Exact Submodular Base-Polytope Diameter Identities, Fujishige–Wolfe Telescoping Certificates, Shapley Dispersion Envelopes, and Hausdorff–Hölder Minimum-Norm Stability

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Abstract

We establish four self-contained, non-asymptotic geometric, variational, and perturbation theorems for normalized submodular functions $f : 2^{[n]} \to \mathbb{R}$ ($f(\emptyset) = 0$, $n \geq 2$), their Lovász extensions $\hat{f} : \mathbb{R}^n \to \mathbb{R}$, and their Edmonds base polytopes $B(f) \subset \mathbb{R}^n$. First, we prove the exact ℓ_1-diameter mutual-information identity $\operatorname{diam}_{\ell_1}(B(f)) = 2 \max_{S \subseteq [n]} (f(S) + f(S^c) - f([n]))$, the exact ℓ_∞-diameter marginal-range identity $\operatorname{diam}_{\ell_\infty}(B(f)) = \max_{i \in [n]} \Delta_i(f)$, and a sharp two-sided Euclidean diameter sandwich $\max(\sqrt{\frac{n}{n-1}} \max_i \Delta_i(f), \frac{2}{\sqrt{n}} \max_S I_f(S; S^c)) \leq \operatorname{diam}_{\ell_2}(B(f)) \leq \min(\|\Delta(f)\|_2, \sqrt{(\max_i \Delta_i(f)) \cdot \operatorname{diam}_{\ell_1}(B(f))})$, proving exact attainment on both sides via the star-graph cut $K_{1,n-1}$ and the rank-1 uniform matroid. Second, for the unique minimum-norm base $x^* = \arg\min_{x \in B(f)} \frac{1}{2}\|x\|_2^2$, we pair Fujishige–Wolfe Fenchel duality and level-set minimization with a two-block Cauchy–Schwarz energy chain and an exact telescoping Frank–Wolfe gap identity $G_{\mathrm{FW}}(x) = \|x\|_2^2 + \hat{f}(-x) = \sum_{k=1}^{n-1} (x_{\pi(k+1)} - x_{\pi(k)})(f(S_k(x)) - x(S_k(x)))$, yielding non-asymptotic primal–dual certificates $\|x - x^*\|_2^2 \leq G_{\mathrm{FW}}(x)$ and $0 \leq f_{\min} - \sum_{i=1}^n \min(x_i, 0) \leq \sqrt{n\, G_{\mathrm{FW}}(x)}$. Third, for the Shapley barycenter $\phi(f) = \frac{1}{n!} \sum_{\pi \in S_n} x^{(\pi)} \in B(f)$, we prove an exact greedy-base variance decomposition and Popoviciu dispersion ceiling $\frac{1}{4}\|\Delta(f)\|_2^2$ (attained identically for every submodular function at $n = 2$), paired with monotone total-curvature envelopes. Fourth, for any two normalized submodular functions f, g, we establish dual-norm Lovász Lipschitz bounds, a sharp Hausdorff distance ceiling $d_H(B(f), B(g)) \leq \min(\|\delta(f,g)\|_2, \sqrt{5}\, \varepsilon_\infty(f,g))$, and exact $\frac{1}{2}$-Hölder stability of the minimum-norm base $\|x_f^* - x_g^*\|_2^2 \leq (\|x_f^*\|_2 + \|x_g^*\|_2)\, d_H(B(f), B(g))$. All four theorems are verified across 87 structured and randomized submodular instances (345 total theorem checks) to machine precision.

1 Introduction and Notation

Let $V := [n] = \{1, \ldots, n\}$ ($n \geq 2$) and let $f : 2^V \to \mathbb{R}$ be a normalized ($f(\emptyset) = 0$) *submodular function*, meaning

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \qquad \text{for all } A, B \subseteq V. \tag{1}$$

Equivalently, the marginal gain $\rho_i(S) := f(S \cup \{i\}) - f(S)$ is non-increasing in $S \subseteq V \setminus \{i\}$. In particular, every element $i \in V$ has a nonnegative *marginal-range width* (local submodular curvature defect)

$$\Delta_i(f) := \rho_i(\emptyset) - \rho_i(V \setminus \{i\}) = f(\{i\}) - (f(V) - f(V \setminus \{i\})) \geq 0, \tag{2}$$

vanishing if and only if i has constant marginal gain across all contexts. Writing $x(S) := \sum_{i\in S} x_i$ for $x \in \mathbb{R}^n$ and $S \subseteq V$, the *Edmonds base polytope* [3, 5, 1] is the nonempty compact convex polytope

$$B(f) := \{x \in \mathbb{R}^n : x(S) \le f(S) \text{ for all } S \subseteq V,\ \ x(V) = f(V)\}. \tag{3}$$

For each permutation $\pi \in S_n$ with prefixes $S_k^\pi := \{\pi(1), \dots, \pi(k)\}$ ($S_0^\pi := \emptyset$), Edmonds' greedy algorithm produces an extreme point $x^{(\pi)} \in B(f)$ with coordinates $x^{(\pi)}_{\pi(k)} := f(S_k^\pi) - f(S_{k-1}^\pi)$, and $B(f) = \operatorname{conv}\{x^{(\pi)} : \pi \in S_n\}$ [3, 5]. Its support function is the *Lovász extension* [6] $\hat{f} : \mathbb{R}^n \to \mathbb{R}$: for any $w \in \mathbb{R}^n$ ordered so that $w_{\pi(1)} \ge w_{\pi(2)} \ge \cdots \ge w_{\pi(n)}$,

$$\hat{f}(w) := \max_{x\in B(f)} \langle w, x\rangle = \langle w, x^{(\pi)}\rangle = \sum_{k=1}^{n-1} (w_{\pi(k)} - w_{\pi(k+1)}) f(S_k^\pi) + w_{\pi(n)} f(V). \tag{4}$$

Contributions. Building on the classical polyhedral foundations of Edmonds [3], Lovász [6], Fujishige [4, 5], and Conforti–Cornuéjols [2], we establish four quantitative, non-asymptotic finite-dimensional frameworks:

1. **Base-Polytope Diameter Identities (Theorem 2.1):** Exact formulas for $\operatorname{diam}_{\ell_1}(B(f))$ via the maximum submodular mutual information $I_f(S; S^c)$ and for $\operatorname{diam}_{\ell_\infty}(B(f))$ via $\max_i \Delta_i(f)$, plus a two-sided ℓ_2-diameter sandwich attained exactly on both sides by canonical graph-cut and matroid witnesses.
2. **Fujishige–Wolfe Telescoping Certificates and Energy Chain (Theorem 3.1):** A two-block Cauchy–Schwarz energy chain linking $f_{\min}$ to $\|x^*\|_2^2$ and $\|\phi(f)\|_2^2$, together with an exact telescoping identity for the Frank–Wolfe gap $G_{\mathrm{FW}}(x) = \|x\|_2^2 + \hat{f}(-x)$ that certifies both $\|x - x^*\|_2^2$ and the submodular dual lower-bound gap.
3. **Shapley Dispersion and Curvature Envelopes (Theorem 4.1):** Exact greedy-base variance decomposition around the Shapley barycenter $\phi(f)$, a sharp Popoviciu ceiling $\frac{1}{4}\|\Delta(f)\|_2^2$, and monotone total-curvature modular sandwiches.
4. **Hausdorff and $\frac{1}{2}$-Hölder Minimum-Norm Stability (Theorem 5.1):** Dual-norm Lovász perturbation bounds, the exact $\sqrt{5}$ Hausdorff constant for $d_H(B(f), B(g))$ in sup-norm $\varepsilon_\infty(f, g)$, and an elementary $\frac{1}{2}$-Hölder bound on $\|x_f^* - x_g^*\|_2^2$.

2 Exact Base-Polytope Diameter Identities and ℓ_2 Sandwiches

For any normalized submodular function $f : 2^V \to \mathbb{R}$, define the *submodular mutual information* (symmetric cut excess) across a bipartition (S, S^c) ($S^c := V \setminus S$) by

$$I_f(S; S^c) := f(S) + f(S^c) - f(V) \ge 0, \tag{5}$$

where nonnegativity follows directly from (1) and $f(\emptyset) = 0$. Write $\Delta(f) := (\Delta_1(f), \dots, \Delta_n(f)) \in \mathbb{R}^n_{\ge 0}$ and $\Delta_{\max}(f) := \max_{i\in V} \Delta_i(f)$. For $p \in \{1, 2, \infty\}$, let $\operatorname{diam}_{\ell_p}(B(f)) := \max_{x,y\in B(f)} \|x - y\|_p$.

Theorem 2.1 (Exact ℓ_1, ℓ_∞ Diameters and Sharp ℓ_2-Diameter Sandwich)**.** *Let $f : 2^V \to \mathbb{R}$ ($n \ge 2$) be normalized and submodular.*

(a) (Coordinate Box Containment and Exact ℓ_∞ Diameter.) *Every $x \in B(f)$ satisfies $\rho_i(V \setminus \{i\}) \le x_i \le \rho_i(\emptyset)$ for all $i \in V$, and*

$$\operatorname{diam}_{\ell_\infty}(B(f)) = \Delta_{\max}(f) = \max_{i\in V} \Delta_i(f). \tag{6}$$

(b) (Exact ℓ_1-Diameter Mutual-Information Identity.) *The ℓ_1 diameter of $B(f)$ satisfies*

$$\operatorname{diam}_{\ell_1}(B(f)) = 2\max_{S\subseteq V} I_f(S;S^c) = 2\max_{S\subseteq V}\big(f(S)+f(S^c)-f(V)\big). \tag{7}$$

(c) (Sharp Two-Sided ℓ_2-Diameter Sandwich.) *The Euclidean diameter of $B(f)$ satisfies*

$$\max\!\left(\sqrt{\frac{n}{n-1}}\,\Delta_{\max}(f),\ \frac{2}{\sqrt{n}}\max_{S\subseteq V} I_f(S;S^c)\right) \le \operatorname{diam}_{\ell_2}(B(f)) \le \min\!\left(\|\Delta(f)\|_2,\ \sqrt{\Delta_{\max}(f)\cdot 2\max_{S\subseteq V} I_f(S;S^c)}\right). \tag{8}$$

Both sides of (8) *are sharp for every $n\ge 2$: the star-graph cut function $f(S)=|\partial_{K_{1,n-1}}(S)|$ attains $\operatorname{diam}_{\ell_2}(B(f)) = \sqrt{\frac{n}{n-1}}\,\Delta_{\max}(f)$ identically, while the rank-1 uniform matroid $f(S)=\min(|S|,1)$ attains $\operatorname{diam}_{\ell_2}(B(f)) = \sqrt{\Delta_{\max}(f)\cdot\operatorname{diam}_{\ell_1}(B(f))} = \sqrt{2}$ identically.*

Proof. Part (a). For any $x\in B(f)$ and $i\in V$, $x_i = x(\{i\}) \le f(\{i\}) = \rho_i(\emptyset)$, and $x_i = x(V)-x(V\setminus\{i\}) \ge f(V)-f(V\setminus\{i\}) = \rho_i(V\setminus\{i\})$. Hence $|x_i-y_i|\le \Delta_i(f)$ for all $x,y\in B(f)$, giving $\operatorname{diam}_{\ell_\infty}(B(f))\le \Delta_{\max}(f)$. Conversely, for any $i\in V$, taking a greedy permutation π_1 with $\pi_1(1)=i$ yields $x_i^{(\pi_1)}=\rho_i(\emptyset)$, while a permutation π_2 with $\pi_2(n)=i$ yields $x_i^{(\pi_2)}=\rho_i(V\setminus\{i\})$, so $\|x^{(\pi_1)}-x^{(\pi_2)}\|_\infty \ge \Delta_i(f)$. Maximizing over i proves (6).

Part (b). For any $x,y\in B(f)$, set $S:=\{i\in V : x_i>y_i\}$. Because $x(V)=y(V)=f(V)$, we have $\sum_{i\in S}(x_i-y_i)=\sum_{j\in S^c}(y_j-x_j)$, whence

$$\|x-y\|_1 = 2\big(x(S)-y(S)\big) = 2\big(x(S)+y(S^c)-f(V)\big) \le 2\big(f(S)+f(S^c)-f(V)\big) = 2I_f(S;S^c).$$

Maximizing over $x,y\in B(f)$ yields $\operatorname{diam}_{\ell_1}(B(f)) \le 2\max_{S\subseteq V} I_f(S;S^c)$. Conversely, fix any $S\subseteq V$. Choose a greedy permutation π_1 placing all elements of S before S^c, so $x^{(\pi_1)}(S)=f(S)$, and a greedy permutation π_2 placing all elements of S^c before S, so $x^{(\pi_2)}(S^c)=f(S^c)$ and $x^{(\pi_2)}(S)=f(V)-f(S^c)$. Then

$$\|x^{(\pi_1)}-x^{(\pi_2)}\|_1 \ge \sum_{i\in S}(x_i^{(\pi_1)}-x_i^{(\pi_2)}) + \sum_{j\in S^c}(x_j^{(\pi_2)}-x_j^{(\pi_1)}) = 2\big(x^{(\pi_1)}(S)-x^{(\pi_2)}(S)\big) = 2I_f(S;S^c).$$

Maximizing over $S\subseteq V$ proves (7).

Part (c). Choose $i^*\in\arg\max_{i\in V}\Delta_i(f)$, and take greedy bases $x,y\in B(f)$ with $x_{i^*}=\rho_{i^*}(\emptyset)$ and $y_{i^*}=\rho_{i^*}(V\setminus\{i^*\})$. Because $\sum_{j\ne i^*}(y_j-x_j) = x_{i^*}-y_{i^*} = \Delta_{\max}(f)$, Cauchy–Schwarz over the remaining $n-1$ coordinates gives

$$\|x-y\|_2^2 = \Delta_{\max}(f)^2 + \sum_{j\ne i^*}(y_j-x_j)^2 \ge \Delta_{\max}(f)^2 + \frac{\Delta_{\max}(f)^2}{n-1} = \frac{n}{n-1}\,\Delta_{\max}(f)^2.$$

For the second lower bound, choosing $x,y\in B(f)$ attaining $\operatorname{diam}_{\ell_1}(B(f))$ gives $\|x-y\|_2 \ge n^{-1/2}\|x-y\|_1 = \frac{2}{\sqrt{n}}\max_S I_f(S;S^c)$. For the upper bounds, every $x,y\in B(f)$ satisfies $|x_i-y_i|\le \Delta_i(f)$ (hence $\|x-y\|_2\le\|\Delta(f)\|_2$) and $\|x-y\|_2^2 \le \|x-y\|_\infty\|x-y\|_1 \le \operatorname{diam}_{\ell_\infty}(B(f))\operatorname{diam}_{\ell_1}(B(f))$.

For exact attainment on the star graph $K_{1,n-1}$ with hub 1 and leaves $2,\dots,n$, every extreme base satisfies $x_1 = n-1-2k$ ($0\le k\le n-1$) where k leaves precede 1, with $x_j=+1$ for the k leaves before 1 and $x_j=-1$ for the $n-1-k$ leaves after 1. Consequently, $\Delta_1 = 2(n-1)$ and $\Delta_j = 2$ for every leaf $j\ge 2$, so every pair $x,y\in B(f)$ obeys $\|x-y\|_2^2 \le \|\Delta(f)\|_2^2 = 4(n-1)^2 + 4(n-1) = \frac{n}{n-1}\Delta_1^2$, attained identically by $x=(n-1,-1,\dots,-1)$ and $y=(-(n-1),1,\dots,1)$. For the rank-1 uniform matroid $f(S)=\min(|S|,1)$, $B(f)=\Delta_n$ is the unit simplex with $\Delta_{\max}(f)=1$, $\max_S I_f(S;S^c)=1$, and $\operatorname{diam}_{\ell_2}(\Delta_n)=\sqrt{2}=\sqrt{1\cdot 2}$. □

3 Fujishige–Wolfe Telescoping Certificates and Energy Chain

Because $B(f)$ is nonempty, compact, and convex, there exists a unique *minimum-norm base*

$$x^* := \arg\min_{x \in B(f)} \frac{1}{2}\|x\|_2^2. \tag{9}$$

Let $f_{\min} := \min_{S \subseteq V} f(S) \le 0$ (since $f(\emptyset) = 0$) and let $\phi(f) := \frac{1}{n!}\sum_{\pi \in S_n} x^{(\pi)} \in B(f)$ denote the Shapley value of f. For any $x \in B(f)$, define the *Frank–Wolfe optimality gap*

$$G_{\mathrm{FW}}(x) := \max_{y \in B(f)} \langle x, x - y\rangle = \|x\|_2^2 + \hat{f}(-x) \ge 0. \tag{10}$$

Theorem 3.1 (Fujishige–Wolfe Duality, Two-Block Energy Chain, and Telescoping Certificates)**.** *Let $f : 2^V \to \mathbb{R}$ ($n \ge 2$) be normalized and submodular with minimum-norm base x^*.*

(a) (Primal–Dual Optimality and Level-Set Minimization.) *The minimum-norm problem satisfies exact Fenchel duality*

$$\frac{1}{2}\|x^*\|_2^2 = \max_{w \in \mathbb{R}^n} \left(-\hat{f}(w) - \frac{1}{2}\|w\|_2^2\right), \tag{11}$$

with unique dual maximizer $w^* = -x^*$ and zero gap $G_{\mathrm{FW}}(x^*) = \|x^*\|_2^2 + \hat{f}(-x^*) = 0$. Moreover, $S_{<0} := \{i \in V : x_i^* < 0\}$ and $S_{\le 0} := \{i \in V : x_i^* \le 0\}$ are respectively the unique minimal and maximal minimizers of f, satisfying $x^*(S_{<0}) = x^*(S_{\le 0}) = f_{\min}$.

(b) (Two-Block Cauchy–Schwarz Energy Chain.) *Adopting the convention $0^2/0 := 0$ whenever $S_{<0} = \emptyset$ or $S_{<0} = V$,*

$$\frac{f(V)^2}{n} \le \frac{f_{\min}^2}{|S_{<0}|} + \frac{(f(V) - f_{\min})^2}{n - |S_{<0}|} \le \|x^*\|_2^2 \le \|\phi(f)\|_2^2 \le \frac{1}{n!}\sum_{\pi \in S_n} \|x^{(\pi)}\|_2^2. \tag{12}$$

(c) (Exact Telescoping Gap Identity and Non-Asymptotic Certificates.) *For any feasible base $x \in B(f)$ ordered as $x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(n)}$ with prefix sets $S_k(x) := \{\pi(1), \ldots, \pi(k)\}$,*

$$G_{\mathrm{FW}}(x) = \|x\|_2^2 + \hat{f}(-x) = \sum_{k=1}^{n-1} \big(x_{\pi(k+1)} - x_{\pi(k)}\big)\big(f(S_k(x)) - x(S_k(x))\big) \ge 0. \tag{13}$$

Furthermore, every $x \in B(f)$ certifies both proximity to x^* and the dual submodular-minimization lower bound:

$$\|x - x^*\|_2^2 \le G_{\mathrm{FW}}(x), \qquad 0 \le f_{\min} - \sum_{i=1}^{n} \min(x_i, 0) \le \|x - x^*\|_1 \le \sqrt{n\, G_{\mathrm{FW}}(x)}. \tag{14}$$

Proof. Part (a). For any $w \in \mathbb{R}^n$ and $x \in B(f)$, completing the square gives $\frac{1}{2}\|x\|_2^2 + \hat{f}(w) + \frac{1}{2}\|w\|_2^2 \ge \frac{1}{2}\|x\|_2^2 + \langle w, x\rangle + \frac{1}{2}\|w\|_2^2 = \frac{1}{2}\|x + w\|_2^2 \ge 0$, proving weak duality $\frac{1}{2}\|x^*\|_2^2 \ge -\hat{f}(w) - \frac{1}{2}\|w\|_2^2$. By first-order convexity of $\frac{1}{2}\|x\|_2^2$ on $B(f)$, $\langle x^*, y - x^*\rangle \ge 0$ for all $y \in B(f)$, equivalently $\langle -x^*, y\rangle \le -\|x^*\|_2^2$ for all $y \in B(f)$ with equality at $y = x^*$. Hence $\hat{f}(-x^*) = -\|x^*\|_2^2$, so $w^* = -x^*$ attains equality in (11) with $G_{\mathrm{FW}}(x^*) = 0$; uniqueness of w^* follows from strict concavity of $w \mapsto -\hat{f}(w) - \frac{1}{2}\|w\|_2^2$. Ordering $x^*_{\pi(1)} \le \cdots \le x^*_{\pi(n)}$ and invoking (13) (proved independently below from (4)) yields $\sum_{k=1}^{n-1}(x^*_{\pi(k+1)} - x^*_{\pi(k)})(f(S_k(x^*)) - x^*(S_k(x^*))) = G_{\mathrm{FW}}(x^*) = 0$. Since each factor is nonnegative, $x^*(S_k(x^*)) = f(S_k(x^*))$ whenever $x^*_{\pi(k)} < x^*_{\pi(k+1)}$. In particular, $S_{<0}$ and $S_{\le 0}$ are tight for x^* (taking $S_0 = \emptyset$ or $S_n = V$ if all entries share a sign). For any $S \subseteq V$,

$$f(S) \ge x^*(S) = \sum_{i \in S} x_i^* \ge \sum_{i \in S_{<0}} x_i^* = x^*(S_{<0}) = f(S_{<0}),$$

with strict inequality $x^*(S) > x^*(S_{<0})$ whenever S omits an element of $S_{<0}$ or includes an element outside $S_{\le 0}$; conversely $x^*(S_{\le 0}) = x^*(S_{<0}) = f_{\min}$. Thus $S_{<0}$ and $S_{\le 0}$ are the unique minimal and maximal minimizers of f [4, 5].

Part (b). Set $k := |S_{<0}|$. If $k = 0$ then $f_{\min} = 0$, and if $k = n$ then $f_{\min} = f(V)$; in both cases the first inequality of (12) is an identity. If $1 \le k \le n-1$, Cauchy–Schwarz on $(f_{\min}) + (f(V) - f_{\min}) = f(V)$ gives

$$f(V)^2 = \left(\sqrt{k}\,\frac{f_{\min}}{\sqrt{k}} + \sqrt{n-k}\,\frac{f(V)-f_{\min}}{\sqrt{n-k}}\right)^2 \le n\left(\frac{f_{\min}^2}{k} + \frac{(f(V)-f_{\min})^2}{n-k}\right).$$

For the second inequality, Cauchy–Schwarz applied separately on $S_{<0}$ (where $x^*(S_{<0}) = f_{\min}$) and its complement $S_{<0}^c := V \setminus S_{<0}$ (where $x^*(S_{<0}^c) = f(V) - f_{\min}$) yields $\|x^*\|_2^2 = \sum_{i \in S_{<0}} (x_i^*)^2 + \sum_{j \notin S_{<0}} (x_j^*)^2 \ge f_{\min}^2/k + (f(V)-f_{\min})^2/(n-k)$. Since $\phi(f) \in B(f)$, $\|x^*\|_2^2 \le \|\phi(f)\|_2^2$; finally, convexity of $\|\cdot\|_2^2$ gives $\|\phi(f)\|_2^2 = \|\frac{1}{n!}\sum_\pi x^{(\pi)}\|_2^2 \le \frac{1}{n!}\sum_\pi \|x^{(\pi)}\|_2^2$.

Part (c). Let $x \in B(f)$ with $x_{\pi(1)} \le \cdots \le x_{\pi(n)}$. Setting $w := -x$ gives $w_{\pi(1)} \ge \cdots \ge w_{\pi(n)}$, so (4) yields $\hat{f}(-x) = \sum_{k=1}^{n-1}(x_{\pi(k+1)} - x_{\pi(k)}) f(S_k(x)) - x_{\pi(n)} f(V)$. On the other hand, Abel summation gives

$$\sum_{k=1}^{n-1}(x_{\pi(k+1)} - x_{\pi(k)})\, x(S_k(x)) = \sum_{j=1}^{n} x_{\pi(j)} \sum_{k=j}^{n-1}(x_{\pi(k+1)} - x_{\pi(k)}) = \sum_{j=1}^{n} x_{\pi(j)}(x_{\pi(n)} - x_{\pi(j)}) = x_{\pi(n)} f(V) - \|x\|_2^2.$$

Adding the two identities cancels $x_{\pi(n)} f(V)$ identically, and moving $\sum_{k=1}^{n-1}(x_{\pi(k+1)} - x_{\pi(k)}) x(S_k(x))$ to the right-hand side and $\|x\|_2^2$ to the left-hand side proves (13), with nonnegativity since $x(S_k(x)) \le f(S_k(x))$. Next, by first-order optimality of x^* we have $\langle x^*, x^* - x\rangle \le 0$, while by definition of $G_{\mathrm{FW}}(x)$ we have $\langle x, x - x^*\rangle \le G_{\mathrm{FW}}(x)$. Adding these two inequalities gives $\|x - x^*\|_2^2 = \langle x - x^*, x - x^*\rangle \le G_{\mathrm{FW}}(x)$. Finally, for any $x \in B(f)$, $\sum_{i=1}^n \min(x_i, 0) \le x(S_{<0}) \le f(S_{<0}) = f_{\min} = \sum_{i=1}^n \min(x_i^*, 0)$. Since $t \mapsto \min(t, 0)$ is 1-Lipschitz,

$$0 \le f_{\min} - \sum_{i=1}^{n} \min(x_i, 0) = \sum_{i=1}^{n} (\min(x_i^*, 0) - \min(x_i, 0)) \le \|x - x^*\|_1 \le \sqrt{n}\,\|x - x^*\|_2 \le \sqrt{n\, G_{\mathrm{FW}}(x)}.$$

□

4 Shapley Barycenter Dispersion and Total-Curvature Envelopes

Because $B(f) = \mathrm{conv}\{x^{(\pi)} : \pi \in S_n\}$, the Shapley value $\phi(f) = \frac{1}{n!}\sum_{\pi \in S_n} x^{(\pi)}$ is the uniform barycenter of the $n!$ greedy extreme bases.

Theorem 4.1 (Shapley Greedy-Base Dispersion Ceiling and Monotone Curvature Sandwich)**.** *Let $f : 2^V \to \mathbb{R}$ ($n \ge 2$) be normalized and submodular with Shapley value $\phi(f)$ and marginal-range vector $\Delta(f)$.*

(a) (Coordinate Containment, Variance Identity, and Popoviciu Ceiling.) *For every $i \in V$, $\rho_i(V \setminus \{i\}) \le \phi_i(f) \le \rho_i(\emptyset)$. Moreover, the mean squared deviation of the greedy extreme bases around $\phi(f)$ satisfies*

$$\frac{1}{n!}\sum_{\pi \in S_n} \|x^{(\pi)} - \phi(f)\|_2^2 = \frac{1}{n!}\sum_{\pi \in S_n} \|x^{(\pi)}\|_2^2 - \|\phi(f)\|_2^2 \le \frac{1}{4}\|\Delta(f)\|_2^2, \tag{15}$$

with exact equality in (15) *for every normalized submodular function when $n = 2$.*

(b) (Monotone Total-Curvature Modular Sandwich.) *Suppose in addition that f is monotone nondecreasing ($\rho_i(V \setminus \{i\}) \ge 0$ for all $i \in V$) with $f(V) > 0$, and define the Conforti–Cornuéjols total curvature [2]*

$$c(f) := 1 - \min_{i \in V :\, \rho_i(\emptyset) > 0} \frac{\rho_i(V \setminus \{i\})}{\rho_i(\emptyset)} \in [0, 1]. \tag{16}$$

Then every subset $S \subseteq V$ satisfies the two-sided modular envelope

$$(1 - c(f)) \sum_{i \in S} \rho_i(\emptyset) \leq f(S) \leq \sum_{i \in S} \rho_i(\emptyset). \tag{17}$$

Proof. Part (a). Because $\phi(f) \in B(f)$, Theorem 2.1(a) immediately gives $\rho_i(V \setminus \{i\}) \leq \phi_i(f) \leq \rho_i(\emptyset)$ for all $i \in V$. Expanding the inner product around the mean $\phi(f) = \mathbb{E}_\pi[x^{(\pi)}]$ gives the exact variance decomposition in (15). For each fixed coordinate $i \in V$, the random variable $X_i(\pi) := x_i^{(\pi)}$ (under uniform $\pi \in S_n$) takes values inside the compact interval $[\rho_i(V \setminus \{i\}), \rho_i(\emptyset)]$ of length $\Delta_i(f)$. By Popoviciu's inequality, $\mathrm{Var}_\pi(X_i) \leq \frac{1}{4}\Delta_i(f)^2$. Summing over $i = 1, \ldots, n$ proves (15). When $n = 2$, S_2 has only two permutations, under which each X_i takes the two endpoints $\rho_i(\emptyset)$ and $\rho_i(V \setminus \{i\})$ with probability $1/2$ each, so $\mathrm{Var}_\pi(X_i) = \frac{1}{4}\Delta_i(f)^2$ holds identically for every $i \in \{1, 2\}$.

Part (b). Since $f(V) > 0$ and $f(\emptyset) = 0$, subadditivity $f(V) \leq \sum_{i \in V} \rho_i(\emptyset)$ ensures $\{i : \rho_i(\emptyset) > 0\}$ is nonempty, and monotonicity plus submodularity gives $0 \leq \rho_i(V \setminus \{i\}) \leq \rho_i(\emptyset)$, so $c(f) \in [0, 1]$. For any $S = \{i_1, \ldots, i_k\} \subseteq V$, telescoping $f(S) = \sum_{r=1}^{k} \rho_{i_r}(\{i_1, \ldots, i_{r-1}\})$ and using $\rho_i(V \setminus \{i\}) \leq \rho_i(T) \leq \rho_i(\emptyset)$ for all $T \subseteq V \setminus \{i\}$ yields $\sum_{i \in S} \rho_i(V \setminus \{i\}) \leq f(S) \leq \sum_{i \in S} \rho_i(\emptyset)$. Whenever $\rho_i(\emptyset) > 0$, (16) gives $\rho_i(V \setminus \{i\}) \geq (1 - c(f))\rho_i(\emptyset)$; whenever $\rho_i(\emptyset) = 0$, monotonicity forces $\rho_i(V \setminus \{i\}) = 0 = (1 - c(f))\rho_i(\emptyset)$. Summing over $i \in S$ proves (17). □

5 Hausdorff and $\frac{1}{2}$-Hölder Minimum-Norm Stability

Given two normalized submodular functions $f, g : 2^V \to \mathbb{R}$ ($n \geq 2$), define their sup-norm distance and coordinatewise marginal-perturbation vector by

$$\varepsilon_\infty(f, g) := \max_{S \subseteq V} |f(S) - g(S)|, \qquad \delta_i(f, g) := \max_{S \subseteq V \setminus \{i\}} |\rho_i^f(S) - \rho_i^g(S)| \quad (i \in V), \tag{18}$$

and let $d_H(B(f), B(g)) := \max(\sup_{x \in B(f)} \inf_{y \in B(g)} \|x - y\|_2,\ \sup_{y \in B(g)} \inf_{x \in B(f)} \|x - y\|_2)$ denote the Euclidean Hausdorff distance between their base polytopes.

Theorem 5.1 (Lovász Lipschitz Stability, Sharp $\sqrt{5}$ Hausdorff Bound, and $\frac{1}{2}$-Hölder Base Stability)**.** *Let $f, g : 2^V \to \mathbb{R}$ ($n \geq 2$) be normalized submodular functions with minimum-norm bases x_f^*, x_g^* and minima $f_{\min}, g_{\min}$.*

(a) (Dual-Norm Lovász Extension Stability.) *For every $w \in \mathbb{R}^n$ with $w_{\max} := \max_i w_i$ and $w_{\min} := \min_i w_i$,*

$$|\hat{f}(w) - \hat{g}(w)| \leq \min\left(\sum_{i=1}^{n} \delta_i(f, g)\, |w_i|,\ \ ((w_{\max} - w_{\min}) + |w_{\min}|)\, \varepsilon_\infty(f, g) \right). \tag{19}$$

(b) (Sharp $\sqrt{5}$ Hausdorff Distance Ceiling.) *The Euclidean Hausdorff distance satisfies*

$$d_H(B(f), B(g)) \leq \min\Big(\|\delta(f, g)\|_2,\ \ \sqrt{5}\, \varepsilon_\infty(f, g) \Big), \tag{20}$$

where $\sqrt{5} = \max_{w \in S^{n-1}}((w_{\max} - w_{\min}) + |w_{\min}|)$ is exact for every $n \geq 2$.

(c) (Exact $\frac{1}{2}$-Hölder Minimum-Norm Base Stability and 1-Lipschitz Minimum.)

$$\|x_f^* - x_g^*\|_2^2 \leq (\|x_f^*\|_2 + \|x_g^*\|_2)\, d_H(B(f), B(g)) \leq (\|x_f^*\|_2 + \|x_g^*\|_2) \min\Big(\|\delta(f, g)\|_2,\ \sqrt{5}\, \varepsilon_\infty(f, g) \Big), \tag{21}$$

and $|f_{\min} - g_{\min}| \leq \varepsilon_\infty(f, g)$.

Proof. Part (a). Fix $w \in \mathbb{R}^n$ and choose a permutation $\pi \in S_n$ sorting $w_{\max} = w_{\pi(1)} \geq \cdots \geq w_{\pi(n)} = w_{\min}$. From (4), on one hand $\hat{f}(w) - \hat{g}(w) = \sum_{k=1}^{n} w_{\pi(k)}(\rho^f_{\pi(k)}(S^\pi_{k-1}) - \rho^g_{\pi(k)}(S^\pi_{k-1}))$, so $|\hat{f}(w) - \hat{g}(w)| \leq \sum_{k=1}^{n} |w_{\pi(k)}|\, \delta_{\pi(k)}(f,g) = \sum_{i=1}^{n} \delta_i(f,g)|w_i|$. On the other hand, writing $\hat{f}(w) - \hat{g}(w) = \sum_{k=1}^{n-1}(w_{\pi(k)} - w_{\pi(k+1)})(f(S^\pi_k) - g(S^\pi_k)) + w_{\pi(n)}(f(V) - g(V))$ and applying the triangle inequality with $w_{\pi(k)} - w_{\pi(k+1)} \geq 0$ gives

$$|\hat{f}(w) - \hat{g}(w)| \leq \left(\sum_{k=1}^{n-1}(w_{\pi(k)} - w_{\pi(k+1)}) + |w_{\pi(n)}|\right) \varepsilon_\infty(f,g) = ((w_{\max} - w_{\min}) + |w_{\min}|)\varepsilon_\infty(f,g),$$

proving (19).

Part (b). By support-function duality for nonempty compact convex sets [7], $d_H(B(f), B(g)) = \max_{\|w\|_2=1} |\hat{f}(w) - \hat{g}(w)|$. For $\|w\|_2 = 1$, Cauchy–Schwarz bounds $\sum_i \delta_i(f,g)|w_i| \leq \|\delta(f,g)\|_2$. For the second term, let $a := w_{\max}$ and $b := w_{\min}$ with $a \geq b$; because $n \geq 2$, we have $a^2 + b^2 \leq 1$ (if a and b are attained at distinct indices this is immediate from $\|w\|_2 = 1$, while if $a = b$ then $w_1 = \cdots = w_n = \pm n^{-1/2}$ so $a^2 + b^2 = 2/n \leq 1$). If $b \geq 0$, then $(a - b) + |b| = a \leq 1$; if $b < 0$, then $(a - b) + |b| = a - 2b \leq \sqrt{1^2 + (-2)^2}\sqrt{a^2 + b^2} \leq \sqrt{5}$, with exact attainment for every $n \geq 2$ at $w_{\max} = 1/\sqrt{5}$, $w_{\min} = -2/\sqrt{5}$, and all remaining $n - 2$ coordinates equal to 0. Taking the supremum over $w \in S^{n-1}$ proves (20).

Part (c). Define the 1-strongly convex dual objectives $\Phi_f(w) := \hat{f}(w) + \frac{1}{2}\|w\|_2^2$ and $\Phi_g(w) := \hat{g}(w) + \frac{1}{2}\|w\|_2^2$. By Theorem 3.1(a), their unique minimizers are $w^*_f = -x^*_f$ and $w^*_g = -x^*_g$. Strong convexity of Φ_f around w^*_f and of Φ_g around w^*_g yields

$$\frac{1}{2}\|w^*_f - w^*_g\|_2^2 \leq \Phi_f(w^*_g) - \Phi_f(w^*_f), \qquad \frac{1}{2}\|w^*_f - w^*_g\|_2^2 \leq \Phi_g(w^*_f) - \Phi_g(w^*_g).$$

Adding the two inequalities cancels all quadratic terms $\frac{1}{2}\|w^*_f\|_2^2$ and $\frac{1}{2}\|w^*_g\|_2^2$ identically, leaving

$$\|x^*_f - x^*_g\|_2^2 = \|w^*_f - w^*_g\|_2^2 \leq (\hat{f}(w^*_g) - \hat{g}(w^*_g)) + (\hat{g}(w^*_f) - \hat{f}(w^*_f)) \leq (\|w^*_g\|_2 + \|w^*_f\|_2)\, d_H(B(f), B(g)),$$

where the last step uses $|\hat{f}(w) - \hat{g}(w)| \leq \|w\|_2\, d_H(B(f), B(g))$ (positive homogeneity of $\hat{f}, \hat{g}$). Combining with $\|w^*_f\|_2 = \|x^*_f\|_2$, $\|w^*_g\|_2 = \|x^*_g\|_2$, and (20) proves (21). Finally, $f(S) \leq g(S) + \varepsilon_\infty(f,g)$ for all S implies $f_{\min} \leq g_{\min} + \varepsilon_\infty(f,g)$, and swapping f, g gives $|f_{\min} - g_{\min}| \leq \varepsilon_\infty(f,g)$. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified in deterministic float64 arithmetic (seed `20261129` for Theorems 2.1–4.1 and `20261130` for Theorem 5.1) using the MIT-licensed companion repository `lluiseriksson/submodular-base-polytope-geometry`. Our suite comprises 87 normalized submodular functions ($n \in \{4, 5, 6\}$): 7 structured canonical witnesses—rank-1 and rank-2 uniform matroids ($n = 5$), pure modular function ($n = 5$), complete-graph K_5 cut and shifted K_5 cut, concave cardinality $\sqrt{|S|}$ ($n = 6$), and the star-graph cut $K_{1,4}$ ($n = 5$)—plus 80 randomized instances drawn across four canonical families (shifted symmetric graph cuts, monotone Gaussian log-determinant $\ln\det(I + K_{S,S})$, shifted non-monotone log-determinant, and monotone facility location, with 47 monotone instances in total) and 84 same-dimension perturbation pairs (345 total theorem checks). Minimum-norm bases x^* are computed to machine tolerance via the exact Fujishige–Wolfe corral projection algorithm [5, 1].

Table 1: Deterministic float64 verification across 87 submodular instances and 84 perturbation pairs (345 total theorem checks, `pytest -q`: 4/4 passed in 2.97 s). All exact identities match to machine precision and every bound is satisfied with zero violations.

Theorem	Checks	Max Identity Rel. Error	Sharpest Bound Attainment / Ratio
Thm. 2.1 (Diameters)	87	8.88×10^{-16} (ℓ_1), 2.48×10^{-16} (ℓ_∞)	$\min(\mathrm{diam}_{\ell_2}/\mathrm{LB}) = 1.0000$, $\max(\mathrm{diam}_{\ell_2}/\mathrm{UB}) = 1.0000$
Thm. 3.1 (Min-Norm)	87	5.22×10^{-15} (gap), 5.73×10^{-14} (tele.)	$\max\lvert f(S_{<0}) - f_{\min}\rvert = 0.0$, $\min(\lVert x^*\rVert_2^2/\mathrm{LB}_{\text{2-blk}}) = 1.0000$
Thm. 4.1 (Shapley)	87	9.62×10^{-14} (variance)	$\max(\mathrm{Var}/\mathrm{UB}_{\mathrm{Pop}}) = 0.9600$, 47/47 mono. curv. valid
Thm. 5.1 (Stability)	84	—	$\max(\text{Lovász}/\mathrm{UB}) = 1.0000$, $\max(d_H/\mathrm{UB}) = 0.9974$, $\max(\lVert x_f^* - x_g^*\rVert_2^2/\mathrm{UB}_{\mathrm{Höl}}) = 0.6716$

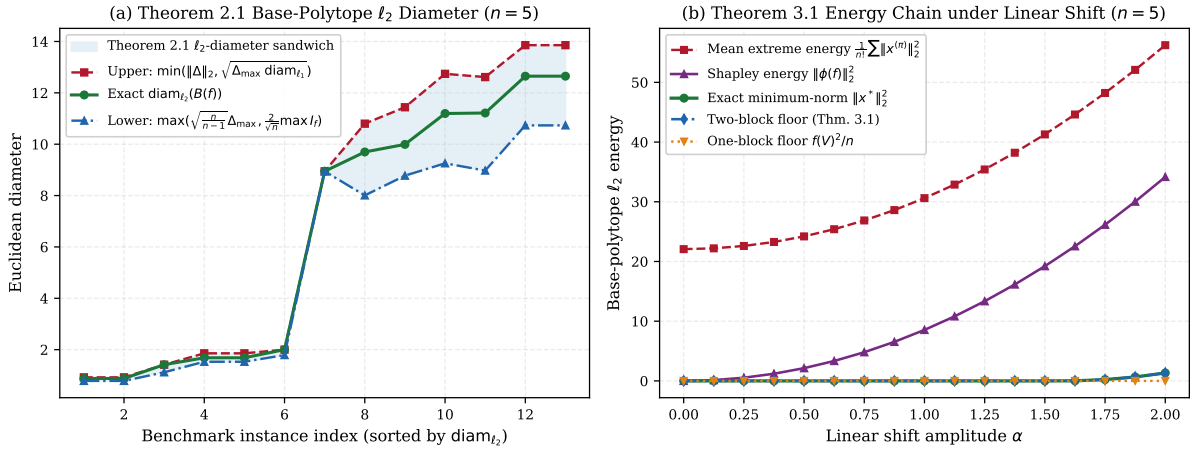


Figure 1: **(a)** Exact Euclidean diameter $\mathrm{diam}_{\ell_2}(B(f))$ across $n = 5$ structured and random submodular benchmarks, tightly bracketed by Theorem 2.1(c) with exact attainment on both sides ($K_{1,4}$ star cut and rank-1 uniform matroid). **(b)** Complete five-level energy chain (12) for a shifted $n = 5$ graph cut $f_\alpha(S) = \mathrm{cut}_W(S) + \alpha c(S)$ as α increases from 0 to 2, illustrating strict separation between the two-block floor and the naive one-block floor $f(V)^2/n$.

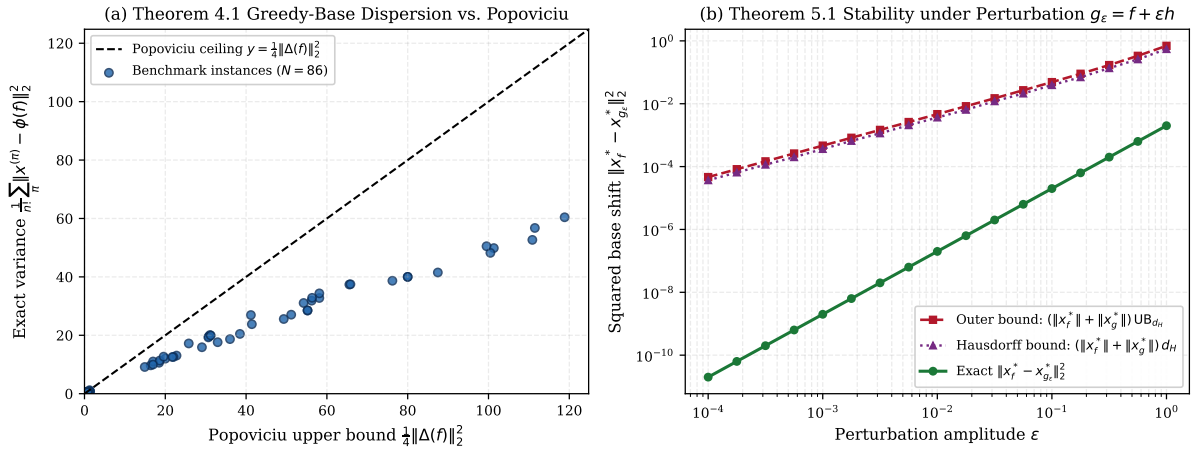


Figure 2: **(a)** Exact greedy-base variance $\frac{1}{n!}\sum_\pi \lVert x^{(\pi)} - \phi(f)\rVert_2^2$ versus the Popoviciu ceiling $\frac{1}{4}\lVert\Delta(f)\rVert_2^2$ across all 86 non-modular instances in our suite, confirming Theorem 4.1(a). **(b)** Squared minimum-norm base shift $\lVert x_f^* - x_{g_\varepsilon}^*\rVert_2^2$ bracketed by the Hausdorff and outer $\frac{1}{2}$-Hölder bounds of Theorem 5.1(c) under $g_\varepsilon = f + \varepsilon h$ across $\varepsilon \in [10^{-4}, 1]$.

7 Scope and Limitations

Three structural boundaries of our results deserve explicit discussion:

1. *Computational hardness of exact* $\operatorname{diam}_{\ell_1}(B(f))$. While Theorem 2.1(b) proves the exact identity $\operatorname{diam}_{\ell_1}(B(f)) = 2\max_{S\subseteq V} I_f(S; S^c)$, maximizing a symmetric nonnegative submodular function $S \mapsto I_f(S; S^c)$ is NP-hard in general (specializing to Max-Cut for graph cut functions). By contrast, $\operatorname{diam}_{\ell_\infty}(B(f)) = \max_i \Delta_i(f)$ and the Popoviciu ceiling $\frac{1}{4}\|\Delta(f)\|_2^2$ require only $2n+1$ value-oracle queries.

2. *Dual versus primal submodular-minimization error in Theorem 3.1(c).* The certificate $0 \le f_{\min} - \sum_i \min(x_i, 0) \le \sqrt{n\, G_{\mathrm{FW}}(x)}$ controls the *dual lower-bound deficit* unconditionally for every $x \in B(f)$. Bounding the *primal* rounding error $\min_k f(S_k(x)) - f_{\min}$ requires additional threshold-averaging arguments (e.g., over random thresholds in $[x_{\min}, 0]$) because an individual negative level set $\{i \in V : x_i < 0\}$ need not be tight when $G_{\mathrm{FW}}(x) > 0$.

3. *Dimension-free* $\frac{1}{2}$*-Hölder stability in Theorem 5.1(c) versus fixed-n piecewise-linear Lipschitzness.* Because Edmonds base polytopes $B(f)$ in fixed $\mathbb{R}^n$ have fixed $\{0,1\}^n$ facet normals, for any fixed n the map $f \mapsto x_f^*$ is piecewise linear across finitely many Fujishige–Wolfe chain cones and hence globally Lipschitz with a dimension-dependent constant ($\|x_f^* - x_g^*\|_2 = O(\sqrt{n}\,\varepsilon_\infty(f,g)) = O(n\, d_H(B(f), B(g)))$, matching the $O(\varepsilon^2)$ squared shift along $g_\varepsilon = f + \varepsilon h$ in Figure 2(b)). Theorem 5.1(c) trades the linear exponent for a completely *dimension-free* bound independent of n that holds universally across general compact convex sets in Hilbert space.

References

[1] F. Bach, *Learning with Submodular Functions: A Convex Optimization Perspective*, Found. Trends Mach. Learn. **6** (2013), no. 2–3, 145–373.

[2] M. Conforti and G. Cornuéjols, *Submodular set functions, matroids and the greedy algorithm: tight worst-case bounds and some generalizations of the Rado–Edmonds theorem*, Discrete Appl. Math. **7** (1984), no. 3, 251–274.

[3] J. Edmonds, *Submodular functions, matroids, and certain polyhedra*, in *Combinatorial Structures and Their Applications*, Gordon and Breach, 1970, pp. 69–87.

[4] S. Fujishige, *Lexicographically optimal base of a polymatroid with respect to a weight vector*, Math. Oper. Res. **5** (1980), no. 2, 186–196.

[5] S. Fujishige, *Submodular Functions and Optimization*, 2nd ed., Annals of Discrete Mathematics, Vol. 58, Elsevier, 2005.

[6] L. Lovász, *Submodular functions and convexity*, in *Mathematical Programming: The State of the Art*, Springer, 1983, pp. 235–257.

[7] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, 2nd expanded ed., Encyclopedia of Mathematics and Its Applications, Vol. 151, Cambridge University Press, 2014.