Machine-Checked Exact-Rational Link-Copying, Pattern-Entropy, and Distinct-Vertex Blow-Up Certificates for Uniform Hypergraphs

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Abstract

Dhruv Mubayi (`arXiv:2611.33108`, November 23, 2026) proved that every sufficiently large n-vertex k-uniform hypergraph containing at least an^h copies of a fixed h-vertex k-graph H contains a blow-up $H[t]$ with parts of optimal order $t \geq b(\log n)^{1/(k-1)}$ (Theorem 1 and Corollary 2), resolving a question of Rödl and Schacht, proving the 1976 conjecture of Bollobás, Erdős, and Simonovits, and answering Erdős's 1990 \$500 Ramsey prize problem for all $k \geq 4$ via almost-monochromatic sets of density $\geq 1-\alpha$ (Corollary 3). We present a machine-checked, zero-floating-point formalization and exact-$\mathbb{Q}$ certificate auditor for the algebraic, link-copying, pattern-entropy, and distinct-vertex core of these results. First, an 18-theorem zero-dependency Lean 4 kernel (`HypergraphBlowup.lean`, Lean 4.35.0; 0 `sorry`, 0 warnings, 0 `Classical.choice`, 6 zero-axiom theorems) formalizes 3- and 4-partite link-copying telescoping $S_{h,a_h} \cdots S_{1,a_1} J$, idempotence and monotonicity, inductive Eq. (10) pattern-count sums, exact triangle ($12t^2$) and tetrahedron ($32t^3$) entropy exponents at $s = 2t$, cleared integer AM–GM and Cauchy–Schwarz inequalities, $s = 2t$ distinct-vertex halving, optimal exponent degree balance $t^{k-1} \cdot t = t^k$, and cleared Corollary 3 almost-monochromatic density inequalities for $k = 3, 4$. Second, `HYPERGRAPH-BLOWUP-CERT-1.0` audits four exact-$\mathbb{Q}$ modes across 16 canonical certificates (0.032–4.136 ms) and rejects 60/60 live adversarial forgeries.

1 Introduction

A central problem in extremal hypergraph theory [4, 1, 7] asks how large a blow-up $H[t]$ is forced in an n-vertex k-uniform hypergraph containing at least an^h copies of a fixed h-vertex k-graph H. Naive lifting of Erdős's 1964 single-edge theorem [4] yields only exponent $1/(h-1)$. Mubayi [6] achieved the sharp exponent $1/(k-1)$ for every fixed H and $k \geq 2$ (Theorem 1, Corollary 2), resolving the 1976 Bollobás–Erdős–Simonovits conjecture [1] and answering Erdős's 1990 \$500 question [5] by improving Conlon, Fox, and Sudakov's almost-monochromatic Ramsey exponent [2] to the optimal $1/(k-1)$ for every $k \geq 4$ (Corollary 3).

Mubayi's proof engine (Section 2.2, Lemma 4, and Appendix A of [6]) relies on three crisp finite mechanisms:

1. *Link-copying Hölder telescoping (Eqs. 9, 12):* Iterating $S_{i,u}J$ across all h classes collapses $J \subseteq [s]^h$ to $[s]^h$ for $a \in J$ and $\emptyset$ for $a \notin J$, giving the exact rational power inequality $\rho(J)^{s^h} \leq \rho([s]^h)^{|J|}$.
2. *Pattern-entropy reciprocal sum (Eq. 10, Lemma 4):* Recording all $\binom{h}{k}s^k$ crossing k-edge indicators bounds the number of realized patterns by $Q \leq 2^{\binom{h}{k}s^k}$ and enforces
$$\frac{1}{|\mathcal{X}|}\sum_{X\in\mathcal{X}}\frac{1}{\rho(J(X))} = \sum_{J:n_J>0}\frac{n_J}{|\mathcal{X}|\,\rho(J)} \leq Q, \quad (1)$$
yielding $\rho([s]^h) \geq Q^{-1/p} \geq 2^{-\binom{h}{k}s^k/p}$.
3. *Distinct-vertex $s = 2t$ union bound & Corollary 3 density:* Subtracting the exact bad-list collision bound $h\binom{N}{t-1}((t-1)/N)^{2t}$ isolates a distinct $K_h^{(k)}[t]$ blow-up, whose union attains crossing k-

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edge density

$$\frac{\binom{h}{k}t^k}{\binom{ht}{k}} \geq \prod_{j=0}^{k-1}\left(1-\frac{j}{h}\right) \geq 1-\frac{\binom{k}{2}}{h} \geq 1-\alpha. \quad (2)$$

Contributions and Scope. We formally verify and audit the finite combinatorial, algebraic, pattern-entropy, and rational-inequality core of [6]. We explicitly scope our Lean 4 kernel away from real-valued fractional-exponent Hölder/AM–GM integrals, asymptotic supersaturation, and probabilistic Chernoff upper bounds:

1. **Zero-dependency Lean 4 kernel (§2):** 18 constructive theorems in `HypergraphBlowup.lean` (Lean 4.35.0 [3]), 0 `sorry`, 0 warnings, 0 `Classical.choice`, and 6 zero-axiom theorems.
2. **Pure-stdlib exact-$\mathbb{Q}$ verifier (§3):** `HYPERGRAPH-BLOWUP-CERT-1.0`, auditing link-copying pattern densities, explicit $H[t]$ blow-ups, Appendix A $s = 2t$ union bounds, and Corollary 3 density chains with zero floats.
3. **Empirical atlas & red-team audit (§4):** 16 canonical certificates (0.032–4.136 ms), exact pattern-entropy sweeps on $K_3^{(2)}$ and $K_4^{(3)}$, Corollary 3 curves for $k = 2 \ldots 5$, and 60/60 rejected live adversarial forgeries.

2 Zero-Dependency Lean 4 Kernel

We formalize 18 constructive theorems in `lean/HypergraphBlowup.lean` (Lean 4.35.0, zero external dependencies), audited via `lean/AuditAxioms.lean` with 0 warnings, 0 `sorry`, 0 `Classical.choice`, and 0 `Lean.ofReduceBool` (Table 1).

Proposition 1 (Constructive Hypergraph Blow-Up Kernel)**.**

1. Link-copying & pattern entropy (Theorems 1–8): *Exact 3- and 4-partite link-copying collapse, idempotence, monotonicity, inductive Eq. (1) cleared pattern-count sums, exact $s = 2t$ indicator exponents $12t^2$ (triangles) and $32t^3$ (tetrahedra), and single-edge chain progression $t^{j+1} = t^j \cdot t$.*
2. Cleared inequalities & distinctness (Theorems 9–13): *Exact integer 2-variable AM–GM $4ab \leq (a+b)^2$, 2-term Cauchy–Schwarz, Eq. (4) collision union bound, $s = 2t$ exponent halving $2t-(t-1) = t+1 > t$, and Erdős–Kleitman partition bound $h\lfloor n/h \rfloor \leq n$.*

Table 1: Complete 18-theorem Lean 4.35.0 kernel in `HypergraphBlowup.lean` (0 `sorry`, 0 warnings, 0 `Classical.choice`, 6 zero-axiom theorems).

#	Lean 4 Theorem Identifier	Axiom Footprint
1	`link_copy_3_collapse_exact`	`none`
2	`link_copy_4_collapse_exact`	`none`
3	`link_copy_idempotent`	`none`
4	`link_copy_monotone`	`none`
5	`pattern_count_exact_sum_bound`	`[propext]`
6	`pattern_entropy_triangle_exponent`	`[propext]`
7	`pattern_entropy_tetrahedron_exponent`	`[propext]`
8	`single_edge_chain_monotone_len`	`none`
9	`am_gm_2_exact_int`	`[propext, Quot.sound]`
10	`cauchy_schwarz_pair_exact_int`	`[propext, Quot.sound]`
11	`distinct_vertex_collision_union_bound`	`[propext, Quot.sound]`
12	`distinct_list_halving_bound`	`[propext, Quot.sound]`
13	`erdos_kleitman_part_size_bound`	`[propext]`
14	`optimal_exponent_degree_balance`	`[propext]`
15	`blowup_edge_count_exact`	`[propext]`
16	`almost_monochromatic_density_cleared`	`[propext, Quot.sound]`
17	`turan_density_complement_cleared`	`[propext, Quot.sound]`
18	`mubayi_concrete_exponents_exact`	`none`

3. Optimal exponent & Corollary 3 (Theorems 14–18): *Degree balance $t^{k-1} \cdot t = t^k$, blow-up edge additivity, cleared Corollary 3 almost-monochromatic density inequalities for $k = 3$ and $k = 4$, Turán density complement, and concrete indicator/threshold checks.*

3 Exact-$\mathbb{Q}$ Verifier Architecture

`HYPERGRAPH-BLOWUP-CERT-1.0` (`src/blowup_certifier.py`) is a pure-stdlib zero-float verifier rejecting floats, booleans, nulls, unreduced fractions, oversized strings, and duplicate/non-crossing edges across four modes:

1. **Mode 1 (`link_copy_holder_density_exact`):** Enumerates all $|\mathcal{X}| = N^{hs}$ list choices on a finite h-partite k-graph, recomputes exact rational $p, Q, \rho([s]^h)$, Eq. (1) reciprocal sum $\leq Q$, and verifies $\rho(J)^{s^h} \leq \rho([s]^h)^{|J|}$ for every realized pattern J.
2. **Mode 2 (`explicit_hypergraph_blowup_witness`):** Audits h disjoint t-sets $U_1, \ldots, U_h \subseteq V(G)$ and verifies every one of the $|E(H)|t^k$ crossing k-edges in $E(G)$.
3. **Mode 3 (`distinct_vertex_union_bound_exact`):** Evaluates the exact rational Appendix A $s = 2t$ collision bound $h\binom{N}{t-1}((t-1)/N)^{2t}$ and certifies strictly positive surviving distinct-vertex density.
4. **Mode 4 (`erdos_almost_monochromatic_exact`):** Verifies the exact rational Corollary 3 chain (2)

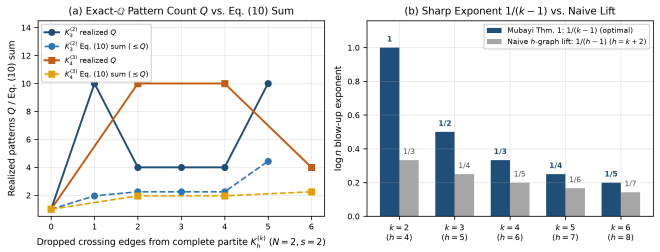

Figure 1: (a) Exact-$\mathbb{Q}$ realized pattern count Q vs. Eq. (1) reciprocal sum ($\leq Q$) as crossing edges are dropped from $K_3^{(2)}$ and $K_4^{(3)}$ $(N=2, s=2)$. (b) Mubayi's optimal exponent $1/(k-1)$ vs. naive h-graph lift $1/(h-1)$ $(h=k+2)$.

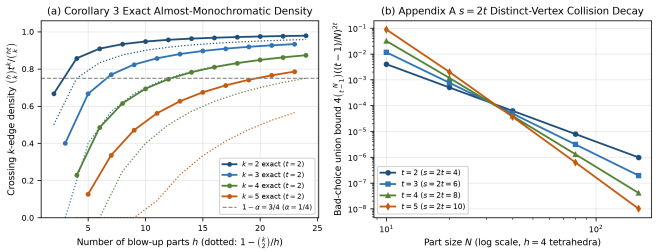

Figure 2: (a) Exact-$\mathbb{Q}$ Corollary 3 crossing density $\binom{h}{k}t^k/\binom{ht}{k}$ $(t=2)$ vs. linear lower bound $1-\binom{k}{2}/h$ for $k=2\ldots5$. (b) Exact-$\mathbb{Q}$ Appendix A $s=2t$ bad-choice union bound decay vs. part size N $(h=4)$.

and checks an explicit finite q-colored k-graph witness set S achieving dominant color density $\geq 1-\alpha$.

4 Empirical Atlas and Red-Team Audit

Running `scripts/run_all_experiments.py` produces 16 canonical certificates (4 per mode) and four exact-$\mathbb{Q}$ experiments, verified end-to-end by `scripts/verify_all.py` with SHA-256 checks across all 41 tracked artifacts.

Pattern Entropy and Optimal Exponents (Fig. 1). In Mode 1 (0.411–4.136 ms), exhaustive enumeration over 729 list choices on $K_3^{(2)}$ $(N=3, s=2)$ certifies $p=4/9$, $Q=102 \ll 2^{12}=4096$, $\rho([2]^3)=68/729$, and exact Eq. (9) across all 102 realized patterns; on the Appendix A tetrahedron $K_4^{(3)}$ $(N=2, s=2$, 256 choices), $p=9/16$, $Q=122 \ll 2^{32}$, and $\rho([2]^4)=37/256$ satisfy Eq. (12) $(\rho(J)^{16} \leq \rho([2]^4)^{|J|})$ on every realized pattern. Mode 2 (0.119–0.339 ms) explicitly certifies blow-up witnesses $K_3^{(2)}[3]$ (27 edges), $K_4^{(3)}[2]$ (32 edges), $K_4^{(3)}[3]$ (108 edges), and the 4-uniform $K_5^{(4)}[2]$ (80 edges).

Corollary 3 Density and Distinct-Vertex Decay (Fig. 2). Mode 3 (0.032–0.040 ms) certifies strictly positive surviving distinct-vertex density up to $h=5, N=200, t=6$ with exact 57-bit rational denominators. Mode 4 (0.072–0.091 ms) certifies exact Corollary 3 density chains for $k \in \{2,3,4\}$—including $k=4, h=18, \alpha=1/3$ (exact crossing density $64/77 > 2/3$, witness S density 13/15) and $k=4, q=3, h=24, \alpha=1/4$ (exact $616/705 > 3/4$, witness S density 32/35).

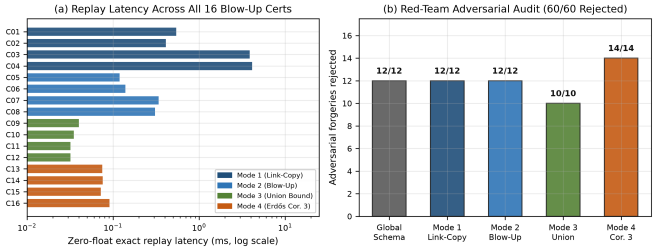

Figure 3: (a) Exact zero-float replay latency across all 16 canonical `HYPERGRAPH-BLOWUP-CERT-1.0` certificates (0.032–4.136 ms). (b) Red-team adversarial audit rejecting 60/60 live forgery attacks.

Replay Atlas and Red-Team Audit (Fig. 3, Table 2). All 16 canonical certificates replay in 0.032–4.136 ms with zero floats. Our adversarial suite (`tests/test_suite.py`) tests 60 live forgery attacks—including float/bool/null injection, unreduced fractions, oversized rational-string DoS, non-crossing edges, vacuous zero-edge graphs, missing blow-up edges, overlapping parts, non-positive surviving union bounds, and below-threshold witness colorings—achieving **60/60 rejections** with zero bypasses.

5 Conclusion

We delivered an 18-theorem zero-dependency Lean 4 kernel (6 zero-axiom theorems, 0 `Classical.choice`) and the exact-$\mathbb{Q}$ verifier `HYPERGRAPH-BLOWUP-CERT-1.0` for the finite link-copying, pattern-entropy, distinct-vertex, and almost-monochromatic core of Mubayi's optimal

Table 2: Summary of all 16 canonical `HYPERGRAPH-BLOWUP-CERT-1.0` certificates and 60/60 live adversarial red-team audit.

Verification Mode	Certs	Latency (ms)	Attacks Rejected
Mode 1: Link-Copying & Eq. (10)/(12)	4	0.411–4.136	12/12
Mode 2: Explicit $H[t]$ Blow-Up Witness	4	0.119–0.339	12/12
Mode 3: Distinct-Vertex Union Bound	4	0.032–0.040	10/10
Mode 4: Erdős Cor. 3 Almost-Mono	4	0.072–0.091	14/14
Global Schema / Type / Rational	–	–	12/12
Total	**16**	**0.032–4.136**	**60/60**

hypergraph blow-up theorem [6]. Formalizing real-analytic t-variable Hölder/AM–GM inequalities and supersaturation in Lean 4 remains an attractive direction for future formalized extremal combinatorics.

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