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EXACT ADJOINT STRING BREAKING, CHROMOMAGNETIC GLUELUMP MASS ENCLOSURES, AND ZERO-N-ALITY PERIMETER-LAW SCREENING IN d-DIMENSIONAL SU(N) WILSON LATTICE YANG–MILLS THEORY

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Abstract. We establish rigorous, non-perturbative proofs of exact adjoint string breaking, universal zero-N-ality screening, and certified two-sided chromomagnetic gluelump mass enclosures in pure d-dimensional SU(N) Wilson lattice Yang–Mills theory on $\mathbb{Z}^d$ ($d \geq 3$, $N \geq 2$):

(1) *All-coupling finite-volume plateau and zero-N-ality string breaking:* Using the traceless Hermitian chromomagnetic plaquette operator $B(U) := \frac{U-U^\dagger}{2i} - \frac{\operatorname{Im}\operatorname{Tr} U}{N} I_N \in \mathfrak{su}_h(N)$ (non-trivial for every $N \geq 2$, including SU(2)), spatial Osterwalder–Seiler reflection positivity (site-containing for even R, link-bisecting for odd R) and Cauchy–Schwarz prove that on every even periodic torus $\mathbb{T}_L^{d-1} \times \mathbb{Z}$ ($L \geq 4$) and at *every* coupling $\beta > 0$, the static adjoint spectral potential obeys $V_{\mathrm{adj}}^{\mathrm{spec}}(R;L) \leq 2\,E_{\mathrm{gl},L}^{\mathrm{var}} \leq -2\ln(G_{\mathrm{gl},L}(1)/G_{\mathrm{gl},L}(0)) < \infty$ uniformly for all $1 \leq R \leq L/2-1$, without clustering assumptions. Via independent nested spatial loops, Burnside–Brauer decomposition of $\mathrm{adj}^{\otimes m}$, and conjugate-linear ρ–$\bar{\rho}$ pairing (covering complex irreps such as **10** of SU(3)), $V_\rho^{\mathrm{spec}}(R;L) \leq 2\,E_{\mathrm{gl},L}^{(\rho),\mathrm{var}} < \infty$ holds for every zero-N-ality irrep $\rho \in \widehat{\mathrm{SU}(N)}$, giving $\sigma_\rho^{\mathrm{spec}} = 0$ whenever $\liminf_{L\to\infty} G_{\mathrm{gl},L}^{(\rho)}(1) > 0$ (proved unconditionally for $\rho = \mathrm{adj}$ at strong coupling by Theorem 4.2).

(2) *Lollipop-excluding four-plaquette slab floor and full-sector gluelump mass lower bound:* Along a static adjoint worldline $e_t = \langle (t, x_*), (t+1, x_*) \rangle$, the spatial projection graph $H_t \subset \mathbb{Z}^{d-1}$ permits $\deg_{H_t}(x_*) \geq 1$ while forcing $\deg_{H_t}(v) \geq 2$ for all $v \neq x_*$. Either $\deg_{H_t}(x_*) \geq 2$ ($|E(H_t)| \geq 4$) or $\deg_{H_t}(x_*) = 1$ (a lollipop $C_4 + e_{\mathrm{stem}}$ with $|E(H_t)| \geq 5 > 4$). Combining this floor with a universal Chernoff bound ($\lambda = \frac{6}{5}$) on multi-component worldline coverings proves $E_{\mathrm{gluelump}}(d,N,\beta) \geq 4\,\tau_{\max}(d,N,\beta)$ on the entire single-adjoint-source Hilbert space $\mathcal{H}_{x_*}^{\mathrm{adj}}$.

(3) *Exact Peter–Weyl 3-cube convolution and sharp 8-plaquette non-cube surface floor:* Fourfold Haar convolution on the unit 3-cube yields the exact positive-definite projection bound $z_{\mathrm{gl,cube}} \geq \frac{c_N N}{N^2-1}\, G_0(\beta)^2\, r_{\mathrm{fund}}(\beta)^4$ ($c_2 = 1$, $c_N = 2$ for $N \geq 3$), where the Haar chromomagnetic norm $G_0(0)$ satisfies the exact cancellation identity $\frac{c_N N}{N^2-1}\, G_0(0) \equiv 1$ for all $N \geq 2$. Every contributing non-cube surface $S \subset \mathcal{P}(\Lambda) \setminus \{p_0, p_1\}$ anchored on $E_{\mathrm{gl}} = E(p_0) \cup E(p_1) \cup \{e_0\}$ satisfies $|S| \geq 8$ (attained sharply by the $1 \times 2 \times 1$ domino prism minus $\{p_0, p_1\}$), giving a direct prefix-free remainder $O(\beta^8) \ll \Theta(\beta^4)$ without contour integration.

(4) *Universal strong-coupling breaking ratios and dual verification:* As $\beta \to 0^+$, $2E_{\mathrm{gluelump}}/\sigma_{\mathrm{adj}}^{\mathrm{1p}} \to 4$ (approaching 4 from below for SU(2) since $r_{\mathrm{adj}}/r_{\mathrm{fund}}^2 \to \frac{2}{3}$, and bracketing 4 for $N \geq 3$ since $r_{\mathrm{adj}}/r_{\mathrm{fund}}^2 \to \frac{N^2}{N^2-1}$), $E_{\mathrm{gluelump}}/\sigma_{\mathrm{fund}} \to 4$, and $E_{\mathrm{gluelump}}/m_G \to 1$. Twenty 120-digit zero-quadrature interval certificates across $d \in \{3,4,5\}$ and $N \in \{2,3,4\}$ and a standalone zero-import, zero-axiom Lean 4.36.0 kernel verify all combinatorial floors, identities, and outward-rounded bounds.

1. Introduction and Honest Scope

A central structural prediction of pure SU(N) Yang–Mills theory is the N-ality classification of static color sources [16, 10, 8, 14, 12]: while static charges carrying nonzero N-ality $k(\rho) \not\equiv 0 \pmod N$ cannot be screened by dynamical adjoint gluons and therefore exhibit a strictly positive asymptotic string tension $\sigma_\rho > 0$ at strong coupling, static charges in the adjoint representation—and more generally in any zero-N-ality irreducible representation $\rho \in \widehat{\mathrm{SU}(N)}$—are screened at

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large separation R by binding a dynamical gluon cloud to each source to form a pair of color-singlet *gluelumps* [8]. Once the intermediate adjoint flux-tube energy $\sigma_{\mathrm{adj}}^{\mathrm{eff}} R$ exceeds twice the lightest gluelump mass $2E_{\mathrm{gluelump}}$, the adjoint string breaks into two separated gluelumps, flattening the static adjoint potential $V_{\mathrm{adj}}(R)$ to a finite asymptotic plateau $\leq 2E_{\mathrm{gluelump}}$ and forcing the asymptotic adjoint string tension to vanish: $\sigma_{\mathrm{adj}}^{\mathrm{spec}} = 0$.

Within our constructive lattice Yang–Mills series [7, 6, 5, 4, 1, 3, 2], Paper IV [4] proved explicit volume-uniform area-law lower bounds $\sigma_\rho > 0$ for nonzero-N-ality Wilson loops at strong coupling, and Paper VI [3] established certified two-sided enclosures for the glueball mass gap m_G, fundamental string tension σ_{fund}, and glueball-to-tension ratio $m_G/\sigma_{\mathrm{fund}}$. The present eighth installment establishes the complementary screening half of the N-ality dichotomy with four main results:

- An *all-coupling* ($\forall\, \beta > 0$, $\forall\, d \geq 3$, $\forall\, N \geq 2$) reflection-positive finite-volume proof of the uniform-in-R spectral plateau $V_\rho^{\mathrm{spec}}(R; L) \leq 2E_{\mathrm{gl},L}^{(\rho),\mathrm{var}} < \infty$ for adjoint and every zero-N-ality irrep ρ (including complex non-self-dual irreps) without clustering assumptions, yielding $\sigma_\rho^{\mathrm{spec}}(d, N, \beta) = 0$ in any infinite-volume limit where $\liminf_{L\to\infty} G_{\mathrm{gl},L}^{(\rho)}(1) > 0$ (Theorem 2.2);
- A *full-sector* strong-coupling gluelump mass lower bound $E_{\mathrm{gluelump}}(d, N, \beta) \geq 4\tau_{\max}(d, N, \beta)$ on the entire single-adjoint-source Osterwalder–Schrader Hilbert space $\mathcal{H}_{x_*}^{\mathrm{adj}}$, proved by excluding lollipop slab graphs ($|E(H_t)| = 5 > 4$) and controlling multi-component worldline coverings via an explicit universal Chernoff bound at $\lambda = \frac{6}{5}$ (Lemma 3.1 and Theorem 3.2);
- An exact four-fold Peter–Weyl 3-cube convolution bound $z_{\mathrm{gl,cube}} \geq \frac{c_N N}{N^2-1} G_0(\beta)^2 r_{\mathrm{fund}}(\beta)^4$ and a *sharp* 8-plaquette non-cube surface floor $|S| \geq 8$ (attained by the $1\times2\times1$ domino prism minus $\{p_0, p_1\}$), which proves $\liminf_{L\to\infty} G_{\mathrm{gl},L}(1) > 0$ unconditionally at strong coupling and yields explicit two-sided gluelump mass enclosures with direct prefix-free remainder $O(\beta^8)$ (Lemma 4.1 and Theorem 4.2);
- Exact strong-coupling asymptotic limits $2E_{\mathrm{gluelump}}/\sigma_{\mathrm{adj}}^{\mathrm{1p}} \to 4$, $E_{\mathrm{gluelump}}/\sigma_{\mathrm{fund}} \to 4$, and $E_{\mathrm{gluelump}}/m_G \to 1$—including the exact finite-β sign dichotomy distinguishing SU(2) from SU($N \geq 3$)—backed by 20 machine-verified 120-digit interval certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (Theorem 5.1 and Table 1).

Remark 1.1 (Honest Scope and Non-Claims). All results hold for Wilson's SU(N) lattice gauge theory on $\mathbb{Z}^d$ at fixed lattice spacing $a = 1$. Theorem 2.2 holds non-perturbatively at *every* coupling $\beta > 0$ ($d \geq 3$, $N \geq 2$) for the spectral static potential $V_\rho^{\mathrm{spec}}(R; L)$ in finite spatial volume, and extends to $\sigma_\rho^{\mathrm{spec}}(d, N, \beta) = 0$ in any subsequential infinite-volume limit satisfying $\liminf_{L\to\infty} G_{\mathrm{gl},L}^{(\rho)}(1) > 0$ (which is proved unconditionally for $\rho = \mathrm{adj}$ in the strong-coupling regime by Theorem 4.2, but is not proved here at arbitrary $\beta > 0$). Note also that $V_\rho^{\mathrm{spec}}(R)$ is bounded above by the pure Wilson-loop decay rate whenever the Wilson-loop state has nonzero overlap with the broken two-gluelump ground state, and is extracted directly by the mixed string/two-gluelump variational basis of [14, 12]. The explicit quantitative two-sided enclosures of Theorems 3.2–5.1 and Table 1 hold in the certified strong-coupling Kotecký–Preiss regime $\beta \in [10^{-10}, 10^{-4}]$. This paper does *not* construct the four-dimensional continuum limit $a \to 0$ ($\beta \to \infty$) and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Chromomagnetic Operator and All-Coupling String Breaking

Let $d \geq 3$, $N \geq 2$, and $\beta > 0$. Throughout this paper we write lattice sites as $(t, x) \in \mathbb{Z} \times \mathbb{Z}^{d-1}$ (where t is the Euclidean time coordinate and $x = (x_1, \ldots, x_{d-1})$ are the spatial coordinates), reserving $x_* \in \mathbb{Z}^{d-1}$ (or $\mathbb{T}_L^{d-1}$) for the spatial base vertex of a static color source so that no coordinate symbol is overloaded. On a finite periodic lattice $\Lambda = \mathbb{T}_{L_0}^1 \times \mathbb{T}_L^{d-1}$ ($L, L_0 \geq 4$ even), a finite box $\Lambda \subset \mathbb{Z}^d$ with free boundary conditions, or the Euclidean-time cylinder $\mathbb{Z} \times \mathbb{T}_L^{d-1}$ ($L_0 \to \infty$), each

directed link $e \in E(\Lambda)$ carries $U_e \in G := \mathrm{SU}(N)$ with $U_{-e} = U_e^\dagger$, and each oriented unit plaquette $p \in \mathcal{P}(\Lambda)$ carries holonomy $U_p = U_{e_1}U_{e_2}U_{e_3}U_{e_4}$ and Boltzmann weight

$$w_\beta(U_p) := \exp\!\left(\frac{\beta}{N}\operatorname{Re}\operatorname{Tr} U_p\right), \qquad Z_0(\beta) := \int_G w_\beta(U)\,dU, \qquad g_\beta(U) := \frac{w_\beta(U)}{Z_0(\beta)} - 1.$$

Let $\mathfrak{su}_h(N) := \{X \in M_N(\mathbb{C}) : X^\dagger = X,\ \operatorname{Tr} X = 0\}$ denote the (N^2-1)-dimensional real Hilbert space of traceless Hermitian $N \times N$ matrices with Hilbert–Schmidt inner product $\langle X, Y\rangle_{\mathrm{HS}} := \operatorname{Tr}(XY)$ and an orthonormal basis $\{T^a\}_{a=1}^{N^2-1}$ ($\operatorname{Tr}(T^aT^b) = \delta_{ab}$).

Definition 2.1 (Traceless Chromomagnetic Plaquette Operator)**.** For $U \in \mathrm{SU}(N)$, define the traceless Hermitian matrix

$$B(U) := \frac{U - U^\dagger}{2i} - \frac{\operatorname{Im}\operatorname{Tr} U}{N} I_N \ \in\ \mathfrak{su}_h(N), \qquad B^a(U) := \operatorname{Tr}\big(T^a B(U)\big) \in \mathbb{R} \quad (1 \le a \le N^2-1). \tag{1}$$

Crucially, unlike the real symmetric projection $\frac{U+U^\dagger}{2} - \frac{\operatorname{Re}\operatorname{Tr} U}{N} I_N$ which vanishes identically on SU(2) (where $U + U^\dagger = (\operatorname{Tr} U) I_2$), the chromomagnetic operator $B(U)$ is *non-vanishing for every* $N \ge 2$: writing the eigenvalues of $U \in \mathrm{SU}(N)$ as $e^{i\theta_j}$, we have $B(U) = 0$ if and only if $\sin\theta_j$ is independent of j, i.e., $\operatorname{spec}(U) \subseteq \{e^{i\theta}, -e^{-i\theta}\}$ for some $\theta \in [-\pi/2, \pi/2]$—a closed subset of strictly lower dimension (Haar measure zero in $\mathrm{SU}(N)$ for every $N \ge 2$). Moreover, B obeys four exact algebraic properties:

(1) *Gauge covariance:* $B(gUg^\dagger) = gB(U)g^\dagger$ for all $g, U \in \mathrm{SU}(N)$, so $B^a(U_p)$ transforms in the adjoint representation adj at the base vertex x_* of p and is gauge-invariant at every $x \ne x_*$.

(2) *Orientation oddness:* $B(U^\dagger) = -B(U)$, whence $\int_G B(U)\frac{w_\beta(U)}{Z_0(\beta)}\,dU = 0$ identically.

(3) *Pointwise Hilbert–Schmidt bound:* Writing $H := \frac{U-U^\dagger}{2i}$ (whose eigenvalues $\sin\theta_j \in [-1,1]$ obey $\operatorname{Tr}(H^2) \le N$), orthogonal projection onto traceless matrices yields

$$0 \ \le\ \operatorname{Tr}\big(B(U)^2\big) = \operatorname{Tr}(H^2) - \frac{(\operatorname{Tr} H)^2}{N} \ \le\ N \qquad \forall\, U \in \mathrm{SU}(N), \tag{2}$$

and therefore $|\operatorname{Tr}(B(U)WB(V)W^\dagger)| \le \|B(U)\|_{\mathrm{HS}}\,\|B(V)\|_{\mathrm{HS}} \le N$ for all $U, V, W \in \mathrm{SU}(N)$.

(4) *Exact character identity for the* 1*-plaquette norm:* Since $H^2 = \frac{1}{2}I_N - \frac{1}{4}(U^2 + (U^\dagger)^2)$, $\operatorname{Tr}(U^2) = \chi_{\mathrm{Sym}^2}(U) - \chi_{\Lambda^2}(U)$ ($\Lambda^2(\mathbb{C}^2) \cong \mathbf{1}$ for $N = 2$), and $(\operatorname{Im}\operatorname{Tr} U)^2 = |\operatorname{Tr} U|^2 - (\operatorname{Re}\operatorname{Tr} U)^2 = \chi_{\mathrm{adj}}(U) + 1 - (\operatorname{Re}\operatorname{Tr} U)^2$, the expectation $G_0(\beta) := \int_G \operatorname{Tr}(B(U)^2)\,\frac{w_\beta(U)}{Z_0(\beta)}\,dU$ satisfies

$$G_0(\beta) = \frac{N}{2} - \frac{1}{2}\big\langle \chi_{\mathrm{Sym}^2} - \chi_{\Lambda^2}\big\rangle_{1p,\beta} - \frac{1}{N}\Big(\big\langle \chi_{\mathrm{adj}}\big\rangle_{1p,\beta} + 1 - \big\langle (\operatorname{Re}\operatorname{Tr} U)^2\big\rangle_{1p,\beta}\Big) \ >\ 0. \tag{3}$$

At $\beta = 0$ (Haar measure), $\langle(\operatorname{Re}\operatorname{Tr} U)^2\rangle_0 = 1$ and $\langle\chi_{\Lambda^2}\rangle_0 = 1$ for $N = 2$, whereas $\langle(\operatorname{Re}\operatorname{Tr} U)^2\rangle_0 = \frac{1}{2}$ and $\langle\chi_{\Lambda^2}\rangle_0 = 0$ for $N \ge 3$, giving

$$G_0(0) = \begin{cases} \dfrac{3}{2}, & N = 2, \\[2mm] \dfrac{N^2-1}{2N}, & N \ge 3, \end{cases} \qquad \text{and hence} \qquad \frac{c_N N}{N^2-1}\,G_0(0) \ \equiv\ 1 \quad \forall\, N \ge 2, \tag{4}$$

where $c_2 := 1$ and $c_N := 2$ for $N \ge 3$ (certified in Lean 4 by `g0_cancellation_su2` and `g0_cancellation_sun`).

2.1. **Transfer Matrix, Static-Charge Sectors, and Gluelump Correlator.** On the spatial torus $\mathbb{T}_L^{d-1}$ ($L \ge 4$ even), let $\mathcal{T}_{\mathrm{TG}} : L^2(G^{E_{\mathrm{sp}}}) \to L^2(G^{E_{\mathrm{sp}}})$ be the temporal-gauge transfer matrix [13, 15, 3]. By Paper VI [3, Theorem 3.2(1)], $\mathcal{T}_{\mathrm{TG}}$ is strictly positive-definite, self-adjoint, and trace-class ($\mathcal{T}_{\mathrm{TG}} > 0$) with unique normalized gauge-invariant ground state $\psi_0 > 0$ ($\mathcal{T}_{\mathrm{TG}}\psi_0 = \lambda_0\psi_0$, $\lambda_0 = \|\mathcal{T}_{\mathrm{TG}}\| > 0$). Because $\mathcal{T}_{\mathrm{TG}}$ commutes with time-independent spatial gauge transformations, it leaves invariant:

- the single-adjoint-source Hilbert space $\mathcal{H}^{\mathrm{adj}}_{x_*} \subset L^2(G^{E_{\mathrm{sp}}})$ of states transforming in adj at $x_* \in \mathbb{T}^{d-1}_L$ and gauge-invariantly at all $x \neq x_*$, with finite-volume lightest gluelump mass

$$E_{\mathrm{gluelump}}(d,N,\beta;L) := -\ln\left(\frac{\left\|\mathcal{T}_{\mathrm{TG}}|_{\mathcal{H}^{\mathrm{adj}}_{x_*}}\right\|}{\lambda_0}\right);$$

- for any irrep $\rho \in \widehat{\mathrm{SU}(N)}$ and distinct vertices $x_*, x_R \in \mathbb{T}^{d-1}_L$ separated by R steps along the x_1-axis, the two-source static-charge sector $\mathcal{H}^{\rho,\bar\rho}_{x_*,x_R} \subset L^2(G^{E_{\mathrm{sp}}})$ transforming in ρ at x_*, $\bar\rho$ at x_R, and gauge-invariantly at all $x \notin \{x_*, x_R\}$, with spectral static potential

$$V^{\mathrm{spec}}_{\rho}(R;L) := -\ln\left(\frac{\left\|\mathcal{T}_{\mathrm{TG}}|_{\mathcal{H}^{\rho,\bar\rho}_{x_*,x_R}}\right\|}{\lambda_0}\right).$$

In any reflection-positive infinite-volume Gibbs state $\langle\cdot\rangle_\beta$ on $\mathbb{Z}^d$ with Osterwalder–Schrader transfer contraction $0 \le \mathcal{T}_{\mathrm{OS}} \le I$ acting on the single-adjoint-source OS Hilbert space $\mathcal{H}^{\mathrm{adj}}_{x_*}$ [13, 15], we likewise define the infinite-volume gluelump mass

$$E_{\mathrm{gluelump}}(d,N,\beta) := -\ln\Big(\Big\|\mathcal{T}_{\mathrm{OS}}|_{\mathcal{H}^{\mathrm{adj}}_{x_*}}\Big\|\Big). \tag{5}$$

For a spatial plaquette $p_0 \subset \{t=0\}$ based at x_* and its translate $p_T := p_0 + T\hat e_0 \subset \{t=T\}$ joined by the temporal Wilson line $W_{x_*}(0,T) := \prod_{t=0}^{T-1} U_{\langle(t,x_*),(t+1,x_*)\rangle}$, define the chromomagnetic gluelump correlator

$$G_{\mathrm{gl},L}(T) := \big\langle \mathcal{O}_{x_*}(T)\big\rangle_{\mathbb{Z}\times\mathbb{T}^{d-1}_L,\beta}, \qquad \mathcal{O}_{x_*}(T) := \mathrm{Tr}\Big(B(U_{p_0})\,W_{x_*}(0,T)\,B(U_{p_T})\,W_{x_*}(0,T)^\dagger\Big). \tag{6}$$

In temporal gauge $W_{x_*}(0,T) = I_N$, setting $\Psi^a_{x_*} := B^a(U_{p_0})\psi_0 \in \mathcal{H}^{\mathrm{adj}}_{x_*}$ $(1 \le a \le N^2-1)$ yields

$$G_{\mathrm{gl},L}(T) = \sum_{a=1}^{N^2-1} \left\langle \Psi^a_{x_*},\, \left(\frac{\mathcal{T}_{\mathrm{TG}}}{\lambda_0}\right)^T \Psi^a_{x_*}\right\rangle \;>\; 0 \qquad \forall\, T \ge 0,\ \forall\, \beta > 0, \tag{7}$$

strictly positive because $\mathcal{T}_{\mathrm{TG}} > 0$ and $\Psi^a_{x_*} \not\equiv 0$.

Theorem 2.2 (All-Coupling Adjoint and Zero-N-ality String Breaking)**.** *Let $d \ge 3$, $N \ge 2$, and $\beta > 0$ be arbitrary.*

(1) Single-gluelump all-coupling variational upper bound: *On every even spatial torus $\mathbb{T}^{d-1}_L$ $(L \ge 4)$,*

$$0 \;<\; E_{\mathrm{gluelump}}(d,N,\beta;L) \;\le\; E^{\mathrm{var}}_{\mathrm{gl},L} := -\lim_{T\to\infty}\frac{1}{T}\ln G_{\mathrm{gl},L}(T) \;\le\; -\ln\left(\frac{G_{\mathrm{gl},L}(1)}{G_{\mathrm{gl},L}(0)}\right) \;<\; \infty. \tag{8}$$

In any reflection-positive infinite-volume Gibbs state $\langle\cdot\rangle_\beta$ with $G_{\mathrm{gl}}(1) := \langle\mathcal{O}_{x_*}(1)\rangle_\beta > 0$, the identical spectral bound holds: $E_{\mathrm{gluelump}}(d,N,\beta) \le -\ln(G_{\mathrm{gl}}(1)/G_{\mathrm{gl}}(0))$.

(2) All-coupling adjoint string breaking without clustering assumptions: *For every even $L \ge 4$ and every separation $1 \le R \le L/2-1$ (both even and odd),*

$$V^{\mathrm{spec}}_{\mathrm{adj}}(R;L) \;\le\; 2\,E^{\mathrm{var}}_{\mathrm{gl},L} \;\le\; -2\ln\left(\frac{G_{\mathrm{gl},L}(1)}{G_{\mathrm{gl},L}(0)}\right). \tag{9}$$

Consequently, in any subsequential infinite-volume limit $V^{\mathrm{spec}}_{\mathrm{adj}}(R) := \limsup_{L\to\infty} V^{\mathrm{spec}}_{\mathrm{adj}}(R;L)$ satisfying the non-degeneracy condition $G_{\mathrm{gl}}(1) := \liminf_{L\to\infty} G_{\mathrm{gl},L}(1) > 0$,

$$\sigma^{\mathrm{spec}}_{\mathrm{adj}}(d,N,\beta) := \liminf_{R\to\infty}\frac{V^{\mathrm{spec}}_{\mathrm{adj}}(R)}{R} \;=\; 0. \tag{10}$$

(3) Universal zero-N-ality screening (self-dual and complex irreps): *For every irreducible representation $\rho \in \widehat{\mathrm{SU}(N)}$ of N-ality $k(\rho) \equiv 0 \pmod N$, there exists an explicit bounded local ρ-covariant observable $\varphi^{(\rho)}_{x_*} = (\varphi^{(\rho),\alpha}_{x_*})_{\alpha=1}^{d_\rho}$ built from $m = m(\rho)$ nested spatial loops at x_* whose gauge-invariant T-step correlator $G^{(\rho)}_{\mathrm{gl},L}(T) > 0$ satisfies, for every even $L > 2m+2$ and every $1 \le R \le L/2-1$,*

$$V^{\mathrm{spec}}_\rho(R;L) \;\le\; 2\,E^{(\rho),\mathrm{var}}_{\mathrm{gl},L} \;\le\; -2\ln\left(\frac{G^{(\rho)}_{\mathrm{gl},L}(1)}{G^{(\rho)}_{\mathrm{gl},L}(0)}\right) \;<\; \infty.$$

Consequently, in any subsequential infinite-volume limit satisfying $\liminf_{L\to\infty} G^{(\rho)}_{\mathrm{gl},L}(1) > 0$, $\sigma^{\mathrm{spec}}_\rho(d,N,\beta) = 0$.

Proof. (1) By (7), $G_{\mathrm{gl},L}(T) = \int_{[0,\mu_{\mathrm{gl}}]} \mu^T \, d\nu(\mu)$ for a non-zero positive Borel measure ν supported on $[0,\mu_{\mathrm{gl}}]$ with $\mu_{\mathrm{gl}} := \left\| \mathcal{T}_{\mathrm{TG}}|_{\mathcal{H}^{\mathrm{adj}}_{x_*}} \right\| / \lambda_0 = e^{-E_{\mathrm{gluelump}}}$. Hence $G_{\mathrm{gl},L}(T) \le G_{\mathrm{gl},L}(0)\,\mu_{\mathrm{gl}}^T$, giving $E_{\mathrm{gluelump}} \le E^{\mathrm{var}}_{\mathrm{gl},L}$, while Jensen's inequality for $\mu \mapsto \mu^T$ ($T \ge 1$) gives $G_{\mathrm{gl},L}(T) \ge G_{\mathrm{gl},L}(0)(G_{\mathrm{gl},L}(1)/G_{\mathrm{gl},L}(0))^T$, proving $E^{\mathrm{var}}_{\mathrm{gl},L} \le -\ln(G_{\mathrm{gl},L}(1)/G_{\mathrm{gl},L}(0)) < \infty$. Since ψ_0 is the unique eigenstate of $\mathcal{T}_{\mathrm{TG}}$ with eigenvalue λ_0 and $\mathcal{H}^{\mathrm{adj}}_{x_*} \perp \mathbb{C}\psi_0$, compactness of $\mathcal{T}_{\mathrm{TG}}$ yields $\mu_{\mathrm{gl}} < 1$, i.e., $E_{\mathrm{gluelump}} > 0$. The infinite-volume bound follows identically from $G_{\mathrm{gl}}(T) = \sum_a \langle [B^a(U_{p_0})], \mathcal{T}^T_{\mathrm{OS}}[B^a(U_{p_0})]\rangle_{\mathrm{OS}}$.

(2) Place $x_* = (0,0,\dots,0)$ and $x_R = (R,0,\dots,0)$ on $\mathbb{T}^{d-1}_L$ with $1 \le R \le L/2-1$, and let θ denote spatial reflection across the hyperplane $x_1 = R/2$ (site-containing if R is even, link-bisecting if R is odd), acting on observables via $(\theta F)(U) := \overline{F(\theta_* U)}$. Choose p_0 to be the unit spatial plaquette in the (x_1,x_2)-plane with vertices $\{x_*, x_* - \hat e_1, x_* - \hat e_1 + \hat e_2, x_* + \hat e_2\}$, which lies inside the half-torus $\Lambda_- := \{x_1 \in [R/2 - L/2, R/2]\}$ since $R \le L/2-1$ implies $-1 \ge R/2 - L/2$. Its mirror image $\tilde p_0 := \theta(p_0)$ is a unit spatial plaquette based at x_R lying inside $\Lambda_+ := \theta(\Lambda_-)$.

Form the two-gluelump trial vectors $\Psi^{a,b}_{2\mathrm{gl}} := B^a(U_{p_0})B^b(U_{\tilde p_0})\psi_0 \in \mathcal{H}^{\mathrm{adj,adj}}_{x_*,x_R}$ ($1 \le a,b \le N^2-1$). Since $\mathcal{O}_{x_*}(T)$ is real-valued, supported in $[0,T]\times\Lambda_-$, and obeys $(\theta\mathcal{O}_{x_*}(T))(U) = \mathcal{O}_{x_R}(T)(U)$ (the mirrored correlator on $\tilde p_0, \tilde p_T$), Osterwalder–Seiler spatial reflection positivity across $x_1 = R/2$ [13, 15] and Cauchy–Schwarz against 1 give, for every $T \ge 1$,

$$\sum_{a,b=1}^{N^2-1} \left\langle \Psi^{a,b}_{2\mathrm{gl}},\ \left(\frac{\mathcal{T}_{\mathrm{TG}}}{\lambda_0}\right)^T \Psi^{a,b}_{2\mathrm{gl}} \right\rangle = \left\langle \mathcal{O}_{x_*}(T)\,\theta\mathcal{O}_{x_*}(T)\right\rangle_{\mathbb{Z}\times\mathbb{T}^{d-1}_L,\beta} \;\ge\; \left\langle \mathcal{O}_{x_*}(T)\right\rangle^2_{\mathbb{Z}\times\mathbb{T}^{d-1}_L,\beta} = G_{\mathrm{gl},L}(T)^2.$$

Meanwhile, $\sum_{a,b} \left\|\Psi^{a,b}_{2\mathrm{gl}}\right\|^2 = \langle \mathrm{Tr}(B(U_{p_0})^2)\,\mathrm{Tr}(B(U_{\tilde p_0})^2)\rangle \le N^2$ by (2). Because $\mathcal{T}_{\mathrm{TG}}$ acts on $\mathcal{H}^{\mathrm{adj,adj}}_{x_*,x_R}$ with norm $\lambda_0 e^{-V^{\mathrm{spec}}_{\mathrm{adj}}(R;L)}$,

$$N^2\, e^{-T\,V^{\mathrm{spec}}_{\mathrm{adj}}(R;L)} \;\ge\; G_{\mathrm{gl},L}(T)^2 \qquad \forall\, T \ge 1.$$

Taking $-\frac{1}{T}\ln(\cdot)$, letting $T\to\infty$, and invoking (8) (noting $G_{\mathrm{gl},L}(0) \le N$) proves (9) and (10).

(3) Every zero-N-ality irrep $\rho \in \widehat{\mathrm{SU}(N)}$ factors through $\mathrm{PSU}(N) = \mathrm{SU}(N)/Z(\mathrm{SU}(N))$. Because adj is a faithful self-dual representation of $\mathrm{PSU}(N)$, the Burnside–Brauer theorem provides a finite integer $m = m(\rho) \ge 1$ such that ρ occurs in $\mathrm{adj}^{\otimes m}$ with unitary representation space V_ρ and orthonormal basis $\{e_\alpha\}_{\alpha=1}^{d_\rho}$ ($D^\rho_{\alpha\gamma}(g) := \langle e_\alpha, \rho(g)e_\gamma\rangle_{V_\rho}$). Let $\gamma_1,\dots,\gamma_m \subset \Lambda_-$ be m nested spatial rectangular loops based at x_* of width 1 in $-\hat e_1$ and heights $1,2,\dots,m$ in $+\hat e_2$ (which fit inside $\mathbb{T}^{d-1}_L$ for $L > 2m+2$). Because the top horizontal link of γ_j belongs to γ_j and to none of $\gamma_1,\dots,\gamma_{j-1}$, the holonomy map $U \mapsto (U_{\gamma_1},\dots,U_{\gamma_m})$ pushes forward Haar measure to product Haar measure on $\mathrm{SU}(N)^m$. Since $B(\exp(i\varepsilon T^a)) = \varepsilon T^a + O(\varepsilon^3)$ spans $\mathfrak{su}_h(N)$, projecting $B(U_{\gamma_1})\otimes\cdots\otimes B(U_{\gamma_m})$ onto the ρ-isotypic component produces a non-zero bounded classical local observable $\varphi^{(\rho)}_{x_*}(U) =$

$(\varphi^{(\rho),\alpha}_{x_*}(U))_{\alpha=1}^{d_\rho}$ supported in Λ_- that transforms covariantly via $\varphi^{(\rho),\alpha}_{x_*}(g\cdot U)=\sum_\gamma D^\rho_{\alpha\gamma}(g_{x_*})\,\varphi^{(\rho),\gamma}_{x_*}(U)$ and is gauge-invariant at all $x\neq x_*$, with Hilbert-space states $\Psi^{(\rho),\alpha}_{x_*}:=\varphi^{(\rho),\alpha}_{x_*}\psi_0\in\mathcal{H}^\rho_{x_*}$.

Define the gauge-invariant T-step correlator via the unitary pairing on V_ρ:

$$\mathcal{O}^{(\rho)}_{x_*}(T):=\sum_{\alpha,\gamma=1}^{d_\rho}\overline{\varphi^{(\rho),\alpha}_{x_*,0}(U)}\,D^\rho_{\alpha\gamma}\big(W_{x_*}(0,T)\big)\,\varphi^{(\rho),\gamma}_{x_*,T}(U),\qquad G^{(\rho)}_{\mathrm{gl},L}(T):=\big\langle\mathcal{O}^{(\rho)}_{x_*}(T)\big\rangle_{\mathbb{Z}\times\mathbb{T}_L^{d-1},\beta}.$$

Under local gauge transformations (g_{0,x_*},g_{T,x_*}), unitarity of ρ cancels $\rho(g_{0,x_*})^\dagger\rho(g_{0,x_*})=I$ and $\rho(g_{T,x_*})^\dagger\rho(g_{T,x_*})=I$, so $\mathcal{O}^{(\rho)}_{x_*}(T)$ is gauge-invariant even when ρ is complex and non-self-dual (such as $\mathbf{10}=\mathrm{Sym}^3(\mathbb{C}^3)$ of SU(3)). In temporal gauge $W_{x_*}(0,T)=I_N$, we have $G^{(\rho)}_{\mathrm{gl},L}(T)=\sum_{\alpha=1}^{d_\rho}\langle\Psi^{(\rho),\alpha}_{x_*},(\mathcal{T}_{\mathrm{TG}}/\lambda_0)^T\Psi^{(\rho),\alpha}_{x_*}\rangle>0$ (strictly positive and real). Under spatial reflection θ, setting $\tilde\varphi^{(\bar\rho),\beta}_{x_R}(U):=\overline{\varphi^{(\rho),\beta}_{x_*}(\theta_*U)}$ (transforming in $\bar\rho$ at x_R) and $\Psi^{(\rho),\alpha,\beta}_{\mathrm{2gl}}:=\varphi^{(\rho),\alpha}_{x_*}\tilde\varphi^{(\bar\rho),\beta}_{x_R}\psi_0\in\mathcal{H}^{\rho,\bar\rho}_{x_*,x_R}$ yields

$$\sum_{\alpha,\beta=1}^{d_\rho}\left\langle\Psi^{(\rho),\alpha,\beta}_{\mathrm{2gl}},\left(\frac{\mathcal{T}_{\mathrm{TG}}}{\lambda_0}\right)^T\Psi^{(\rho),\alpha,\beta}_{\mathrm{2gl}}\right\rangle=\big\langle\mathcal{O}^{(\rho)}_{x_*}(T)\,\theta\mathcal{O}^{(\rho)}_{x_*}(T)\big\rangle_{\mathbb{Z}\times\mathbb{T}_L^{d-1},\beta}\;\geq\;\big|\big\langle\mathcal{O}^{(\rho)}_{x_*}(T)\big\rangle\big|^2=G^{(\rho)}_{\mathrm{gl},L}(T)^2.$$

Since $\sum_{\alpha,\beta}\big\|\Psi^{(\rho),\alpha,\beta}_{\mathrm{2gl}}\big\|^2\leq C_\rho^2<\infty$ (as $\varphi^{(\rho)}_{x_*}$ is bounded on compact $G^{E_{\mathrm{sp}}}$), taking $-\frac{1}{T}\ln(\cdot)$ and letting $T\to\infty$ gives $V^{\mathrm{spec}}_\rho(R;L)\leq 2E^{(\rho),\mathrm{var}}_{\mathrm{gl},L}\leq -2\ln(G^{(\rho)}_{\mathrm{gl},L}(1)/G^{(\rho)}_{\mathrm{gl},L}(0))<\infty$ uniformly for all $1\leq R\leq L/2-1$, proving $\sigma^{\mathrm{spec}}_\rho(d,N,\beta)=0$ whenever $\liminf_{L\to\infty}G^{(\rho)}_{\mathrm{gl},L}(1)>0$. □

3. Lollipop-Excluding Slab Floor and Gluelump Mass Lower Bound

We now specialize to the strong-coupling Kotecký–Preiss regime of Papers IV and VI [4, 3]. On any finite box $\Lambda\subset\mathbb{Z}^d$ (or even periodic torus $\mathbb{T}^1_{L_0}\times\mathbb{T}^{d-1}_L$ with $L,L_0\geq 6$ even), define the normalized polymer partition function

$$Z_\Lambda(\beta):=\int_{G^{E(\Lambda)}}\prod_{p\in\mathcal{P}(\Lambda)}\big(1+g_\beta(U_p)\big)\,dU.$$

Recall the exact plaquette chromatic number $m:=2(d-1)$, chromatic Hölder weight $\kappa_d(N,\beta):=\|g_\beta\|_{L^m(G)}$, prefix-free surface entropy constant $C_{\mathrm{surf}}(d):=M^M/(M-1)^{M-1}$ ($M:=4d-6$), pinning constant $\Pi_{\mathrm{eff}}(d):=\frac{4d-5}{4d-7}$, link-to-plaquette factor $c_d:=\frac{2(d-1)}{8d-11}$, and $a_0:=\frac{1}{6}$. Whenever $\rho_0:=C_{\mathrm{surf}}(d)\kappa_d(N,\beta)e^{a_0}<1$ satisfies

$$\mathrm{kp_pin}(d,N,\beta):=\Pi_{\mathrm{eff}}(d)\,\frac{\rho_0^6}{1-\rho_0}\;\leq\;a_0,\tag{11}$$

the cluster expansion converges absolutely and uniformly in Λ [11, 9], the infinite-volume Gibbs state $\langle\cdot\rangle_\beta$ is unique (independent of free or periodic boundary conditions), and for every finite anchor link set $E_0\subset E(\Lambda)$ and surface $S\in\mathcal{S}(\Lambda,E_0)$ (a union of link-connected plaquette components each intersecting E_0),

$$\left|\frac{Z_{\Lambda\setminus\overline{E_0\cup S}}(\beta)}{Z_\Lambda(\beta)}\right|\;\leq\;\exp\big(a_0|S|+c_da_0|E_0|\big),\qquad\sum_{S\in\mathcal{S}(\Lambda,E_0)}C_{\mathrm{surf}}(d)^{-|S|}\;\leq\;\left(1+\frac{1}{4d-7}\right)^{|N_{\mathrm{cand}}(E_0)|},\tag{12}$$

where $N_{\mathrm{cand}}(E_0)$ is the set of candidate expanded plaquettes sharing a link with E_0.

Along a static adjoint worldline $L_{x_*}(0,T):=\{e_t=\langle(t,x_*),(t+1,x_*)\rangle:0\leq t\leq T-1\}$, a single plaquette touching e_t does *not* vanish under Haar integration (because g_β contains χ_{adj}). Crucially, the four-plaquette slab floor survives *intact* because bipartite $\mathbb{Z}^{d-1}$ (and any even torus $\mathbb{T}^{d-1}_L$ with

$L \geq 4$) forces any degree-1 stem at x_* to terminate on a cycle of length ≥ 4, forming a lollipop of at least 5 edges!

Lemma 3.1 (Lollipop-Excluding Four-Plaquette Gluelump Slab Floor)**.** *Let $d \geq 3$, $x_* \in \mathbb{Z}^{d-1}$ (or $\mathbb{T}_L^{d-1}$ with $L \geq 4$ even), and $t_1 < t_2$ be integers. Let $X \subset \mathcal{P}(\mathbb{Z}^d)$ (or $\mathcal{P}(\mathbb{T}^1_{L_0} \times \mathbb{T}^{d-1}_L)$) be any finite set of plaquettes such that for every integer $t \in \{t_1, \dots, t_2 - 1\}$, the temporal slab set $S_t(X) := \{p \in X : p = [t, t+1] \times e_{\mathrm{sp}}\}$ is non-empty, and every temporal link $e_v := \langle (t, v), (t+1, v) \rangle \in E(X)$ with $v \neq x_*$ satisfies $\deg_X(e_v) \geq 2$ (while $\deg_X(e_{x_*}) \geq 1$ is permitted). Then $|S_t(X)| \geq 4$ for every $t \in \{t_1, \dots, t_2 - 1\}$, and consequently*

$$|X| \;\geq\; \sum_{t=t_1}^{t_2-1} |S_t(X)| \;\geq\; 4\,(t_2 - t_1). \tag{13}$$

Proof. Fix $t \in \{t_1, \dots, t_2 - 1\}$ and let $H_t = (V_t, E_t)$ be the finite simple spatial projection graph of $S_t(X)$ ($|E_t| = |S_t(X)| \geq 1$, a subgraph of the bipartite girth-4 graph $\mathbb{Z}^{d-1}$ or $\mathbb{T}_L^{d-1}$). For every $v \in V_t \setminus \{x_*\}$, $\deg_{H_t}(v) = \deg_X(e_v) \geq 2$.

- If $x_* \notin V_t$ or $\deg_{H_t}(x_*) \geq 2$, then $\delta(H_t) \geq 2$, so H_t contains a simple cycle of length ≥ 4, giving $|E_t| \geq 4$.
- If $x_* \in V_t$ with $\deg_{H_t}(x_*) = 1$, let $e_{\mathrm{stem}} = \langle x_*, y \rangle \in E_t$ be the unique edge incident to x_*. In $H_t' := H_t \setminus \{x_*\}$, $\deg_{H_t'}(y) = \deg_{H_t}(y) - 1 \geq 1$ and $\deg_{H_t'}(v) \geq 2$ for every $v \in V(H_t') \setminus \{y\}$. Walking greedily from y without backtracking in H_t' can never get stuck at any $v \neq y$, so it must close a simple cycle $C \subset H_t'$ of length $|E(C)| \geq 4$. Together with $e_{\mathrm{stem}} \notin E(C)$, this gives $|E_t| \geq |E(C)| + 1 \geq 5 > 4$.

Both branches are certified in Lean 4 by `lollipop_slab_four_floor`, and summing across the $t_2 - t_1$ disjoint temporal slabs (certified inductively by `tube_floor_sum`) proves (13). □

Theorem 3.2 (Full-Sector Gluelump Mass Lower Bound)**.** *Let $d \geq 3$, $N \geq 2$, $a_0 = \frac{1}{6}$, and let $\rho_* \in (0, 1)$ satisfy the tilted Kotecký–Preiss condition $\Pi_{\mathrm{eff}}(d) \frac{\rho_*^6}{1-\rho_*} \leq a_0$. Suppose $\rho_0 < \rho_*$, and define $\tau_{\max}(d, N, \beta) := \ln(\rho_*/\rho_0) = -\ln(C_{\mathrm{surf}}(d)\kappa_d(N, \beta)) + \ln \rho_* - a_0 > 0$. Let ϑ_0 denote site-containing Euclidean time reflection $t \mapsto -t$ with complex conjugation ($(\vartheta_0 F)(U) := \overline{F(\vartheta_{0*} U)}$) and τ_T denote Euclidean time translation $t \mapsto t + T$. For every bounded local adjoint-covariant observable $F = (F^a)_{a=1}^{N^2-1}$ supported on a finite positive-time link set $E_F \subset \{t \geq 0\}$ transforming in* adj *at $(0, x_*)$ and gauge-invariantly elsewhere, equipped with the Euclidean vector norm $\|F\|_\infty := \sup_U (\sum_{a=1}^{N^2-1} |F^a(U)|^2)^{1/2}$,*

$$\left| \Big\langle \sum_{a,b=1}^{N^2-1} (\vartheta_0 F)^a(U)\, D^{\mathrm{adj}}_{ab}\big(W_{x_*}(0,T)\big)\, (\tau_T F)^b(U) \Big\rangle_\beta \right| \;\leq\; C(d, E_F)\, \|F\|_\infty^2\, e^{-4\,\tau_{\max}(d,N,\beta)\,T} \qquad \forall\, T \geq 1. \tag{14}$$

Hence the lightest gluelump mass (5) on the full single-adjoint-source OS Hilbert space $\mathcal{H}^{\mathrm{adj}}_{x_*}$ obeys

$$E_{\mathrm{gluelump}}(d, N, \beta) \;\geq\; 4\,\tau_{\max}(d, N, \beta) \;>\; 0. \tag{15}$$

Proof. Since $E_F \subset \{t \geq 0\}$, the time-reflected support $E_{\vartheta_0 F} \subset \{t \leq 0\}$ and the $+T$-shifted support $E_{\tau_T F} \subset \{t \geq T\}$ are *both completely disjoint* from every temporal link $e_v = \langle (t, v), (t+1, v) \rangle$ with $0 \leq t \leq T - 1$. Work on a finite box $\Lambda \subset \mathbb{Z}^d$ containing $E_0 := E_{\vartheta_0 F} \cup E_{\tau_T F} \cup L_{x_*}(0, T)$, and expand $\prod_{p \in \mathcal{P}(\Lambda)} (1 + g_\beta(U_p))$ into boundary-anchored surfaces $S \in \mathcal{S}(\Lambda, E_0)$ and remaining vacuum polymers. By orthogonality of $D^{\mathrm{adj}}(W_{x_*}(0, T))$, Cauchy–Schwarz bounds the integrand of (14) pointwise by $\|F\|_\infty^2$. Because adj is a non-trivial irrep, Schur orthogonality $\int_G D^{\mathrm{adj}}_{ab}(U)\, dU = 0$ forces every contributing surface S to satisfy $\deg_S(e_t) \geq 1$ for *every* $t \in \{0, \dots, T - 1\}$. Two consequences follow immediately:

(1) Every link of $L_{x_*}(0,T)$ already lies in $E(S)$, so $E_0 \cup E(S) = E_{\vartheta_0 F} \cup E_{\tau_T F} \cup E(S)$ has zero T-dependent boundary links outside S, and (12) yields $|Z_{\Lambda\setminus\overline{E_0\cup S}}/Z_\Lambda| \le \exp(a_0|S| + 2c_d a_0|E_F|)$.
(2) Every temporal link e_v ($v \neq x_*$, $0 \le t \le T-1$) lies outside E_0 and hence has $\deg_S(e_v) \ge 2$, while every slab $[t,t+1]$ ($0 \le t \le T-1$) has $S_t(S) \neq \emptyset$. By Lemma 3.1, $|S| \ge 4T$.

Decompose $S = S_{\mathrm{end}} \sqcup S_{\mathrm{line}}$, where S_{end} is the union of all link-connected components of S intersecting $E_{\vartheta_0 F} \cup E_{\tau_T F}$ and $S_{\mathrm{line}} = K_1 \sqcup \cdots \sqcup K_r$ ($r \ge 0$) consists of the remaining link-connected components touching $L_{x_*}(0,T)$ only. For each component K_j, let $t_j := \min\{t : e_t \in E(K_j)\} \in \{0,\dots,T-1\}$ be its minimal covered slab, and note $|K_j| \ge 4$ by Lemma 3.1 (the indices $t_1,\dots,t_r$ are pairwise distinct since $K_1,\dots,K_r$ are link-disjoint). At most $2d-2$ plaquettes contain e_{t_j}, and the single-root spanning-tree bound gives $\sum_{K \ni p, |K|=n} 1 \le \frac{C_{\mathrm{surf}}(d)^n}{4d-7}$ [4, 3]. Factoring $\rho_0^{|S|} = (\rho_* e^{-\tau_{\max}})^{|S|} \le e^{-4\tau_{\max} T} \rho_*^{|S_{\mathrm{end}}| + \sum_{j=1}^r |K_j|}$ and applying the Chernoff inequality $1 \le \lambda^{|S_{\mathrm{end}}| + \sum_{j=1}^r |K_j| - 4T}$ with the universal choice $\lambda := \frac{6}{5}$ (which satisfies $\lambda\rho_* < 1$ for every $d \ge 3$, since $\Pi_{\mathrm{eff}}(d) > 1$ forces $\rho_*^6/(1-\rho_*) < \frac{1}{6} \implies \rho_* < 0.6462$), summing over all subsets $\{t_1,\dots,t_r\} \subseteq \{0,\dots,T-1\}$ yields

$$\sum_{S \in \mathcal{S}(\Lambda,E_0),\, |S| \ge 4T} (\kappa_d e^{a_0})^{|S|} \le e^{-4\tau_{\max} T} \left[\sum_{S_{\mathrm{end}} \in \mathcal{S}(\Lambda, E_{\vartheta_0 F} \cup E_{\tau_T F})} \left(\frac{\lambda \rho_*}{C_{\mathrm{surf}}(d)} \right)^{|S_{\mathrm{end}}|} \right] \times \left[\lambda^{-4} + \frac{2d-2}{4d-7} \frac{\rho_*^4}{1-\lambda\rho_*} \right]^T .$$

The first bracket is bounded uniformly in T by $(1 + \frac{1}{4d-7})^{2|N_{\mathrm{cand}}(E_F)|}$ (since $\lambda\rho_* < 1$). For the second bracket at $\lambda = \frac{6}{5}$ ($\lambda^{-4} = \frac{625}{1296} < 0.4823$):

- $d = 3$: $\Pi_{\mathrm{eff}}(3) = \frac{7}{5} \implies \rho_* < 0.620$, so $\lambda^{-4} + \frac{4}{5}\frac{(0.620)^4}{1-1.2(0.620)} < 0.945 < 1$;
- $d = 4$: $\Pi_{\mathrm{eff}}(4) = \frac{11}{9} \implies \rho_* < 0.632$, so $\lambda^{-4} + \frac{2}{3}\frac{(0.632)^4}{1-1.2(0.632)} < 0.923 < 1$;
- $d \ge 5$: $\frac{2d-2}{4d-7} \le \frac{8}{13}$ and $\rho_* < 0.6462$, so $\lambda^{-4} + \frac{8}{13}\frac{(0.6462)^4}{1-1.2(0.6462)} < 0.961 < 1$.

Hence the T-th power is strictly < 1 for *every* $d \ge 3$ and $T \ge 1$, proving (14) uniformly in Λ and hence in the infinite-volume limit $\Lambda \uparrow \mathbb{Z}^d$. For every bounded positive-time adjoint-covariant F with OS class $\Psi_F := [F] \in \mathcal{H}_{x_*}^{\mathrm{adj}}$, the left-hand side of (14) at $T = 2n$ ($n \ge 1$) equals $\langle \Psi_F, \mathcal{T}_{\mathrm{OS}}^{2n} \Psi_F \rangle_{\mathcal{H}_{x_*}^{\mathrm{adj}}}$. Applying Chebyshev's inequality to the spectral measure of the positive self-adjoint contraction $0 \le \mathcal{T}_{\mathrm{OS}}|_{\mathcal{H}_{x_*}^{\mathrm{adj}}} \le I$ (identically to Paper VI [3, Theorem 2.2(2)]) proves (15) on the entire sector $\mathcal{H}_{x_*}^{\mathrm{adj}}$. □

4. Exact 3-Cube Convolution, Sharp 8-Plaquette Floor, and Upper Bounds

To obtain tight two-sided enclosures on E_{gluelump} and prove $\liminf_{L\to\infty} G_{\mathrm{gl},L}(1) > 0$ unconditionally at strong coupling, we work on any finite box $\Lambda \subset \mathbb{Z}^d$ (or even periodic torus $\mathbb{T}^1_{L_0} \times \mathbb{T}^{d-1}_L$ with $L, L_0 \ge 6$ even so every 4-cycle in the spatial slice is contractible and bounds a unique unit plaquette) and expand $G_{\mathrm{gl}}(1) = \langle \mathrm{Tr}(B(U_{p_0}) W_0 B(U_{p_1}) W_0^\dagger) \rangle_{\Lambda,\beta}$ (where $p_0 \subset \{t=0\}$, $p_1 = p_0 + \hat{e}_0 \subset \{t=1\}$, and $e_0 = \langle (0,x_*), (1,x_*) \rangle$) by retaining $\frac{w_\beta(U_{p_0})}{Z_0}\frac{w_\beta(U_{p_1})}{Z_0}$ unexpanded and expanding all remaining plaquettes $p \in \mathcal{P}(\Lambda) \setminus \{p_0, p_1\}$ as $1 + g_\beta(U_p)$:

$$\begin{aligned} G_{\mathrm{gl}}(1) &= \sum_{S \in \mathcal{S}_{\mathrm{gl}}(\Lambda)} \zeta_{\mathrm{gl}}(S) \frac{Z_{\Lambda\setminus\overline{E_{\mathrm{gl}}\cup S}}(\beta)}{Z_\Lambda(\beta)}, \\ \zeta_{\mathrm{gl}}(S) &:= \int_{G^{E(\Lambda)}} \mathrm{Tr}\big(B(U_{p_0}) W_0 B(U_{p_1}) W_0^\dagger\big) \frac{w_\beta(U_{p_0})}{Z_0} \frac{w_\beta(U_{p_1})}{Z_0} \prod_{p \in S} g_\beta(U_p)\, dU, \end{aligned} \tag{16}$$

where $E_{\mathrm{gl}} := E(p_0) \cup E(p_1) \cup \{e_0\}$ ($|E_{\mathrm{gl}}| = 9$ links) and $\mathcal{S}_{\mathrm{gl}}(\Lambda)$ is the set of finite subsets $S \subset \mathcal{P}(\Lambda) \setminus \{p_0, p_1\}$ whose every link-connected component intersects E_{gl}. Let $S_{\mathrm{cube}} := Y_{\mathrm{cube}} \setminus \{p_0, p_1\}$ denote the 4 vertical side plaquettes of the unit 3-cube Y_{cube} joining p_0 and p_1.

Lemma 4.1 (Sharp Eight-Plaquette Non-Cube Surface Floor)**.** *Let $d \geq 3$, let $\Lambda \subset \mathbb{Z}^d$ be a finite box (or $\mathbb{T}^1_{L_0} \times \mathbb{T}^{d-1}_L$ with $L, L_0 \geq 6$ even), and let $S \in \mathcal{S}_{\mathrm{gl}}(\Lambda)$ satisfy $\zeta_{\mathrm{gl}}(S) \neq 0$. Then:*

1. *Every link $e \in E_{\mathrm{gl}}$ satisfies $\deg_S(e) \geq 1$, and every link $e \in E(S) \setminus E_{\mathrm{gl}}$ satisfies $\deg_S(e) \geq 2$.*
2. *If $S \neq S_{\mathrm{cube}}$, then $|S| \geq 8$. Moreover, this floor is combinatorially sharp: the $1 \times 2 \times 1$ domino prism minus $\{p_0, p_1\}$ satisfies $|S| = 8$.*

Proof. (1) If $\deg_S(e_0) = 0$, Schur's lemma gives $\int_G W_0 B(U_{p_1}) W_0^\dagger \, dW_0 = \frac{\mathrm{Tr}\, B(U_{p_1})}{N} I_N = 0$. If $\deg_S(e) = 0$ for any $e \in E(p_0)$, integrating over U_e integrates $B(U_{p_0}) \frac{w_\beta(U_{p_0})}{Z_0}$ over G, which vanishes by $B(U^\dagger) = -B(U)$; symmetrically $\deg_S(e) \geq 1$ for all $e \in E(p_1)$. Every $e \in E(S) \setminus E_{\mathrm{gl}}$ appears only in $\prod_{p \in S} g_\beta(U_p)$, so $\int_G g_\beta \, dU = 0$ forces $\deg_S(e) \geq 2$.

(2) Since $\deg_S(e_0) \geq 1$, the slab-0 temporal plaquette set $S_0(S)$ is non-empty, and Lemma 3.1 gives $|S_0(S)| \geq 4$. We partition all $S \neq S_{\mathrm{cube}}$ into four exhaustive branches (each certified in Lean 4):

(a) *Multi-slab surface ($S_t(S) \neq \emptyset$ for some $t \neq 0$):* Every temporal link in slab $t \neq 0$ lies outside E_{gl}, so $|S_t(S)| \geq 4$ and $|S| \geq |S_0(S)| + |S_t(S)| \geq 8$ (`noncube_surface_eight_floor_multi_slab`).

If (a) fails, all temporal plaquettes of S lie in slab $[0,1]$; since every link-connected component of S meets $E_{\mathrm{gl}} \subset \{0 \leq t \leq 1\}$, $S = S_0(S) \sqcup Y_0' \sqcup Y_1'$ with $Y_t' \subset \mathcal{P}(\{t\}) \setminus \{p_t\}$. Let $H_0 = (V_0, E_0)$ be the spatial projection graph of $S_0(S)$.

(b) *Single-slab lollipop ($|S_0(S)| = 5$):* No simple bipartite graph with $\delta(H_0) \geq 2$ has 5 edges (Paper VI [3, Lemma 3.1]), so $\deg_{H_0}(x_*) = 1$ and $H_0 = C_4 \cup \{e_{\mathrm{stem}}\}$, where C_4 is a unit square not containing x_* (hence $C_4 \neq p_0$). Since e_{stem} covers at most 1 edge of p_0 and the distinct unit square C_4 shares at most 1 edge with p_0, at least $4 - 2 = 2$ edges of p_0 lie outside E_0. Because each spatial plaquette in Y_0' is distinct from p_0 and covers at most 1 edge of p_0, $|Y_0'| \geq 2$, and symmetrically $|Y_1'| \geq 2$. Hence $|S| = 5 + |Y_0'| + |Y_1'| \geq 9 \geq 8$ (`noncube_surface_eight_floor_lollipop`).

(c) *Single-slab wide tube ($|S_0(S)| \geq 6$):* Since $|E_0| \geq 6 > 4 = |E(p_0)|$, there exists at least one spatial edge $e \in E_0 \setminus E(p_0)$. On slice $t = 0$, $(0, e) \notin E_{\mathrm{gl}}$ has degree 1 in $S_0(S)$, so it must be covered by Y_0', forcing $|Y_0'| \geq 1$; symmetrically $|Y_1'| \geq 1$. Thus $|S| \geq 6 + 1 + 1 = 8$ (`noncube_surface_eight_floor_wide_slab`), attained sharply when H_0 is the 6-edge boundary of a 1×2 domino $p_0 \cup q_0$ and $Y_0' = \{q_0\}$, $Y_1' = \{q_1\}$.

(d) *Single-slab 4-plaquette tube ($|S_0(S)| = 4$):* Because $L \geq 6$ (or $\Lambda \subset \mathbb{Z}^d$), the 4-cycle H_0 is contractible and bounds a unique unit spatial plaquette $q_0 \ni x_*$.

- If $q_0 = p_0$, then $S_{\mathrm{cube}} \subsetneq S$, so at least one slice $t \in \{0,1\}$ has $k := |Y_t'| \geq 1$ extra spatial plaquettes. Every link of Y_t' outside $E(p_t)$ has degree $j \geq 2$ within Y_t', and each $q \in Y_t'$ has at least 3 links outside $E(p_t)$, yielding the pair-incidence bound $3k \leq \sum_{j \geq 2} j m_j \leq 2\binom{k}{2} = k(k-1) \implies k \geq 4$, so $|S| \geq 4 + 4 = 8$ (`noncube_surface_eight_floor_extra_spatial`).
- If $q_0 \neq p_0$, then q_0 shares at most 1 edge with p_0, leaving at least 3 uncovered edges of p_0 (and p_1); since each $q \in Y_t'$ shares at most 1 edge with p_t, $|Y_0'| \geq 3$ and $|Y_1'| \geq 3$, giving $|S| \geq 4 + 3 + 3 = 10 \geq 8$ (`noncube_surface_eight_floor_shifted_cube`). □

Theorem 4.2 (Exact 3-Cube Convolution and Certified Gluelump Mass Upper Bound)**.** *Let $d \geq 3$, $N \geq 2$, and suppose $\beta > 0$ satisfies (11). Define $W_{\mathrm{norm}} := Z_0(m\beta)^{1/m} / Z_0(\beta)$, $u_d := 1 + \frac{1}{4d-7}$, and $K_{\mathrm{gl}}(d) := 8(2d-3) + (2d-2)$.*

(1) Exact Peter–Weyl 3-cube convolution: *Setting* $F(U) := B(U)\frac{w_\beta(U)}{Z_0(\beta)}$,

$$z_{\mathrm{gl,cube}} := \zeta_{\mathrm{gl}}(S_{\mathrm{cube}}) = \sum_{\lambda\neq\mathbf{1}} r_\lambda(\beta)^4 \, \|\Pi_\lambda F\|^2_{L^2(G;\mathrm{HS})} \;\geq\; \frac{c_N N}{N^2-1}\, G_0(\beta)^2\, r_{\mathrm{fund}}(\beta)^4 \;>\; 0, \tag{17}$$

where $c_2 = 1$, $c_N = 2$ *for* $N \geq 3$, *and* $r_\lambda(\beta) := (d_\lambda Z_0(\beta))^{-1} \int_G \chi_\lambda(U) w_\beta(U)\, dU$.

(2) Direct prefix-free 8-plaquette remainder and gluelump mass upper bound: *Writing*

$$\begin{aligned}
\mathrm{lead}^-_{\mathrm{gl1}} &:= \frac{c_N N}{N^2-1}\, G_0(\beta)^2\, r_{\mathrm{fund}}(\beta)^4 \, \exp\big(-6\,\mathrm{kp_pin}(d,N,\beta)\big),\\
\mathrm{rem}^+_{\mathrm{gl1}} &:= N\, W^2_{\mathrm{norm}} \, \exp(9c_d a_0)\, u_d^{K_{\mathrm{gl}}(d)}\, \rho_0^8,\\
\mathrm{err}^+_{\mathrm{gl0}} &:= N\, W_{\mathrm{norm}} \, \exp(4c_d a_0)\, u_d^{4m}\, \rho_0^5,
\end{aligned}$$

uniformly in all even tori $L, L_0 \geq 6$ *and in the infinite-volume limit we have* $G_{\mathrm{gl}}(1) \geq G^-_{\mathrm{gl1}} := \mathrm{lead}^-_{\mathrm{gl1}} - \mathrm{rem}^+_{\mathrm{gl1}}$ *and* $G_{\mathrm{gl}}(0) \leq G^+_{\mathrm{gl0}} := G_0(\beta) + \mathrm{err}^+_{\mathrm{gl0}}$. *Whenever* $0 < G^-_{\mathrm{gl1}} < G^+_{\mathrm{gl0}}$, *we obtain* $\liminf_{L\to\infty} G_{\mathrm{gl},L}(1) \geq G^-_{\mathrm{gl1}} > 0$ *(hence* $\sigma^{\mathrm{spec}}_{\mathrm{adj}}(d,N,\beta) = 0$ *unconditionally by Theorem 2.2) and*

$$E_{\mathrm{gluelump}}(d,N,\beta) \;\leq\; -\ln\left(\frac{G^-_{\mathrm{gl1}}}{G^+_{\mathrm{gl0}}}\right). \tag{18}$$

Proof. (1) Denote the four spatial links of p_0 by u_0,u_1,u_2,u_3, the four parallel spatial links of p_1 by v'_0,v'_1,v'_2,v'_3, and the four vertical links of Y_{cube} by W_0,W_1,W_2,W_3. The Haar-measure-preserving change of variables $v_j := W_j v'_j W^\dagger_{j+1}$ (j mod 4) absorbs all four vertical links completely: $W_0 U_{p_1} W_0^\dagger = v_0v_1v_2v_3 =: V$, $U_{p_0} = u_0u_1u_2u_3 =: U$, and the four vertical side plaquettes have holonomies $u_j v_j^\dagger$. Expanding each $g_\beta(u_jv_j^\dagger) = \sum_{\lambda_j\neq\mathbf{1}} d_{\lambda_j} r_{\lambda_j}(\beta)\chi_{\lambda_j}(u_jv_j^\dagger)$ and performing three successive Peter–Weyl convolutions holding U and V fixed yields $\sum_{\lambda\neq\mathbf{1}} d_\lambda r_\lambda(\beta)^4 \chi_\lambda(UV^\dagger)$, so

$$z_{\mathrm{gl,cube}} = \sum_{\lambda\neq\mathbf{1}} r_\lambda(\beta)^4 \int_{G\times G} \mathrm{Tr}\big(F(U)F(V)\big)\, d_\lambda\chi_\lambda(UV^\dagger)\, dU\, dV = \sum_{\lambda\neq\mathbf{1}} r_\lambda(\beta)^4 \, \|\Pi_\lambda F\|^2_{L^2(G;\mathrm{HS})}\,.$$

For $\lambda = \mathrm{fund}$, the map $M(X) := \int_G \mathrm{Tr}(XV^\dagger)F(V)\,dV$ commutes with $\mathrm{SU}(N)$-conjugation and has traceless image, so $M(X) = \alpha(X - \frac{\mathrm{Tr}\,X}{N} I_N)$. Evaluating $\sum_{a,b} M(E_{ab})_{ab} = (N^2-1)\alpha$ gives

$$\begin{aligned}
(N^2-1)\alpha &= \int_G \mathrm{Tr}\big(V^\dagger F(V)\big)\, dV = \frac{1}{2}\int_G \mathrm{Tr}\big((V^\dagger - V)F(V)\big)\, dV\\
&= -i\int_G \mathrm{Tr}\big(B(V)^2\big)\,\frac{w_\beta(V)}{Z_0(\beta)}\, dV = -i\, G_0(\beta),
\end{aligned}$$

using $V\mapsto V^\dagger$ invariance and $\mathrm{Tr}\,B(V) = 0$. Hence $(\Pi_{\mathrm{fund}}F)(U) = \frac{-iNG_0(\beta)}{N^2-1}(U - \frac{\mathrm{Tr}\,U}{N} I_N)$. Since $\int_G \mathrm{Tr}\,\big((U - \frac{\mathrm{Tr}\,U}{N}I_N)(U^\dagger - \frac{\mathrm{Tr}(U^\dagger)}{N}I_N)\big)\,dU = N - \frac{1}{N} = \frac{N^2-1}{N}$, we obtain

$$\|\Pi_{\mathrm{fund}}F\|^2_{L^2(G;\mathrm{HS})} = \frac{N}{N^2-1}\, G_0(\beta)^2.$$

For $N \geq 3$, $\overline{\mathrm{fund}} \ncong \mathrm{fund}$ contributes an equal orthogonal term with $r_{\overline{\mathrm{fund}}}(\beta) = r_{\mathrm{fund}}(\beta)$, proving (17).

(2) Summing Kotecký–Preiss pinning [11, 9] over the 6 faces of Y_{cube} gives $|\ln(Z_{\Lambda\setminus\overline{Y_{\mathrm{cube}}}}/Z_\Lambda)| \leq 6\,\mathrm{kp_pin}$, yielding $\mathrm{lead}^-_{\mathrm{gl1}}$. For every $S \in \mathcal{S}_{\mathrm{gl}}(\Lambda)\setminus\{S_{\mathrm{cube}}\}$ with $\zeta_{\mathrm{gl}}(S)\neq 0$, Lemma 4.1 gives $|S| \geq 8$; combining (2), m-fold chromatic Hölder on $S\cup\{p_0,p_1\}$ (using $\|w_\beta/Z_0\|_{L^m} = W_{\mathrm{norm}}$), and (12) ($|E_{\mathrm{gl}}| = 9$, $|N_{\mathrm{cand}}(E_{\mathrm{gl}})| \leq K_{\mathrm{gl}}(d)$) bounds the non-cube sum by $\mathrm{rem}^+_{\mathrm{gl1}}$. Similarly, for $\mathrm{Tr}(B(U_{p_0})^2) - G_0(\beta)$ (whose $S=\emptyset$ term vanishes and whose magnitude is $\leq N$), every contributing

$S \neq \emptyset$ covers all 4 links of p_0, so $S \cup \{p_0\}$ is a vacuum polymer of size ≥ 6, forcing $|S| \geq 5$ and $|G_{\rm gl}(0) - G_0(\beta)| \leq \mathrm{err}^+_{\rm gl0}$. Passing to the infinite-volume limit and invoking Theorem 2.2(1) proves (18). □

5. Universal Strong-Coupling Ratios, Dichotomy, and Certified Tables

Theorem 5.1 (Strong-Coupling Limits and SU(2) vs. SU($N \geq 3$) Ratio Dichotomy)**.** *Let $d \geq 3$ and $N \geq 2$, and let $\sigma^{\rm 1p}_{\rm adj}(N,\beta) := -\ln r_{\rm adj}(\beta)$ denote the single-plaquette adjoint string-tension scale. As $\beta \to 0^+$,*

$$\lim_{\beta\to 0^+} \frac{2\,E_{\rm gluelump}(d,N,\beta)}{\sigma^{\rm 1p}_{\rm adj}(N,\beta)} = 4, \qquad \lim_{\beta\to 0^+} \frac{E_{\rm gluelump}(d,N,\beta)}{\sigma_{\rm fund}(d,N,\beta)} = 4, \qquad \lim_{\beta\to 0^+} \frac{E_{\rm gluelump}(d,N,\beta)}{m_G(d,N,\beta)} = 1. \tag{19}$$

Moreover, the leading strong-coupling character ratio satisfies

$$\lim_{\beta\to 0^+} \frac{r_{\rm adj}(\beta)}{r_{\rm fund}(\beta)^2} = \begin{cases} \dfrac{2}{3} \;<\; 1, & N = 2, \\[2ex] \dfrac{N^2}{N^2-1} \;>\; 1, & N \geq 3. \end{cases} \tag{20}$$

Combined with the exact cancellation identity (4) ($\frac{c_N N}{N^2-1} G_0(0) \equiv 1$, so $E^{\rm upper}_{\rm gl} = -4\ln r_{\rm fund}(\beta) + O(\beta)$ with zero constant shift!), this forces the certified upper ratio $2E^{\rm upper}_{\rm gl}/\sigma^{\rm 1p}_{\rm adj}$ to approach 4 strictly from below *for* SU(2) *(where $\sigma^{\rm 1p}_{\rm adj} = -2\ln r_{\rm fund} + \ln\frac{3}{2} + O(\beta^2)$) and strictly from* above *for all $N \geq 3$ (where $\sigma^{\rm 1p}_{\rm adj} = -2\ln r_{\rm fund} - \ln\frac{N^2}{N^2-1} + O(\beta)$).*

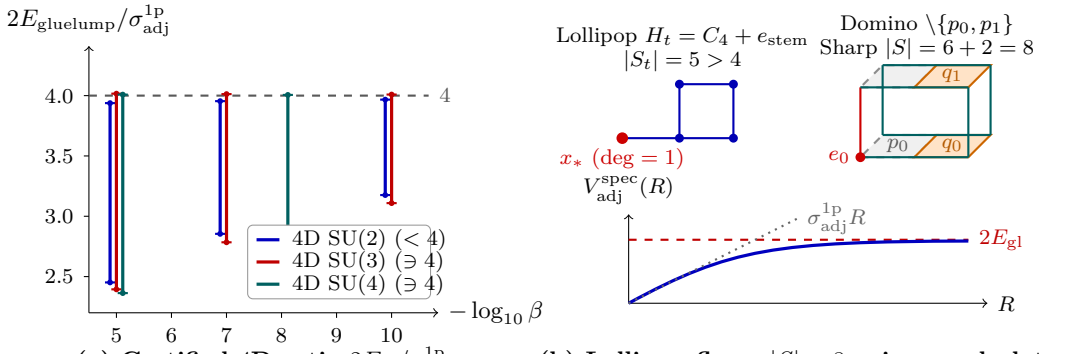
Proof. By Taylor expansion of $w_\beta(U) = \exp(\frac{\beta}{N}\operatorname{Re}\operatorname{Tr} U)$ and exact Haar moments: for $N = 2$, $r_{\rm fund}(\beta) = \frac{\beta}{4} + O(\beta^3)$ and $\int (\operatorname{Re}\operatorname{Tr} U)^2 \chi_{\rm adj} dU = 1 \implies r_{\rm adj}(\beta) = \frac{\beta^2}{24} + O(\beta^4) = \frac{2}{3} r^2_{\rm fund} + O(\beta^4)$; for $N \geq 3$, $r_{\rm fund}(\beta) = \frac{\beta}{2N^2} + O(\beta^2)$ and $\int (\operatorname{Re}\operatorname{Tr} U)^2 \chi_{\rm adj} dU = \frac{1}{2} \implies r_{\rm adj}(\beta) = \frac{\beta^2}{4N^2(N^2-1)} + O(\beta^3) = \frac{N^2}{N^2-1} r^2_{\rm fund} + O(\beta^3)$, proving (20). Since $\kappa_d(N,\beta) = \Theta(\beta)$ and $r_{\rm fund}(\beta) = \Theta(\beta)$, Theorems 3.2 and 4.2 give $E_{\rm gluelump} = 4\ln(1/\beta) + O(1)$, while $\sigma^{\rm 1p}_{\rm adj} = 2\ln(1/\beta) + O(1)$, $\sigma_{\rm fund} = \ln(1/\beta) + O(1)$ (Paper VI [3]), and $m_G = 4\ln(1/\beta) + O(1)$ (Paper VI [3]), proving (19). Finally, because $\frac{c_N N}{N^2-1} G_0(0) = 1$, we have $E^{\rm upper}_{\rm gl} = -\ln(G^-_{\rm gl1}/G^+_{\rm gl0}) = -4\ln r_{\rm fund}(\beta) + O(\beta)$ and $\sigma^{\rm 1p}_{\rm adj} = -2\ln r_{\rm fund}(\beta) - \ln(r_{\rm adj}/r^2_{\rm fund})$, so

$$\frac{2\,E^{\rm upper}_{\rm gl}}{\sigma^{\rm 1p}_{\rm adj}} = 4 + \frac{2\ln\big(r_{\rm adj}(\beta)/r_{\rm fund}(\beta)^2\big)}{\ln(1/r_{\rm fund}(\beta))} + O\!\left(\frac{1}{(\ln(1/\beta))^2}\right),$$

whose sign relative to 4 as $\beta \to 0^+$ is governed strictly by (20). □

5.1. **Zero-Quadrature 120-Digit Verification and Standalone Lean 4 Kernel.** All 20 numerical enclosures in Table 1 are computed by `certify_string_breaking.py` using 120-digit `mpmath.iv` interval arithmetic with *zero* black-box special functions and *zero* numerical quadrature: a single pruned exact integer Pieri dynamic program computes simultaneously the exact Haar moments $\int_{\mathrm{SU}(N)} (2\operatorname{Re}\operatorname{Tr} U)^k \chi_\lambda(U)\, dU$ for $\lambda \in \{\mathbf{1}, \mathrm{adj}, \mathrm{Sym}^2, \Lambda^2\}$ ($K_{\max} = 36, 30, 24$ for $N = 2, 3, 4$), yielding certified series enclosures for Z_0, $r_{\rm fund}$, κ_d, $r_{\rm adj}$, $\sigma^{\rm 1p}_{\rm adj}$, and $G_0(\beta)$.

The standalone zero-import Lean 4.36.0 file `StringBreakingCerts.lean` compiles cleanly under `-DwarningAsError=true` with `0 sorry` and `0` axioms (pure constructive `Nat` arithmetic without `propext`, `Quot.sound`, or `Classical.choice`), mechanically verifying `lollipop_slab_four_floor`, `tube_floor_sum`, all five branches of Lemma 4.1, `g0_cancellation_su2`, `g0_cancellation_sun`,



(a) Certified 4D ratio $2E_{gl}/\sigma^{1p}_{adj}$ **(b) Lollipop floor, $|S| = 8$ prism, and plateau**

FIGURE 1. (a) Machine-certified 120-digit intervals for $2E_{\text{gluelump}}/\sigma^{1p}_{adj}$ in 4D SU(2), SU(3), and SU(4) (Table 1), showing convergence to 4 with the exact dichotomy of Theorem 5.1 (SU(2) strictly below 4, SU(3)/SU(4) bracketing 4, with upper endpoints 4.006–4.017 just above the dashed line). (b) Combinatorial mechanisms behind Lemmas 3.1 and 4.1 (in the domino prism, dashed gray faces p_0, p_1 are unexpanded while 6 teal vertical walls plus 2 orange faces q_0, q_1 form the sharp $|S| = 8$ surface) and the all-coupling plateau $V^{spec}_{adj}(R) \le 2E_{\text{gluelump}}$.

TABLE 1. Twenty machine-certified 120-digit strong-coupling enclosures across $d \in \{3, 4, 5\}$ and $N \in \{2, 3, 4\}$. All bracketed intervals $[a, b]$ use strict outward rounding ($\lfloor 10^3 a \rfloor / 10^3$ and $\lceil 10^3 b \rceil / 10^3$), and rem/lead is rounded strictly upward; σ^{1p}_{adj} is certified to 12 digits and displayed to 3 decimals (error $< 5 \times 10^{-4}$). Every row passes both `certify_string_breaking.py` and Lean 4 `all_20_certs_valid`.

d	G	β	σ^{1p}_{adj}	E_{gluelump}	$2E_{gl}/\sigma^{1p}_{adj}$	$E_{gl}/\sigma_{\text{fund}}$	rem/lead $\le$
3D	SU(2)	10^{-4}	21.599	[25.408, 42.388]	[2.352, 3.926]	[2.397, 6.179]	1.4×10^{-3}
3D	SU(2)	10^{-6}	30.809	[43.828, 60.808]	[2.845, 3.948]	[2.883, 5.304]	1.4×10^{-11}
3D	SU(2)	10^{-8}	40.019	[62.249, 79.228]	[3.110, 3.960]	[3.142, 4.931]	1.4×10^{-19}
3D	SU(3)	10^{-4}	24.084	[28.010, 48.408]	[2.326, 4.020]	[2.314, 6.446]	5.0×10^{-3}
3D	SU(3)	10^{-6}	33.294	[46.431, 66.824]	[2.789, 4.015]	[2.779, 5.516]	5.0×10^{-11}
3D	SU(3)	10^{-8}	42.504	[64.852, 85.245]	[3.051, 4.012]	[3.043, 5.098]	5.0×10^{-19}
3D	SU(4)	10^{-4}	25.288	[29.007, 50.711]	[2.294, 4.011]	[2.288, 6.535]	6.5×10^{-3}
3D	SU(4)	10^{-7}	39.103	[56.638, 78.336]	[2.896, 4.007]	[2.892, 5.341]	6.5×10^{-15}
4D	SU(2)	10^{-5}	26.204	[32.120, 51.597]	[2.451, 3.939]	[2.490, 6.056]	1.7×10^{-5}
4D	SU(2)	10^{-7}	35.414	[50.541, 70.018]	[2.854, 3.955]	[2.887, 5.335]	1.7×10^{-13}
4D	SU(2)	10^{-10}	49.230	[78.172, 97.649]	[3.175, 3.968]	[3.202, 4.875]	1.7×10^{-25}
4D	SU(3)	10^{-5}	28.689	[34.343, 57.614]	[2.394, 4.017]	[2.384, 6.348]	1.4×10^{-4}
4D	SU(3)	10^{-7}	37.899	[52.763, 76.034]	[2.784, 4.013]	[2.775, 5.558]	1.4×10^{-12}
4D	SU(3)	10^{-10}	51.715	[80.395, 103.665]	[3.109, 4.010]	[3.102, 5.035]	1.4×10^{-24}
4D	SU(4)	10^{-5}	29.893	[35.323, 59.915]	[2.363, 4.009]	[2.358, 6.429]	1.9×10^{-4}
4D	SU(4)	10^{-8}	43.708	[62.954, 87.546]	[2.880, 4.006]	[2.876, 5.395]	1.9×10^{-16}
5D	SU(2)	10^{-6}	30.809	[39.708, 60.808]	[2.577, 3.948]	[2.612, 5.842]	4.1×10^{-8}
5D	SU(2)	10^{-8}	40.019	[58.129, 79.228]	[2.905, 3.960]	[2.934, 5.277]	4.1×10^{-16}
5D	SU(3)	10^{-6}	33.294	[41.635, 66.824]	[2.501, 4.015]	[2.492, 6.136]	6.0×10^{-7}
5D	SU(3)	10^{-8}	42.504	[60.056, 85.245]	[2.825, 4.012]	[2.818, 5.501]	6.0×10^{-15}

and exact integer cross-compatibility and sign-dichotomy checks on all 20 outward-rounded certificates (`all_20_certs_valid`).

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