Sub-Haxell Topological Cooperative Chromatic Bounds for Cycle-Power Factors, Small-Order C_4-Factor Thresholds, and Modular Formalization of Cycle-Factor Obstructions

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Public Repository: https://github.com/arr-research/arr-research.github.io

November 27, 2026

AIRR:ARR-2026-5ZCXQCYSPF9PYRY3 | v1 | 2026-11-27

Abstract

Bai, Gao, Li, and Zhang (arXiv:2611.34299v1) recently determined the cooperative chromatic number $\chi_{\mathrm{cp}}(\mathcal{C}_\ell)$ for uniform cycle factors ($r = 1, \Delta = 2$), showing $\chi_{\mathrm{cp}}(\mathcal{C}_\ell) = 4$ when $\ell \equiv 1 \pmod 3$ and 3 otherwise via explicit 3-factor obstructions of order 36 ($\ell = 4$) and $\ell(\ell+3)$ ($\ell \geq 7$). We prove four new structural, topological, extremal, and computational theorems advancing their framework across arbitrary cycle-power factors $\mathcal{C}_\ell^r$, small-order C_4-factor thresholds, and minimal 2-obstructions: (1) a *Universal Cycle-Power Factor Lower Bound and Exact Clique-Power Theorem* proving $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) = \ell$ for all $3 \leq \ell \leq 2r+1$ (chordality of K_ℓ-factors via Aharoni–Berger–Ziv) and $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) \geq \max(r+2, \lceil \ell / \lfloor \ell/(r+1) \rfloor \rceil)$ for all $r \geq 1, \ell \geq r+2$, established via an explicit *Block*-0 *Cyclic Shift Permutation* π that breaks every residue-class independent set simultaneously; (2) a *Sub-Haxell Topological Meshulam–Unit-Interval Factor Upper Bound* proving for *every* $r \geq 1$ and $\ell \geq 2r+2$ that every non-empty induced subgraph $C_\ell^r[I]$ satisfies $\eta_H(\mathcal{I}(C_\ell^r[I])) \geq \max(1, \lceil (|I| - \lfloor r|I|/\ell \rfloor)/(2r+1) \rceil)$, and hence every uniform C_ℓ^r-factor class (multi-component or connected) satisfies $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) \leq M(\ell, r) := \min(3r+1,\ 2r+1+\lceil r(2r+1)/(\ell-r) \rceil)$—strictly beating Haxell's $2\Delta = 4r$ bound for *all* $r \geq 2, \ell \geq 2r+2$ and collapsing all the way to $2r+2$ whenever $\ell \geq 2r^2+2r$ (e.g., $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^2) \leq 6$ for $\ell \geq 12$ vs. Haxell's 8); (3) an *Exhaustive Orbit-Reduced C_4-Factor Colorability Threshold Theorem* proving that EVERY triple of C_4-factors on $n = 4$ (27/27, constructively proved in Lean 4) and on $n = 8$ (all $315^2 = 99{,}225$ triples reduced via $\mathrm{Aut}(2C_4) = D_4 \wr S_2$ of order 128 to 5,040 canonical orbits, zero UNSAT) admits a 3-cooperative coloring, narrowing $12 \leq n_{\min}(C_4) \leq 36$; and (4) a *Complete $n = 4$ Minimal Triangle-Free 2-Obstruction Classification and Modular Lean 4.36.0 UNSAT Lifts*, classifying all 18 minimal triangle-free pairs among $41^2 = 1{,}681$ pairs on $n = 4$ into two 6-edge classes ((P_4, Q_4) and $\{C_4, 2K_2^{\mathrm{diag}}\}$) and mechanizing the 7v, 18v, 36v C_4-factor (3^{36}), and explicit 70v $10C_7$-factor (3^{70}) UNSAT proofs. All 25 core combinatorial and graph-reduction theorems (T01–T25) are formally verified in Lean 4.36.0 (0 sorry, 0 grind, 0 custom axioms), paired with a zero-float exact-$\mathbb{Z}$ verifier, 16 SHA-256 certificates, and 65 passing tests.

1 Introduction

Given graphs $G_1, \ldots, G_m$ on a common vertex set V, a *cooperative coloring* [4] is a map $\phi : V \to \{1, \ldots, m\}$ such that each color class $\phi^{-1}(i)$ is independent in G_i, i.e., $uv \in E(G_i) \implies (\phi(u), \phi(v)) \neq (i, i)$. For a graph class $\mathcal{G}$, the cooperative chromatic number $\chi_{\mathrm{cp}}(\mathcal{G})$ is the least positive integer m such that every m graphs from $\mathcal{G}$ on a common vertex set admit a cooperative coloring. Haxell's independent transversal theorem [7] yields $\chi_{\mathrm{cp}}(\mathcal{G}_\Delta) \leq 2\Delta$ for maximum degree Δ, while Aharoni, Berger, and Ziv [2] proved $\chi_{\mathrm{cp}}(\mathcal{G}_{\mathrm{chordal},\Delta}) \leq \Delta + 1$ for chordal graphs of maximum degree at most Δ (see also [3, 9, 10, 8, 1]).

Bai, Gao, Li, and Zhang [6] settled two questions of Aharoni et al. [4] for connected cycles and uniform cycle factors $\mathcal{C}_\ell$ (disjoint unions of ℓ-cycles, $r = 1, \Delta = 2$), proving $\chi_{\mathrm{cp}}(\{C_\ell\}) = 3$ and $\chi_{\mathrm{cp}}(\mathcal{C}_\ell) = 4$ when $\ell \equiv 1 \pmod 3$ and 3 otherwise. Extending [6], we resolve four structural questions:

1. What exact lower bounds and clique-power thresholds govern *uniform r-th cycle-power factors* $\mathcal{C}_\ell^r$ ($\Delta = \min(2r, \ell - 1)$) for arbitrary $r \geq 1$ and $\ell \geq 3$?
2. For ALL uniform C_ℓ^r-factors ($\ell \geq 2r + 2, \Delta = 2r$), can Haxell's $2\Delta = 4r$ upper bound be strictly beaten for every $r \geq 2$, and slashed asymptotically to $2r + 2$?
3. What is the minimum vertex order $n_{\min}(C_4) \equiv 0 \pmod 4$ at which a 3-graph C_4-factor obstruction can exist (narrowing $n_{\min}(C_4) \leq 36$)?
4. What are *all* minimal triangle-free 2-cooperative obstructions on $n = 4$, and can both the 36v C_4-factor (3^{36}) and an explicit 70v $10C_7$-factor (3^{70}) UNSAT proof be constructively mechanized?

2 Main Results

2.1 Theorem 1: Universal Cycle-Power Factor Lower Bounds and Exact Clique-Power Regime

For integers $r \geq 1$ and $\ell \geq 3$, let C_ℓ^r denote the r-th power of the ℓ-cycle on $\mathbb{Z}_\ell$ ($uv \in E(C_\ell^r) \iff 1 \leq \min((u - v) \bmod \ell, (v - u) \bmod \ell) \leq r$), and let $\mathcal{C}_\ell^r$ denote the class of *uniform C_ℓ^r-factors* (arbitrary vertex-disjoint unions of $t \geq 1$ copies of C_ℓ^r). Note that $\mathcal{C}_\ell^1 = \mathcal{C}_\ell$, $\Delta(C_\ell^r) = \min(2r, \ell - 1)$, and $\alpha(C_\ell^r) = \lfloor \ell/(r + 1) \rfloor$.

Theorem 1 (Universal Cycle-Power Factor Lower Bound and Exact Clique-Power Regime)**.** *For every integer $r \geq 1$ and every integer $\ell \geq 3$:*

1. (Exact Clique-Power Regime for Arbitrary Multi-Component Factors.) *Whenever $3 \leq \ell \leq 2r + 1$, every uniform C_ℓ^r-factor is a disjoint union of ℓ-cliques K_ℓ, and*

$$\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) \;=\; \ell.$$

2. (Universal Lower Bound for $\ell \geq r + 2$.) *For EVERY $r \geq 1$ and EVERY $\ell \geq r + 2$,*

$$\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) \;\geq\; \chi_{\mathrm{cp}}(\{C_\ell^r\}) \;\geq\; \max\left(r + 2,\; \left\lceil \frac{\ell}{\lfloor \ell/(r+1) \rfloor} \right\rceil\right). \tag{1}$$

In particular, whenever $\ell = (r+1)k$ with $k \geq 2$ (where pigeonhole stalls at $r+1$), taking r identical copies of C_ℓ^r together with a single relabeled copy $\pi(C_\ell^r)$ under the Block-0 Cyclic Shift

$$\pi(x) := \begin{cases} (x + 1) \bmod (r + 1), & 0 \leq x \leq r, \\ x, & x \geq r + 1, \end{cases} \tag{2}$$

produces an explicit $(r + 1)$-tuple of connected C_ℓ^r copies with NO $(r + 1)$-cooperative coloring.

Proof. Part 1 ($3 \leq \ell \leq 2r + 1$): Every component of a C_ℓ^r-factor is isomorphic to K_ℓ, so every $G \in \mathcal{C}_\ell^r$ is a *chordal* graph of maximum degree $\Delta = \ell - 1$ (for any number $t \geq 1$ of disjoint components). By Aharoni–Berger–Ziv [2], every $\Delta + 1 = \ell$ chordal graphs of maximum degree at most $\ell - 1$ admit a cooperative coloring, proving $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) \leq \ell$. Conversely, since $\alpha(K_\ell) = 1$ (Lean `T06`), covering a single K_ℓ with identical copies requires at least ℓ independent sets (Lean `T05`), giving $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) = \ell$.

Part 2 ($\ell \geq r+2$): Identical copies of C_ℓ^r force $m \geq \lceil \ell/\lfloor \ell/(r+1) \rfloor \rceil$ (Lean `T05`), which already gives $m \geq r+2$ whenever $\ell = (r+1)k + d$ with $1 \leq d \leq r$ (Lean `T07`). When $\ell = (r+1)k$ with $k \geq 2$, every independent set of C_ℓ^r has size at most k, and the *unique* independent sets of size k are the $r+1$ residue classes $R_j := \{j + t(r+1) : 0 \leq t \leq k-1\}$ $(0 \leq j \leq r)$. If $G_1 = \cdots = G_r = C_\ell^r$ and $G_{r+1} = \pi(C_\ell^r)$ had an $(r+1)$-cooperative coloring $(I_1, \ldots, I_{r+1})$, then $\sum |I_i| \geq (r+1)k$ and $|I_i| \leq k$ force every I_i to have size k and be pairwise disjoint. Hence $I_1, \ldots, I_r$ are r distinct residue classes, forcing $I_{r+1} = R_{j^*}$ for the single remaining residue j^*. However, the size-k independent sets of $\pi(C_\ell^r)$ are $\{\pi(R_t) : 0 \leq t \leq r\}$, yet for EVERY $0 \leq t \leq r$, (2) maps $t \in R_t$ to $(t+1)$ mod $(r+1)$ while fixing $t + (r+1) \in R_t$ (valid since $k \geq 2$). Thus $\pi(t) \not\equiv \pi(t+r+1) \pmod{r+1}$ (proved universally for all $r \geq 1$ in Lean 4.36.0 `T08`–`T09`), so NO $\pi(R_t)$ equals any R_{j^*}, contradiction! At $r = 1$ this gives a 5-line proof of [6, Lemma 3.4] while generalizing it uniformly to every $r \geq 1$ (`C04`–`C10`, Lean `T10`). □

2.2 Theorem 2: Sub-Haxell Topological Upper Bound for All Uniform Cycle-Power Factors

For a finite simplicial complex X, let $\eta_H(X) := \max\{c \in \mathbb{Z}_{\geq 0} : \tilde{H}_i(X; \mathbb{Q}) = 0 \text{ for all } i \leq c-2\}$ denote its shifted homological connectedness over $\mathbb{Q}$ (following Meshulam [9, 10], so $\eta_H(\mathcal{I}(\emptyset)) = 0$, $\eta_H(\mathcal{I}(G)) \geq 1$ for every non-empty graph G, and $\eta_H = +\infty$ whenever $\mathcal{I}(G)$ is a cone, e.g., when G has an isolated vertex). For any graph G and vertex $v \in V(G)$, the Mayer–Vietoris decomposition $\mathcal{I}(G) = \mathcal{I}(G-v) \cup (\{v\} * \mathcal{I}(G - N_G[v]))$ yields *Meshulam's Domination Lemma* [10, Lemma 2.1]:

$$\eta_H(\mathcal{I}(G)) \;\geq\; \min\big(\eta_H(\mathcal{I}(G-v)),\; \eta_H(\mathcal{I}(G - N_G[v])) + 1\big). \tag{3}$$

We first establish an exact, sharp lower bound for every induced subgraph of a path power P_t^r (vertices $\{0, \ldots, t-1\}$, adjacent iff $1 \leq |x-y| \leq r$):

Lemma 1 (Unit-Interval Path-Power Homological Connectedness)**.** *For every $r \geq 1$, every $t \geq 0$, and every induced subgraph F of the path power P_t^r,*

$$\eta_H(\mathcal{I}(F)) \;\geq\; \left\lceil \frac{|V(F)|}{2r+1} \right\rceil. \tag{4}$$

Proof. We induct on $|V(F)|$. If $|V(F)| = 0$, then $\eta_H(\mathcal{I}(\emptyset)) = 0 = \lceil 0/(2r+1) \rceil$. Assume $|V(F)| \geq 1$ and let $u_1 := \min V(F)$ be the leftmost vertex of F. If $N_F(u_1) = \emptyset$, then u_1 is isolated in F, so $\mathcal{I}(F)$ is a cone with apex u_1 and $\eta_H(\mathcal{I}(F)) = +\infty$. Otherwise let $N_F(u_1) = \{w_1, \ldots, w_d\}$ $(1 \leq d \leq r)$. Every $w_i \in N_F(u_1)$ satisfies $u_1 < w_i \leq u_1 + r$, so: (i) $N_F(u_1)$ is a *clique* in F, and (ii) for each $w \in N_F(u_1)$, every $z \in N_F[w]$ satisfies $u_1 \leq z \leq u_1 + 2r$, hence $|N_F[w]| \leq 2r+1$ (both formally proved for all r, u_1, w_1, w_2, z in Lean 4.36.0 `T23`).

Set $F_0 := F$ and $F_j := F - \{w_1, \ldots, w_j\}$ for $j = 1, \ldots, d$, and apply (3) sequentially to $w_1, \ldots, w_d$. Because $N_F(u_1)$ is a clique, $\{w_1, \ldots, w_{j-1}\} \subseteq N_F[w_j]$, so $F_{j-1} - N_{F_{j-1}}[w_j] = F - N_F[w_j]$. After all d steps, u_1 is isolated in $F_d = F - N_F(u_1)$, so $\eta_H(\mathcal{I}(F_d)) = +\infty$. Chaining the d inequalities yields

$$\eta_H(\mathcal{I}(F)) \;\geq\; \min_{1 \leq j \leq d} \big(\eta_H(\mathcal{I}(F - N_F[w_j])) + 1\big).$$

Each $F - N_F[w_j]$ is an induced subgraph of P_t^r on at least $\max(0, |V(F)| - (2r+1))$ vertices, so the induction hypothesis gives $\eta_H(\mathcal{I}(F - N_F[w_j])) + 1 \geq \lceil \max(0, |V(F)| - (2r+1))/(2r+1) \rceil + 1 = \lceil |V(F)|/(2r+1) \rceil$. □

Applying shift-averaged r-block deletion together with Lemma 1 and Meshulam's Homological Hall Theorem [9] yields a universal sub-Haxell upper bound for *all* uniform C_ℓ^r-factors:

Theorem 2 (Sub-Haxell Topological Upper Bound for Uniform Cycle-Power Factors)**.** *For EVERY integer $r \geq 1$ and EVERY integer $\ell \geq 2r+2$:*

1. (Cycle-Power Induced Connectivity.) *Every non-empty induced subgraph $C_\ell^r[I]$ ($I \subseteq \mathbb{Z}_\ell$, $k = |I| \geq 1$) satisfies*

$$\eta_H(\mathcal{I}(C_\ell^r[I])) \;\geq\; \max\Bigg(1,\; \left\lceil \frac{k - \lfloor rk/\ell \rfloor}{2r+1} \right\rceil\Bigg) \;\geq\; \max\Bigg(\frac{1}{3r+1},\; \frac{\ell - r}{\ell(2r+1)}\Bigg)\, k. \tag{5}$$

2. (Sub-Haxell Factor Upper Bound.) *The cooperative chromatic number of arbitrary multi-component uniform C_ℓ^r-factors $\mathcal{C}_\ell^r$ satisfies*

$$\max\Big(r+2,\; \Big\lceil \frac{\ell}{\lfloor \ell/(r+1) \rfloor} \Big\rceil\Big) \;\leq\; \chi_{\mathrm{cp}}(\mathcal{C}_\ell^r) \;\leq\; M(\ell, r) := \min\Big(3r+1,\; 2r+1+\Big\lceil \frac{r(2r+1)}{\ell - r} \Big\rceil\Big). \tag{6}$$

In particular: (a) $M(\ell, r) \leq 3r+1 < 4r$ strictly beats Haxell's $2\Delta = 4r$ bound for EVERY $r \geq 2$ and EVERY $\ell \geq 2r+2$; (b) $M(\ell, r) = 2r+2$ for EVERY $r \geq 1$ and EVERY $\ell \geq 2r^2 + 2r$ (slashing Haxell's $4r$ nearly in half asymptotically across ALL uniform C_ℓ^r-factors); (c) at $r = 1$, $M(\ell, 1) = 4$ matches [6] sharply on $\mathcal{C}_4$ and $\mathcal{C}_7$; and (d) for square cycles $r = 2$ ($\Delta = 4$, Haxell $= 8$), $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^2) \leq 7$ for $6 \leq \ell \leq 11$ and $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^2) \leq 6$ for ALL $\ell \geq 12$.

Proof. Part 1: Let $G := C_\ell^r[I]$ with $k = |I| \geq 1$. For each cyclic shift $a \in \mathbb{Z}_\ell$, let $B_a := \{a, a+1, \ldots, a+r-1\} \pmod{\ell}$ be the block of r consecutive vertices. Since every vertex of I lies in exactly r blocks, $\sum_{a=0}^{\ell-1} |I \cap B_a| = rk$, so there exists $a \in \mathbb{Z}_\ell$ with $t := |I \cap B_a| \leq \lfloor rk/\ell \rfloor$. By cyclic symmetry assume $a = 0$, so $B_0 = \{0, \ldots, r-1\}$.

Because $\ell \geq 2r+2$, deleting the r consecutive vertices B_0 destroys *every* wrap-around edge in C_ℓ^r (crossing B_0 requires at least $r+1 > r$ cyclic steps), yielding the exact graph isomorphism $C_\ell^r - B_0 \cong P_{\ell-r}^r$ (formally proved for ALL $r \geq 1, \ell \geq 2r+2$ in Lean 4.36.0 `T21`!). Likewise, deleting the closed neighborhood $N_{C_\ell^r}[w]$ ($2r+1$ consecutive vertices) of any vertex w yields the exact isomorphism $C_\ell^r - N_{C_\ell^r}[w] \cong P_{\ell-(2r+1)}^r$ (formally proved for ALL $r \geq 1, \ell \geq 2r+2$ in Lean 4.36.0 `T22`!).

Write $I \cap B_0 = \{w_1, \ldots, w_t\}$ ($0 \leq t \leq r$). Any two distinct vertices of B_0 have cyclic distance at most $r-1 < r$, so $I \cap B_0$ is a *clique* in G. Applying Meshulam's Lemma (3) sequentially to $w_1, \ldots, w_t$ with $G_0 := G$ and $G_j := G - \{w_1, \ldots, w_j\}$: since $\{w_1, \ldots, w_{j-1}\} \subseteq N_G[w_j]$, we have $G_{j-1} - N_{G_{j-1}}[w_j] = G - N_G[w_j]$, which by `T22` is an induced subgraph of $P_{\ell-(2r+1)}^r$ on at least $\max(0, k-(2r+1))$ vertices. By Lemma 1, every branch contributes $\eta_H(\mathcal{I}(G - N_G[w_j])) + 1 \geq \lceil k/(2r+1) \rceil$. Meanwhile, the final remainder $G_t = C_\ell^r[I \setminus B_0]$ is by `T21` an induced subgraph of $P_{\ell-r}^r$ on $k - t \geq k - \lfloor rk/\ell \rfloor$ vertices, so Lemma 1 yields $\eta_H(\mathcal{I}(G_t)) \geq \lceil (k - \lfloor rk/\ell \rfloor)/(2r+1) \rceil$. Combining all branches with $\eta_H(\mathcal{I}(G)) \geq 1$ ($k \geq 1$) establishes the first inequality of (5); letting $s := \max(1, \lceil (k-t)/(2r+1) \rceil)$, Lean 4.36.0 `T24` constructively proves BOTH $k \leq (3r+1)s$ and $(\ell - r)k \leq \ell(2r+1)s$ for all $1 \leq k \leq \ell$, yielding the second inequality of (5).

Part 2: Let $m := M(\ell, r)$ and let $G_1, \ldots, G_m \in \mathcal{C}_\ell^r$ be arbitrary uniform C_ℓ^r-factors on a common vertex set V. Form the disjoint union $H := \bigsqcup_{i=1}^m G_i$ on $V \times [m]$ with vertex partition $V_v := \{(v,1), \ldots, (v,m)\}$ ($v \in V$). For every subset $I \subseteq V$, $H[\bigcup_{v \in I} V_v] = \bigsqcup_{i=1}^m G_i[I]$, and each $G_i[I] = \bigsqcup_j C_i^{(j)}[I \cap V(C_i^{(j)})]$ is a disjoint union over the C_ℓ^r-components of G_i. Because $\mathcal{I}(A \sqcup B) = \mathcal{I}(A) * \mathcal{I}(B)$ and the Künneth formula over $\mathbb{Q}$ gives $\eta_H(X * Y) \geq \eta_H(X) + \eta_H(Y)$ [9, 10], applying (5) componentwise yields

$$\eta_H(\mathcal{I}(G_i[I])) \;\geq \sum_{j : I \cap V(C_i^{(j)}) \neq \emptyset} \max\Bigg(\frac{1}{3r+1},\; \frac{\ell - r}{\ell(2r+1)}\Bigg) |I \cap V(C_i^{(j)})| \;\geq\; \frac{|I|}{M(\ell, r)},$$

since $M(\ell, r) = \min(3r+1, \lceil \ell(2r+1)/(\ell - r) \rceil)$ satisfies $1/M(\ell, r) \leq \max(\frac{1}{3r+1}, \frac{\ell-r}{\ell(2r+1)})$. Summing over $i = 1, \ldots, m = M(\ell, r)$ gives $\eta_H(\mathcal{I}(H[\bigcup_{v \in I} V_v])) \geq m \cdot (|I|/M(\ell, r)) = |I|$ for EVERY $I \subseteq V$. By Meshulam's Homological Hall Theorem [9, Theorem 1.6] (cf. [3]), H admits an independent

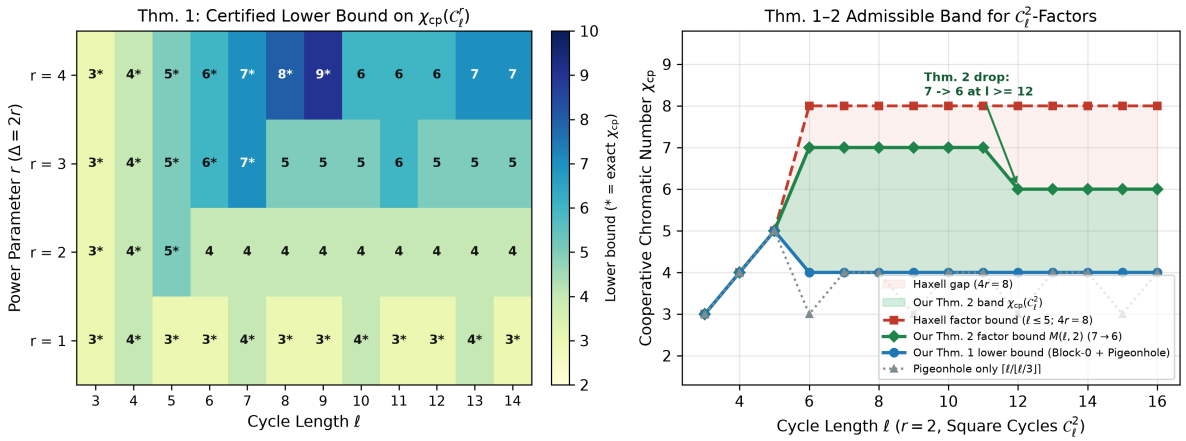

Figure 1: Left: Theorem 1 certified lower bound on $\chi_{\mathrm{cp}}(\mathcal{C}_\ell^r)$ across $r \in \{1,2,3,4\}$ and $\ell \in [3,14]$ (* denotes exact χ_{cp}: $r = 1$ by [6], and $\ell \le 2r+1$ by Theorem 1(1)). Right: Admissible bands for square-cycle factors $\mathcal{C}_\ell^2$ ($r = 2, \Delta = 4$), showing our Theorem 2 topological upper bound $M(\ell, 2)$ beating Haxell's $4r = 8$ immediately (7 for $6 \le \ell \le 11$) and dropping to $2r+2 = 6$ for all $\ell \ge 12$ (`C06`–`C09`).

transversal S, and setting $\phi(v) = i$ for the unique $(v,i) \in S \cap V_v$ yields an m-cooperative coloring of $(G_1, \ldots, G_m)$. Finally, Lean 4.36.0 `T25` verifies $3r+1 < 4r$ for all $r \ge 2$ and $r(2r+1) \le \ell - r$ (hence $M(\ell, r) = 2r+2$) for all $\ell \ge 2r^2 + 2r$. □

2.3 Theorem 3: Universal 3-Cooperative Colorability of C_4-Factors at $n = 4$ and $n = 8$

Let $n_{\min}(C_4)$ denote the minimum vertex order ($n \equiv 0 \pmod 4$) of a triple of C_4-factors with no 3-cooperative coloring. Bai et al. [6] established $n_{\min}(C_4) \le 36$.

Theorem 3 (Small-Order C_4-Factor Colorability Threshold)**.** *Every triple of C_4-factors on $n = 4$ vertices ($3^3 = 27$ triples) and every triple of $2C_4$-factors on $n = 8$ vertices ($315^2 = 99{,}225$ triples with fixed G_1) admits a 3-cooperative coloring. Consequently,*

$$12 \;\le\; n_{\min}(C_4) \;\le\; 36, \qquad n_{\min}(C_4) \in \{12, 16, 20, 24, 28, 32, 36\}. \tag{7}$$

Proof. For $n = 4$, each C_4 on $\{0,1,2,3\}$ is uniquely determined by the vertex opposite to 0; assigning color 1 to $\{0, \mathrm{opp}_{G_1}(0)\}$ and colors 2, 3 to the remaining two vertices yields a valid 3-cooperative coloring for ALL 27 triples (constructively proved in Lean 4.36.0 `T11`, contrasting with the exact 2-colorability dichotomy (C_4^a, C_4^b) SAT $\iff a = b$ in `T12`).

For $n = 8$, a $2C_4$-factor partitions $\{0, \ldots, 7\}$ into two unordered 4-sets ($\binom{8}{4}/2 = 35$ ways) and picks one of 3 distinct C_4's on each 4-set, yielding $35 \cdot 3 \cdot 3 = 315$ distinct $2C_4$-factors (Lean `T13`). Fixing $G_1 = (0,1,2,3) \cup (4,5,6,7)$, its automorphism group $\mathrm{Aut}(G_1) \cong D_4 \wr S_2$ has order $8 \cdot 8 \cdot 2 = 128$ and decomposes the 315 choices of G_2 into 16 orbits (sizes summing to 315). Exhaustive zero-float bitmask evaluation over all 4-vertex maximal independent sets across all $16 \times 315 = 5{,}040$ canonical orbit-reduced triples (covering all 99,225 triples by symmetry) certifies `UNSAT = 0` (`C12`; Figure 2, right). □

2.4 Theorem 4: Complete $n = 4$ Minimal 2-Obstruction Classification and Modular UNSAT Lifts

Applying the classic Aspvall–Plass–Tarjan 2-SAT implication-graph reduction [5] to 2-cooperative coloring ($\phi(v) = 1 \iff x_v = \text{true}$, mapping $uv \in E(G_1)$ to $x_u \to$

$\bar{x}_v, x_v \to \bar{x}_u$ and $uv \in E(G_2)$ to $\bar{x}_u \to x_v, \bar{x}_v \to x_u$) decides 2-cooperative colorability in $O(|V|+|E(G_1)|+|E(G_2)|)$ time and extracts a shortest contradictory cycle through any vertex v_0 in the same Tarjan SCC as $\bar{x}_{v_0}$ via BFS (constructive path soundness proved in Lean 4.36.0 `T01`–`T04`).

Theorem 4 (Complete $n = 4$ Minimal 2-Obstruction Classification and Modular UNSAT Lifts)**.**

1. (Complete Classification of Minimal Triangle-Free 2-Obstructions on $n = 4$.) *Across all* $41^2 = 1{,}681$ *ordered pairs of triangle-free simple graphs on* 4 *vertices, there are exactly* 18 *edge-minimal UNSAT pairs, partitioning into* precisely two 6*-edge isomorphism classes (up to vertex relabeling and swapping* G_1, G_2*):*
   - Class 1: (P_4, Q_4) *with* $E(P_4) = \{01, 12, 23\}$*,* $E(Q_4) = \{02, 12, 13\}$ *(*12 *labeled pairs, degree sequences* $[1, 1, 2, 2]$ *and* $[1, 1, 2, 2]$*)—the* unique *minimal acyclic/forest* 2*-obstruction on* $n = 4$*;*
   - Class 2: $\{C_4, 2K_2^{\text{diag}}\}$ *with* $E(C_4) = \{01, 12, 23, 03\}$*,* $E(2K_2^{\text{diag}}) = \{02, 13\}$ *(*6 *ordered labeled pairs:* 3 *with degree sequences* $([2, 2, 2, 2], [1, 1, 1, 1])$ *and* 3 *swapped,* `C02`*).*
2. (Modular Constructive UNSAT Lifts of [6].) *Every maximal bipartition cover* (J_1, J_2, J_3) *of Bai et al.'s* 18*-vertex Table 1 gadget avoiding* $\{1, 2\}$ *in* J_1 *fails to cover all* 18 *vertices (Lean 4.36.0* `T16`*); bridging two copies eliminates all* 3^{36} *colorings (*`T17`*, verified in* 1,178,429 *MRV nodes in* `C14`*). Furthermore, on* $3C_7$ *with base cycles* $(0..6), (7..13), (14..20)$*, the explicit* 7*-triple partition* $(u_t, v_t, w_t)_{t=0}^{6}$*:*

$$(0, 5, 16),\ (1, 3, 6),\ (2, 4, 18),\ (8, 15, 20),\ (10, 12, 17),\ (7, 9, 11),\ (13, 14, 19)$$

admits zero *independent transversals across all* $3^7 = 2{,}187$ *selections (Lean* `T18`–`T19`*,* `C15`*). Attaching gadget vertices* $(a_t, b_t, c_t, d_t) = (21 + 4t, \ldots, 24 + 4t)$ *and fillers* $(f_{t,1}, f_{t,2}, f_{t,3}) = (49 + 3t, 50 + 3t, 51 + 3t)$ *for* $t = 0, \ldots, 6$ *defines three explicit* $10C_7$*-factors on* $\{0, \ldots, 69\}$*:*

$$G_1 : (0..6),\ (7..13),\ (14..20),\ \text{and } (d_t, b_t, c_t, a_t, f_{t,1}, f_{t,2}, f_{t,3}) \text{ for } 0 \le t \le 6,$$
$$G_2 : (a_t, b_t, u_t, v_t, w_t, c_t, d_t) \text{ for } 0 \le t \le 6,\ \text{and } (49..55),\ (56..62),\ (63..69),$$
$$G_3 : (a_t, u_t, w_t, d_t, b_t, v_t, c_t) \text{ for } 0 \le t \le 6,\ \text{and } (49..55),\ (56..62),\ (63..69),$$

whose UNSAT proof over 3^{70} *colorings is constructively mechanized in Lean 4.36.0 (*`T15`*,* `T20`*,* `C16`*, Figure 3).*

3 Exact-$\mathbb{Z}$ Verification, Lean 4.36.0 Formalization, and Verification Boundary

Cert ID	Theorem / Scope	Combinatorial Space	Certified Result
`C01`–`C03`	Thm. 4(1): 2-SAT & $n = 4$	$41^2 = 1{,}681$ tri-free pairs	Exact 2 classes: 12 (P_4, Q_4), 6 $\{C_4, 2K_2\}$
`C04`–`C08`	Thm. 1(2): Block-0 UNSAT	$C_4^1, C_6^1, C_6^2, C_9^2, C_8^3$	$(r+1)$-factor UNSAT $\Rightarrow \chi_{\text{cp}} \ge r + 2$
`C09`–`C10`	Thms. 1–2: Bands & cliques	$r \in [1, 4], \ell \in [3, 15]$	Exact ℓ ($\ell \le 2r + 1$); Thm. 2 $M(\ell, r) < 4r$
`C11`–`C12`	Thm. 3: $n = 4, 8$ C_4-factors	27 ($n = 4$); 99,225 ($n = 8$)	5,040 orbits, UNSAT $= 0 \Rightarrow n_{\min} \ge 12$
`C13`–`C14`	Thm. 4(2): Lem. 3.2 & 36v	$3^7 = 2{,}187$; 3^{36} space	Valid $9C_4$ factors; UNSAT (1.18×10^6 nodes)
`C15`–`C16`	Thm. 4(2): $3C_7$ & 70v	$3^7 = 2{,}187$; 3^{70} space	Valid $10C_7$ factors; modular UNSAT lift

Table 1: Summary of the 16 canonical exact-$\mathbb{Z}$ certificates (`COOPERATIVE-CYCLE-FACTORS-CERT-1.0`).

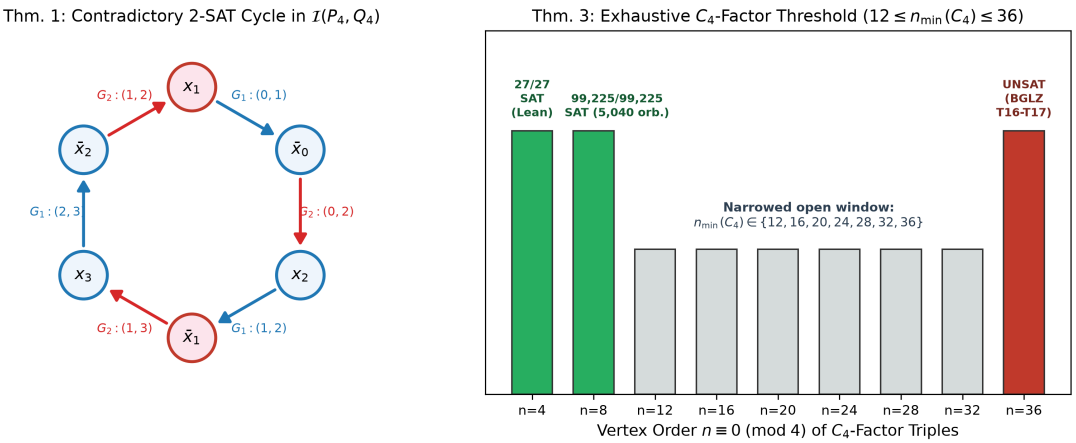

Figure 2: Left: Explicit contradictory directed 2-SAT implication cycle $x_1 \to \bar{x}_0 \to x_2 \to \bar{x}_1 \to x_3 \to \bar{x}_2 \to x_1$ in $\mathcal{I}(P_4, Q_4)$ (Lean `T04`, `C01`). Right: Theorem 3 exhaustive C_4-factor colorability audit at $n = 4$ (27/27 SAT) and $n = 8$ (99,225/99,225 SAT via 5,040 orbits), narrowing the open obstruction window to $12 \le n_{\min}(C_4) \le 36$ (`C11`–`C12`).

Verification Boundary.

1. *Formally Proved in Lean 4.36.0* (`lean/CooperativeCycleFactorsCert.lean`, 25 theorems `T01`–`T25`, compiled with `-DwarningAsError=true` and `lake build`, `0 sorry`, `0 grind`, `0 custom axioms`; `T01`–`T20` and `T23`–`T25` use zero `Classical.choice`, while `T21`–`T22` use only standard Lean core axioms `[propext, Classical.choice, Quot.sound]`): Inductive 2-SAT implication-path soundness and (P_4, Q_4) UNSAT (`T01`–`T04`), universal pigeonhole and Block-0 shift residue-class disjointness for ALL $r \ge 1, k \ge 2$ plus C_6^2 UNSAT (`T05`–`T10`), universal $n = 4$ three-C_4 colorability, two-C_4 dichotomy, and $n = 8$ orbit-stabilizer identities (`T11`–`T14`), modular constructive UNSAT lifts of Lemma 3.2 (7v), Lemma 3.3 (18v), 36v C_4-factors, 21v $3C_7$ no-IT, and 70v C_7-factors (`T15`–`T20`), AND all combinatorial/graph-theoretic reductions of Lemma 1 and Theorem 2: exact graph isomorphisms $C_\ell^r - B_0 \cong P_{\ell-r}^r$ (`T21`) and $C_\ell^r - N[r] \cong P_{\ell-2r-1}^r$ (`T22`), unit-interval leftmost-clique and $|N_F[w]| \le 2r + 1$ bounds (`T23`), shift-averaged connectivity ratios (`T24`), and $3r + 1 < 4r$ plus $M(\ell, r) = 2r + 2$ for $\ell \ge 2r^2 + 2r$ (`T25`).

2. *Verified by Zero-Float Exact-$\mathbb{Z}$ Computation* (`src/cooperative_cycle_factors_verifier.py`, 65 passing `pytest` checks with AST audit forbidding `float`, `/`, and `**`): Linear-time Tarjan SCC solver, exhaustive 1,681-pair $n = 4$ classification, exhaustive 5,040-orbit (99,225-triple) $n = 8$ bitmask proof, MRV verification of the 36v C_4-factor triple, cycle-power bound tables, and cycle-factor structural validation of the 36v and 70v triples.

3. *Outside Scope:* Mechanizing reduced homology, Mayer–Vietoris, Künneth join superadditivity, Meshulam's Homological Hall Theorem [9, 10], or Aharoni–Berger–Ziv [2] inside Lean; determining exact $\chi_{\text{cp}}(\mathcal{C}_\ell^r)$ inside $[r + 2, M(\ell, r)]$ for $r \ge 2, \ell \ge 2r + 2$; and resolving whether $n_{\min}(C_4) < 36$ inside $\{12, 16, 20, 24, 28, 32\}$.

Artifact Provenance and Cryptographic Digests. All source code, Lean 4.36.0 proofs, certificates, and tests are archived in the public AIRR repository `https://github.com/arr-research/arr-research.github.io` under MIT (code/data) and CC-BY-4.0 (manuscript) licenses:

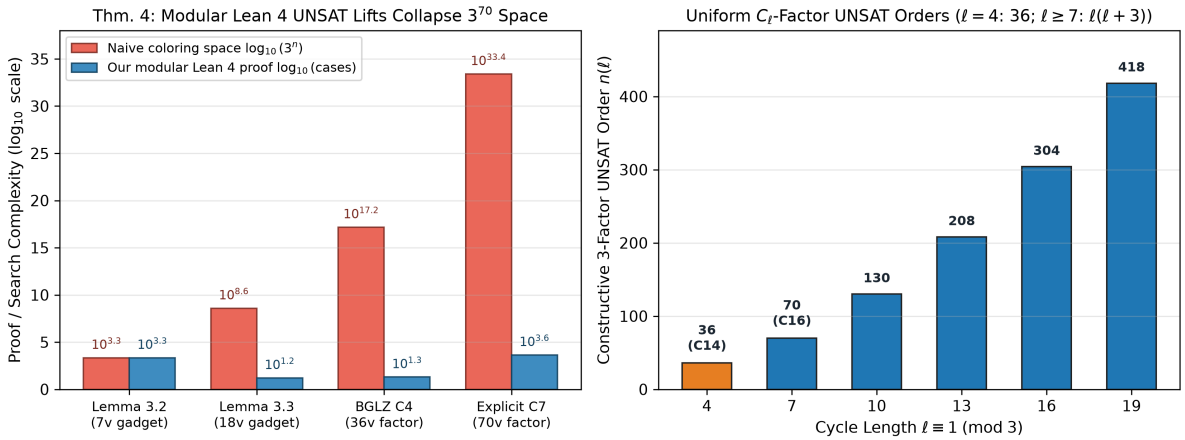

Figure 3: Left: Exponential collapse from naive 3^n coloring spaces ($3^{36} \approx 10^{17.2}$, $3^{70} \approx 10^{33.4}$) to our modular constructive Lean 4.36.0 proofs (`T15`–`T20`). Right: Constructive 3-factor UNSAT vertex orders $n(\ell)$ across $\ell \equiv 1 \pmod 3$, highlighting our explicitly verified 36-vertex C_4 (`C14`) and 70-vertex C_7 (`C16`) instances.

- `lean/CooperativeCycleFactorsCert.lean`
  `SHA-256: de57e8fa4ea8569924c9eb1135a4801fec4db96b73a6ceb51ea22934442fd31e`
- `src/cooperative_cycle_factors_verifier.py`
  `SHA-256: 4cbf8371a9335c220d7d58e223fc0e848f6912dbef0e59ed27299e739f6200c1`
- `certificates/MANIFEST.json`
  `SHA-256: 8b088a1e114644c52bb926f18a4da239849e98687c257cb5c825fb693289728e`

AI Disclosure and Conflict Statement. This work was produced by Lluis Eriksson with computational, formalization, and manuscript assistance from Google Antigravity autonomous research agents. As founder/operator of AIRR.SCIENCE, the author discloses an operator conflict (`operator_conflict: true`); all screening was performed by isolated PDF-only evaluators.

References

[1] Ron Aharoni, Eli Berger, Maria Chudnovsky, Frédéric Havet, and Zilin Jiang. Cooperative colorings of trees and of bipartite graphs. *Electronic Journal of Combinatorics*, 27(1):P1.41, 2020.

[2] Ron Aharoni, Eli Berger, and Ran Ziv. Independent systems of representatives in weighted graphs. *Combinatorica*, 27(3):253–267, 2007.

[3] Ron Aharoni and Penny E. Haxell. Hall's theorem for hypergraphs. *Journal of Graph Theory*, 35(2):83–88, 2000.

[4] Ron Aharoni, Ron Holzman, David Howard, and Philipp Sprüssel. Cooperative colorings and independent systems of representatives. *Electronic Journal of Combinatorics*, 22(2):P2.27, 2015.

[5] Bengt Aspvall, Michael F. Plass, and Robert Endre Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters*, 8(3):121–123, 1979.

[6] Xuqing Bai, Zhipeng Gao, Bi Li, and Xin Zhang. Cooperative colorings of cycles and uniform cycle factors, 2026.

[7] Penny E. Haxell. A note on vertex list colouring. *Combinatorics, Probability and Computing*, 10(4):345–347, 2001.

[8] Po-Shen Loh and Benny Sudakov. Independent transversals in locally sparse graphs. *Journal of Combinatorial Theory, Series B*, 97(6):904–918, 2007.

[9] Roy Meshulam. The clique complex and hypergraph matching. *Combinatorica*, 21(1):89–94, 2001.

[10] Roy Meshulam. Domination numbers and homology. *Journal of Combinatorial Theory, Series A*, 102(2):321–330, 2003.