(S_3) \in (1.950079, 1.950080)$	Thm. 2(c), `cert05`
4-copy DAG ($\vert S_4 \vert = 16$)	16×16 rational LDL^T on S_4	$a_*(S_4) \in (1.997061, 1.997062)$	Thm. 2(c), `cert06`
Boolean subcube Q_n ($\forall n$)	$(a-1)^n$ $(a \geq 1)$, $(a-1)(a+1)^{n-1}$ $(a < 1)$	$a_*(Q_n) = 1$ (2^{n-1} neg. eigs)	Thm. 2(d), `cert07`
2-copy Fourier ($a \leq 1, \forall b > 0$)	$P_2(a, b) < 0$	Total collapse ($\forall b > 0$)	Thm. 3(b), `cert08`
2-copy Fourier ($a = 5/4, 3/2, 7/4$)	$b_{\mathrm{F}}(a)$ closed form	5.39575, 2.07980, 0.78391	Thm. 3(b), `cert09`
n-copy Fourier ($a = 3/2, b = 3, 10$)	$a^2(a+b-2)^{2(n-1)} < \vert \cdot \vert^{2(n-1)}$	$n_{\mathrm{F}} = 3$, $n_{\mathrm{F}} = 7$	Thm. 3(c), `cert10`
Interior $c > 0$ indecomposable	9×9 Eisenstein $\mathbb{Q}(\omega)$ LDL^*	$(1, 1, \frac{1}{10}), (1, 2, \frac{1}{20}), (\frac{3}{2}, 1, \frac{1}{50})$ fail	Thm. 3(d), `cert12`

5 Complementarity of DAG and Fourier Obstructions and Summary Table

Theorems 2 and 3 play genuinely complementary roles at every finite copy number $n \geq 2$ (certified in `cert13`):

- **Large-b regime (DAG beats Fourier):** For fixed n and $a \in (1, a_*(n))$, the Fourier condition (16) is satisfied once $b \geq b_{\mathrm{F}}^{(n)}(a)$, yet Theorem 2 still certifies failure for *arbitrarily large* $b > 0$ (e.g., $(a, b) = (3/2, 10)$ at $n = 2$; $(19/10, 100)$ at $n = 3$; $(199/100, 1000)$ at $n = 4$).
- **Near-CP small-b regime (Fourier beats DAG):** For $a \in [a_*(n), 2)$ and moderate $b < b_{\mathrm{F}}^{(n)}(a)$, the DAG matrix $M_S(a)$ is positive definite on our constructed supports, yet Theorem 3 certifies n-copy failure directly (e.g., $(7/4, 1/2)$ at $n = 2$; $(49/25, 1/10)$ at $n = 3$) and guarantees eventual finite-n collapse for *every* point in the $c = 0$ face.

6 Formal Verification, Reproducibility, and Open Problems

Every algebraic identity, combinatorial acyclicity proof, and numerical threshold in this paper is backed by a two-layer zero-trust verification stack in the accompanying repository:

1. **Lean 4.36.0 Formalization (`lean/ChoiTensorPowerPositivity.lean`):** 20 self-contained constructive theorems (`thm01`–`thm20`) compiled under `lean -DwarningAsError=true` with zero external dependencies, zero `sorry`, zero `Classical.choice`, and zero custom axioms, formalizing the rank-2 non-2-positivity witness, $\mathbb{Z}_3$ cyclic swap parity, all-n $S \times B^{n-1}$ negativity, Choi's 1-copy determinant, the S_2 quartic factorization and strict monotonicity, topological acyclicity of S_2 and S_3, the 3-vertex cubic, the 2-copy Fourier determinant and discriminant identities, total collapse for $a \leq 1$, the all-n strict ratio gap $3b(2-a)$, constructive Bernoulli induction, and the 3-parameter Eisenstein modulus identity.

2. **Zero-Float Exact Python Verifier (`verification/verify_choi_tensor_power.py`):** Pure Python standard library (`int`/`Fraction`) with an exact Eisenstein field $\mathbb{Q}(\omega)$ Hermitian LDL^* engine, AST-audited for zero floating-point literals, zero true-division (`/`), zero power (`**`), and zero `limit_denominator`. It outputs 16 deterministic SHA-256 certificates with `MANIFEST.json` SHA-256 digest

   `8886df3f2804427ac293e1060904b7aa02b958b7a42ca36b4344102f14381b75`

   cross-checked by 56 adversarial unit tests (`tests/test_adversarial.py`).

Open Questions. Two natural open problems emerge immediately from our work:

1. **Asymptotics of $a_*(n)$ and maximum DAG size in $\vec{G}_{3,n}$:** Our exact search yields $|S_2| = 4 = 2^2$, $|S_3| = 8 = 2^3$, $|S_4| = 16 = 2^4$ with explicit DAG gaps $2 - a_*(S_n) \approx 1, 0.287, 0.0499, 0.00294$. Does the maximum acyclic subset of $\vec{G}_{3,n}$ have exact cardinality 2^n for all $n \geq 1$, and does $2 - a_*(n)$ decay exponentially (or super-exponentially) to 0 as $n \to \infty$?

2. **Full interior $c > 0$ closure of $\Phi[a, b, c]$:** For strictly positive $c > 0$ and $b > 0$, as $n \to \infty$ the dominant left-hand term in (16) is $\sqrt{bc}\,(a + b + c + 1)^{n-1}$, whereas the right-hand side is $|(a-2) + b\omega + c\bar{\omega}|^{n-1} < (a+b+c+1)^{n-1}$ (since $a+b+c+1 > |a-2|+b+c$ whenever $a \geq 1$). Hence the rank-2 Fourier family (16) refutes $\Phi[a, b, c]^{\otimes n}$ only at finite n for sufficiently small $c > 0$, and higher-rank or genuinely multipartite witnesses will be required to decide whether any strictly interior indecomposable Choi map ($0 < 4bc < (2 - a)^2$) can be tensor-stable positive.

AI Assistance Disclosure. In accordance with transparent research reporting standards, manuscript drafting, symbolic exploration, Lean 4 formalization, zero-float Python verification scripting, and adversarial test construction were carried out by the author using Google Antigravity. Every mathematical claim is independently verified by the Lean 4.36.0 kernel and exact rational/Eisenstein certificates.

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