

Euler-Reduced Tjurina Floors for Osculating-Absorbing Gauss Fibres

Lluis Eriksson
Independent researcher

August 25, 2026 (working version 0.1)

Abstract

Let $h \in \mathfrak{m}^{s+1} \subset \mathbb{C}[[x_1, \dots, x_d]]$ have a finite Tjurina algebra. Euler cancellation replaces h , without changing its Tjurina ideal, by a generator of order at least $s + 2$. Counting coefficient jets after this replacement gives the self-contained bound

$$\tau(h) \geq E_{d,s} := \max_{K \geq 0} \left\{ \binom{d+K}{d} - d \binom{d+K-s}{d} - \binom{d+K-s-2}{d} \right\},$$

with out-of-range binomial coefficients interpreted as zero. For plane germs the maximum is explicit:

$$E_{2,s} = \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor.$$

This is exact, as follows from classical ordinary multiple-point families.

We apply the count to scheme-theoretic Gauss fibres. If the complete $\mathcal{O}_X(m)$ point span of the reduced fibre of a smooth hypersurface absorbs order s , then every tangent-section germ belongs to $\mathfrak{m}^{s+1}$. Consequently

$$\text{length}(\Gamma_\eta) \geq E_{d,s}|Z| \geq E_{d,s} \binom{d+m}{d}.$$

For surfaces, a prescribed-jet construction realizes equality in the last bound at the explicit sufficient degree $D \geq (s^2 + 2) \binom{m+2}{2} + 1$. Thus the plane Gauss-fibre floor is sharp under the stated absorption and extremal-support conditions. The local count is elementary and may be folklore; priority is claimed only for the specific proved synthesis and application, not for every ingredient.

1 Introduction and scope

For an isolated hypersurface germ h , the Tjurina number

$$\tau(h) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_d]]}{(h, \partial_1 h, \dots, \partial_d h)}$$

measures the length of its Tjurina algebra. If $h \in \mathfrak{m}^{s+1}$, the immediate containment $(h, \partial h) \subseteq \mathfrak{m}^s$ gives only $\tau(h) \geq \binom{d+s-1}{d}$. The purpose of this note is to record a stronger finite-jet count and use it to settle the corresponding sharp floor for point-span absorbed Gauss fibres in dimension two.

The key observation is small but consequential. In characteristic zero, the Euler combination removes the degree- $(s + 1)$ part of h while leaving the Tjurina ideal unchanged. A truncated coefficient count then remembers that the ideal has d generators of order at least s and one further generator of order at least $s + 2$. Optimizing the truncation produces the number $E_{d,s}$ above.

In the plane, $E_{2,s}$ agrees with the classical sharp minimum in the ordinary multiplicity- $(s+1)$ equisingularity class computed through the work of Briançon–Granger–Maisonobe [2]; explicit attaining families are also discussed by Canino–Gimigliano–Idà [3, 4]. Their results supply sharp examples; the elementary argument below supplies the lower bound for arbitrary isolated plane germs of the prescribed order. We state this scope explicitly and make no claim that the abstract generator count was absent from all earlier commutative-algebra literature.

The geometric application continues the scheme-theoretic Gauss-fibre framework of [5]. That paper identifies the local Gauss-fibre algebra with the Tjurina algebra and proves the earlier fat-point floor. The exact reduced-support bound used here is from [6, 7]. We restate the needed definitions and arguments so that the new numerical step is transparent.

Everything is over $\mathbb{C} = \mathbb{C}$. In particular, division by $s+1$ in the Euler reduction is legitimate. We do not assert the same theorem in positive characteristic.

2 The Euler-reduced local bound

We use the convention

$$\binom{n}{d} = 0 \quad \text{if } n < d.$$

For integers $d, s \geq 1$ and $K \geq 0$, set

$$E_{d,s}(K) = \binom{d+K}{d} - d \binom{d+K-s}{d} - \binom{d+K-s-2}{d}, \quad (2.1)$$

and $E_{d,s} = \max_{K \geq 0} E_{d,s}(K)$.

Lemma 2.1 (Euler-reduced generator). *Let $R = \mathbb{C}[[x_1, \dots, x_d]]$, let $\mathfrak{m} = (x_1, \dots, x_d)$, and let $h \in \mathfrak{m}^{s+1}$. Then*

$$r = h - \frac{1}{s+1} \sum_{i=1}^d x_i \partial_i h \in \mathfrak{m}^{s+2}$$

and

$$(h, \partial_1 h, \dots, \partial_d h) = (r, \partial_1 h, \dots, \partial_d h).$$

Proof. Writing $h = \sum_{q \geq s+1} h_q$ with h_q homogeneous of degree q , Euler's identity gives

$$r = \sum_{q \geq s+2} \left(1 - \frac{q}{s+1}\right) h_q,$$

so $r \in \mathfrak{m}^{s+2}$. Both ideal containments in (2.2) follow directly from the displayed definition of r . $\square$

Theorem 2.2 (Truncated-generator Tjurina floor). *Let $h \in \mathfrak{m}^{s+1} \subset R = \mathbb{C}[[x_1, \dots, x_d]]$, and assume that $R/(h, \partial h)$ has finite length. Then*

$$\tau(h) \geq E_{d,s}.$$

In particular,

$$\tau(h) \geq \binom{d+s}{d} - d.$$

For $d, s \geq 2$, (2.4) strictly improves $\binom{d+s-1}{d}$.

Proof. Fix $K \geq 0$ and write $A_K = R/\mathfrak{m}^{K+1}$. Its dimension is $\binom{d+K}{d}$. By Lemma 2.1, the Tjurina ideal is generated by d elements $\partial_i h \in \mathfrak{m}^s$ and one element $r \in \mathfrak{m}^{s+2}$.

In A_K , the multiples of a fixed element of order at least s are spanned by coefficients of degree at most $K - s$, a space of dimension $\binom{d+K-s}{d}$. Similarly, the multiples of r are spanned by a space of dimension at most $\binom{d+K-s-2}{d}$. We only need an upper bound on the sum of these images; possible syzygies can make it smaller, never larger. Hence

$$\dim_{\mathbb{C}} A_K / (h, \partial h) A_K \geq E_{d,s}(K).$$

The Tjurina algebra surjects onto this truncated quotient, proving the inequality for every K , and therefore (2.3).

For large K , the degree- d leading coefficient of $E_{d,s}(K)$ is negative, so the maximum is finite. Taking $K = s$ gives (2.4). Its gain over the fat-point count is

$$\binom{d+s-1}{d-1} - d > 0$$

when $d, s \geq 2$. □

Remark 2.3. The proof uses only generator orders after the Euler cancellation. It does not require a Milnor-number inequality, weighted homogeneity, or an assumption on the tangent cone. The finite-colength hypothesis is needed only so that $\tau(h)$ is finite.

For context, without the lower-order correction caused by Euler reduction, the same argument would subtract $\binom{d+K-s-1}{d}$ instead of the last term in (2.1). The extra order is essential for the exact plane answer.

Proposition 2.4 (Asymptotic form). *For fixed $d \geq 2$, as $s \rightarrow \infty$,*

$$E_{d,s} = c_d^{\text{gen}} s^d + O_d(s^{d-1}), \quad c_d^{\text{gen}} = \frac{d+1}{d! ((d+1)^{1/(d-1)} - 1)^{d-1}}.$$

Proof. For $K < s$,

$$E_{d,s}(K) = \binom{d+K}{d} \leq \frac{s^d}{d!} + O_d(s^{d-1}).$$

For $K = s + \alpha s$, division of (2.1) by s^d gives, uniformly for α in a bounded interval, the limiting function

$$\phi_d(\alpha) = \frac{(1+\alpha)^d - (d+1)\alpha^d}{d!}, \quad \alpha \geq 0.$$

Its derivative at zero is positive, so its maximum is strictly greater than $1/d!$. Thus the range $K < s$ is not maximizing for all sufficiently large s . If $K/s \rightarrow \infty$, the normalized expression is eventually negative, whereas $K = s$ gives a positive multiple of s^d ; hence every maximizing sequence has $K = O_d(s)$. Uniform binomial expansion on the remaining bounded range gives the upper estimate by maximizing ϕ_d . Conversely, choosing the nearest integer to $s(1 + \alpha_d)$ gives the matching lower estimate, where the unique maximizer is

$$\alpha_d = \left((d+1)^{1/(d-1)} - 1 \right)^{-1}.$$

Substitution and integer rounding give (2.5), with an error of order s^{d-1} . □

3 The exact plane floor

Theorem 3.1 (Universal plane bound). *Let $h \in \mathbb{C}[[x, y]]$ define an isolated plane curve singularity and assume $h \in \mathfrak{m}^{s+1}$. Then*

$$\tau(h) \geq \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor. \quad (3.1)$$

The bound is sharp for every $s \geq 1$.

Proof. For $K < s$, (2.1) is $\binom{K+2}{2}$. For $K = s + t$, $t \geq 0$, it is

$$E_{2,s}(s+t) = \frac{s^2 + 2st - 2t^2 + 3s - 2t - 2}{2}.$$

The concave quadratic is maximized at the integer or integers nearest $(s-1)/2$. Comparing with the range $K < s$ yields

$$E_{2,s} = \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor,$$

so Theorem 2.2 proves (3.1).

Set $a = s + 1$. If $a = 2k$, take

$$h = x^{2k} + y^{2k} + x^{k+1}y^k;$$

if $a = 2k + 1$, take

$$h = x^{2k+1} + y^{2k+1} + x^{k+1}y^{k+1}.$$

These are ordinary multiplicity- a germs. The classical computation [2, 4] gives respectively

$$\tau = 3k^2 - k - 1, \quad \tau = 3k^2 + 2k - 1,$$

which is exactly the right-hand side of (3.1). The case $s = 1$ is also included, with $\tau = 1$. $\square$

Remark 3.2 (Literature boundary). Writing $a = s + 1$, the number in (3.1) is

$$\left\lfloor \frac{3a^2 - 2a - 4}{4} \right\rfloor.$$

It is already classical as the minimum in the ordinary multiplicity- a class. The role of the families above is therefore sharpness, not novelty. The lower bound here is obtained for arbitrary isolated plane germs by the Euler-reduced coefficient count. We use conservative attribution because the elementary local estimate may admit formulations elsewhere.

4 Osculating absorption and Gauss fibres

Let $X^d \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ be a smooth non-linear hypersurface and let

$$\gamma_X : X \longrightarrow X^\vee$$

be its Gauss morphism. For a tangent hyperplane $\eta = [W] \in X^\vee$, let Γ_η be the scheme-theoretic fibre and $Z = (\Gamma_\eta)_{\text{red}}$. Fix $m \geq 1$, put $H = \mathcal{O}_X(1)$, and define

$$S_Z = \text{Im} \left(H^0(Z, H^m|_Z)^* \longrightarrow H^0(X, H^m)^* \right).$$

The affine order- s osculating block is

$$\widehat{\text{Osc}}_p^s(H^m) = \text{Im} \left(H^0((s+1)p, H^m|_{(s+1)p})^* \longrightarrow H^0(X, H^m)^* \right).$$

Definition 4.1. For $1 \leq s \leq m$, the fibre support Z is *point-span s -osculating-absorbing* if

$$\widehat{\text{Osc}}_p^s(H^m) \subseteq S_Z \quad (p \in Z).$$

We use two established facts from [5, 6]. First, if h_p is a local equation of the tangent section $X \cap W$ at p , then

$$\widehat{\mathcal{O}}_{\Gamma_\eta, p} \cong \mathbb{C}[[z_1, \dots, z_d]]/(h_p, \partial h_p). \quad (4.1)$$

Second, point-span order- s absorption implies

$$h_p \in \mathfrak{m}_p^{s+1} \quad (p \in Z), \quad |Z| \geq N_{d,m} := \binom{d+m}{d}. \quad (4.2)$$

For completeness, the first assertion in (4.2) follows by multiplying a linear equation of W by a section of H^{m-1} that is a unit at p , and then reversing the osculating-span containment under annihilators.

Theorem 4.2 (Euler-reduced Gauss-fibre floor). *Assume that Z is point-span s -osculating-absorbing in the complete H^m embedding, where $1 \leq s \leq m$. Then*

$$\text{length}(\Gamma_\eta) \geq E_{d,s}|Z| \geq E_{d,s} \binom{d+m}{d}. \quad (4.3)$$

For $d = 2$, this becomes

$$\text{length}(\Gamma_\eta) \geq \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor \binom{m+2}{2}. \quad (4.4)$$

Proof. Equations (4.1)–(4.2) and Theorem 2.2 give $\text{length}(\Gamma_\eta) = \sum_{p \in Z} \tau(h_p) \geq E_{d,s}|Z|$. The support floor in (4.2) proves the second inequality. Formula (4.4) follows from Theorem 3.1. $\square$

Already $K = s$ in the local theorem replaces the former factor $\binom{d+s-1}{d}$ by $\binom{d+s}{d} - d$. Thus (4.3) is strictly stronger than the earlier fat-point floor whenever $d, s \geq 2$.

5 Sharp global realization for surfaces

We now show that (4.4) is an exact geometric floor at a conservative degree. The interpolation architecture follows [5]; the new choice is to prescribe the sharp germs from Theorem 3.1.

Theorem 5.1 (Sharp absorbed surface fibres). *Let m, s be integers with $1 \leq s \leq m$, put*

$$N = \binom{m+2}{2}, \quad T_s = \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor,$$

and suppose

$$D \geq (s^2 + 2)N + 1. \quad (5.1)$$

Then there exist a smooth integral surface $X \subset \mathbb{P}^3$ of degree D , a plane W , and $\eta = [W] \in X^\vee$ such that:

1. $Z = (\gamma_X^{-1}(\eta))_{\text{red}}$ consists of exactly N points;
2. $\dim S_Z = N < h^0(X, H^m)$, and Z is point-span s -osculating-absorbing;
3. every local tangent-section germ has multiplicity $s + 1$ and Tjurina number T_s ;

4.

$$\text{length}(\Gamma_\eta) = T_s N.$$

Consequently the bound (4.4) is sharp. The degree threshold (5.1) is sufficient and is not asserted to be minimal.

Proof. Choose N distinct points $Z \subset W \cong \mathbb{P}^2$ for which evaluation is an isomorphism

$$H^0(W, \mathcal{O}_W(m)) \xrightarrow{\sim} H^0(Z, \mathcal{O}_Z(m)). \quad (5.2)$$

Such a set exists by choosing a basis among point-evaluation functionals.

At every point prescribe the germ from Theorem 3.1 with the appropriate parity of $s + 1$. It has Milnor number s^2 , because it is an ordinary multiplicity- $(s + 1)$ plane singularity, and it has Tjurina number T_s . An isolated hypersurface germ is contact $(\tau + 1)$ -determined [8, Corollary 2.24]. It is therefore enough to prescribe jets through order

$$a = s^2 + 1.$$

For distinct points $p_1, \dots, p_N$ and nonnegative jet orders a_i , the elementary separator product gives surjectivity

$$H^0(\mathbb{P}^2, \mathcal{O}(k)) \longrightarrow \bigoplus_i \mathcal{O}_{\mathbb{P}^2, p_i} / \mathfrak{m}_{p_i}^{a_i+1}$$

whenever $k \geq \sum_i (a_i + 1) - 1$: multiply an order- a_i local interpolant by $\prod_{j \neq i} \ell_{ij}^{a_j+1}$, where ℓ_{ij} vanishes at p_j and is nonzero at p_i . Hence a degree- D plane equation with all prescribed a -jets exists under (5.1).

Let f_* be one such equation and

$$V = H^0(W, \mathcal{I}_{(a+1)Z}(D)), \quad \mathcal{A} = f_* + V.$$

For $u \in W \setminus Z$, the product

$$A_u = \prod_{p \in Z} \ell_{p,u}^{a+1}$$

has degree $(a + 1)N = (s^2 + 2)N$, vanishes to the required orders at Z , and is a unit at u . Since (5.1) leaves at least one further degree, V generates first jets at u . The incidence of members singular at some $u \in W \setminus Z$ has fibre codimension $3 > 2 = \dim W$. Consequently the closure of its image in $\mathcal{A}$ is proper. Thus a general $f \in \mathcal{A}$ is smooth away from Z , and finite contact determinacy gives exactly the prescribed germs at Z .

Embed $W = V(y)$ in $\mathbb{P}^3$, lift f independently of y , and put

$$F_G = f + yG, \quad G \in H^0(\mathbb{P}^3, \mathcal{O}(D - 1)).$$

A general G , chosen nonzero at every point of Z , makes $X = V(F_G)$ smooth: on $W \setminus Z$ this follows from the choice of f ; at Z , $dF_G = G(p) dy$; and off W the first-jet incidence has fibre codimension $4 > 3 = \dim \mathbb{P}^3$, so the closure of its image in the space of G 's is proper. The smooth hypersurface is connected and hence integral.

The section $X \cap W = V(f)$ has singular locus exactly Z . A point of X maps to $[W]$ precisely when it lies on W and is singular on that section, so the reduced Gauss fibre is exactly Z . Locally on X , $y = -f/G$, which is contact equivalent to the prescribed germ.

Because (5.1) implies $D > m$, restriction identifies degree- m ambient forms with $H^0(X, H^m)$. If such a form vanishes on Z , (5.2) says its restriction to W is zero, hence it is divisible by y . Since $y|_X$ has order $s + 1$ at every support, the reverse-annihilator argument gives point-span order- s absorption. Equation (5.2) also gives $\dim S_Z = N$, and this span is proper because $h^0(X, H^m) = \binom{m+3}{3} > N$. Finally, (4.1) and the prescribed Tjurina numbers yield $\text{length}(\Gamma_\eta) = T_s N$. $\square$

6 Comparisons, reproducibility, and limitations

Liu proved $\mu/\tau \leq d$ for isolated hypersurface singularities [9]; together with $\mu \geq s^d$, this gives $\tau \geq \lceil s^d/d \rceil$. For plane curves, Almirón proved the strict inequality $\mu/\tau < 4/3$ [1]. These are important comparisons, but they are not used in the proof of Theorem 2.2. The plane bound (3.1) retains an additional linear term that the ratio estimate alone does not recover.

A recent preprint of Ma–Zuo [10] announces sharper universal dimension-dependent asymptotic ratios in several dimensions. We do not use that preprint, and we do not treat it as externally consolidated. The present estimate is instead a direct, finite-jet generator count and gives an exact geometric answer in dimension two.

The accompanying replays use exact integer combinatorics, exact $\mathbb{Q}$ -Gröbner bases, and sparse modular linear algebra. They check the optimized formula for $1 \leq d \leq 8$, $1 \leq s \leq 20$, verify strict improvement over the former fat-point floor on that grid, and evaluate local quotient fixtures over two large prime fields. For the sharp plane family, exact Gröbner bases over $\mathbb{Q}$ verify $1 \leq s \leq 12$. These computations certify only the displayed fixtures; the proofs above establish the general statements.

There are three deliberate limitations. First, the exact global realization is proved only for surfaces; the higher-dimensional floor need not be sharp. Second, the degree in (5.1) is a transparent separator bound, not an optimized interpolation threshold. Third, no positive-characteristic version is claimed because Euler reduction divides by $s + 1$.

7 Conclusion

Euler cancellation exposes one further vanishing order in the Tjurina ideal. Optimized after truncation, that single order raises the local and global Gauss-fibre floors substantially. In the plane it yields the exact value

$$\left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor$$

per support, and prescribed jets realize equality simultaneously at the smallest support allowed by point-span absorption. The higher-dimensional quantity $E_{d,s}$, its possible sharpness, and lower degree realizations remain open.

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