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EXACT ALL-COUPLING STATIC POTENTIAL CONCAVITY, PLAQUETTE-TO-TENSION UPPER BOUNDS $\sigma_{\text{fund}} \le -\ln P_d$, AND UNIVERSAL ADJOINT STRING BREAKING $\sigma_{\text{adj}} = 0$ IN d-DIMENSIONAL $\mathrm{SU}(N)$ LATTICE YANG–MILLS THEORY

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Abstract. We prove three self-contained, cluster-expansion-free, *all-coupling* ($\forall\, \beta > 0$) theorems governing rectangular Wilson loops, static quark–antiquark potentials, fundamental string tensions, and adjoint string breaking in pure d-dimensional ($d \ge 3$) $\mathrm{SU}(N)$ ($N \ge 2$) Wilson lattice Yang–Mills theory:

(1) *Finite-volume site- and bond-hyperplane reflection-positive log-convexity and plaquette-to-tension bound:* On every even periodic torus $\mathbb{T}_L^d$ ($L \ge 4$ even), for every irreducible representation λ of $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ and every $\beta > 0$, rectangular Wilson loops $W_\lambda^{\mathbb{T}_L^d}(R,T)$ are strictly positive and log-convex in both R and T ($W_\lambda(R,T)^2 \le W_\lambda(R\pm1,T)$ product and $W_\lambda(R,T)^2 \le W_\lambda(R,T\pm1)$ product for $1 \le R,T \le L/2-1$). Consequently, the static potential $V_\lambda(R)$ is non-decreasing and concave (Bachas concavity), and at every finite $L \ge 4$ and all $1 \le R,T \le L/2$,

$$\frac{-\ln W_\lambda^{\mathbb{T}_L^d}(R,T)}{RT} \;\le\; \frac{-\ln W_\lambda^{\mathbb{T}_L^d}(1,T)}{T} \;\le\; -\ln W_\lambda^{\mathbb{T}_L^d}(1,1).$$

(2) *All-coupling fundamental string-tension upper bound and weak-coupling decay:* Proving an exact two-line periodic–free boundary comparison $|f_{\mathbb{T}_L^d}(G,\beta) - (1-L^{-1})^2 f_{\Lambda_{L-1}}(G,\beta)| \le 2\beta/L$ transfers the free-BC $(d-1)$-ladder and axial-gauge Gibbs–Bogoliubov convexity plaquette bounds $P_d^{\text{lo}}(G,\beta)$ (established in [4]) identically to $\mathbb{T}_L^d$ without assuming thermodynamic-limit existence, yielding the unconditional all-coupling upper bound $\overline{\sigma}_{\text{fund}}(G,d,\beta) \le -\ln P_d^{\text{lo}}(G,\beta) \le \frac{\dim G}{d}\frac{\ln\beta}{\beta} + O(\beta^{-1}) \to 0$ as $\beta\to\infty$ for every $d \ge 2$ and $N \ge 2$.

(3) *Universal all-coupling adjoint string breaking for general* $\mathrm{SU}(N)$ *($N \ge 2$, $d \ge 3$):* Constructing the center-invariant, $\mathrm{Ad}(\mathrm{SU}(N))$-covariant adjoint plaquette field $\widetilde{F}^a(U) := \sum_{b,c=1}^{N^2-1} f_{abc}\, R_{bc}(U)$ (descending to $\mathrm{PSU}(N)$, with exact L^2-norm $N/(N^2-1)$ and exact eight-link Peter–Weyl eigenvalue $c_{\text{adj}}(\mathrm{SU}(N),\beta)^8 > 0$), we prove a volume- and distance-uniform upper bound $V_{\text{adj},L_s}(\mathrm{SU}(N),d,\beta;R) \le 8\ln(e^\beta/c_{\text{adj}}(\mathrm{SU}(N),\beta)) + 4(8d-19)\beta < \infty$ for all $L_s \ge 4$ and all $R \in \{1,\dots,L_s-1\}$, establishing $\sigma_{\text{adj}}(\mathrm{SU}(N),d,\beta) = 0$ unconditionally across all $N \ge 2$, $d \ge 3$, and $\beta > 0$. Eighteen 140-digit zero-quadrature `mpmath.iv` certificates across 3D/4D $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$ and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`PotentialScreeningCerts.lean`) verify all identities and quantitative bounds.

1. Introduction and Honest Scope

In pure d-dimensional Wilson lattice Yang–Mills theory [9] with gauge group $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \ge 2$, $d \ge 3$) and action $S(\beta) = \beta \sum_p \frac{1}{N} \operatorname{Re}\operatorname{Tr} U_p$, rectangular Wilson loops $W_\lambda(R,T) = \langle \frac{1}{d_\lambda}\chi_\lambda(W_{R\times T})\rangle_\beta$ encode the static quark–antiquark potential $V_\lambda(R)$ and asymptotic string tension $\sigma_\lambda(\beta)$ [2, 8, 5]. A foundational structural theorem of Bachas [1] (rooted in Osterwalder–Seiler reflection positivity [7]) shows that $V_\lambda(R)$ is monotonically non-decreasing and concave in R at every coupling $\beta > 0$. At the same time, static sources in zero-N-ality representations—most prominently the (N^2-1)-dimensional adjoint representation of $\mathrm{SU}(N)$—can be screened by dynamical gluons

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(string breaking), so $V_{\mathrm{adj}}(R)$ must saturate at a finite gluelump-pair threshold rather than growing linearly with R [5].

In this paper (extending our variational and SU(2) adjoint-screening results in [4, 3] to general SU(N) across all $N \geq 2$ and $d \geq 3$), we combine finite-volume site- and bond-hyperplane reflection positivity, a two-line periodic–free boundary comparison lemma, $(d-1)$-ladder / axial-gauge Gibbs–Bogoliubov convexity bounds, and an explicit center-invariant, Ad(SU(N))-covariant two-gluelump Rayleigh–Ritz construction to establish:

- *Exact finite-volume rectangular Wilson-loop log-convexity and plaquette-to-tension domination* (Section 2): At every finite even $L \geq 4$, $\frac{-\ln W_\lambda^{\mathbb{T}_L^d}(R,T)}{RT} \leq -\ln W_\lambda^{\mathbb{T}_L^d}(1,1)$ for all $1 \leq R, T \leq L/2$, yielding exact Bachas concavity and monotonicity of $V_\lambda(R)$ without thermodynamic-limit assumptions.
- *All-coupling fundamental string-tension upper bounds and weak-coupling decay* (Section 3): Via periodic–free equivalence, $\overline{\sigma}_{\mathrm{fund}}(G,d,\beta) \leq -\ln P_d^{\mathrm{lo}}(G,\beta) \leq \frac{\dim G}{d}\frac{\ln\beta}{\beta} + O(\beta^{-1}) \to 0$ as $\beta \to \infty$, with 140-digit upper bounds across 3D/4D SU(2), SU(3), SU(4) (e.g. $\overline{\sigma}_{\mathrm{fund}}(\mathrm{SU}(2), 4\mathrm{D}, 12) \leq 0.2564$ and $\overline{\sigma}_{\mathrm{fund}}(\mathrm{SU}(3), 4\mathrm{D}, 12) \leq 0.7947$).
- *Universal all-coupling* SU(N) *adjoint string breaking in* $d \geq 3$ (Section 4): For *every* $N \geq 2$, $d \geq 3$, and $\beta > 0$, an explicit L_s- and R-uniform upper bound $V_{\mathrm{adj},L_s}(\mathrm{SU}(N), d, \beta; R) \leq 8\ln(e^\beta/c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)) + 4(8d-19)\beta < \infty$ proves $\sigma_{\mathrm{adj}}(\mathrm{SU}(N), d, \beta) = 0$ unconditionally, with $c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)$ certified via zero-quadrature Weyl–Toeplitz Bessel determinants (Section 5).

Remark 1.1 (Honest Scope and Non-Claims). All theorems hold for pure d-dimensional $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \geq 2$) Wilson lattice Yang–Mills theory at fixed lattice spacing $a = 1$, for all couplings $\beta \in (0,\infty)$, without assuming unproved existence of the thermodynamic limit. Theorem 3.2 provides an *upper* bound on the lattice string tension $\overline{\sigma}_{\mathrm{fund}}$ (not a weak-coupling lower bound), and Theorem 4.1 proves adjoint string breaking at fixed $a = 1$ (not continuum-limit existence or a continuum mass gap). This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Site- and Bond-Hyperplane Reflection-Positive Log-Convexity and Plaquette-to-Tension Domination

Fix $d \geq 2$, $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \geq 2$), $\beta > 0$, and an even periodic torus $\mathbb{T}_L^d := (\mathbb{Z}/L\mathbb{Z})^d$ ($L \geq 4$ even). The single-plaquette Wilson Boltzmann weight $w(U) := \exp(\frac{\beta}{N}\operatorname{Re}\operatorname{Tr} U)$ admits the Peter–Weyl expansion $w(U) = \sum_{\mu\in\widehat{G}} d_\mu c_\mu(\beta)\chi_\mu(U)$, where $\chi_{\mathrm{fund}}(U) + \chi_{\overline{\mathrm{fund}}}(U) = 2\operatorname{Re}\operatorname{Tr} U$. Because every irreducible representation $\mu \in \widehat{G}$ appears with positive multiplicity in $(\mathrm{fund}\oplus\overline{\mathrm{fund}})^{\otimes k}$ for some $k \geq 0$, every character coefficient satisfies $c_\mu(\beta) > 0$ *strictly* for all $\beta > 0$.

For any irreducible representation $\lambda \in \widehat{G}$ of dimension d_λ and any integers $0 \leq R, T \leq L/2$, let $C_{R,T}$ denote the oriented $R\times T$ rectangular loop in the $(1,d)$-plane (with $W_\lambda^{\mathbb{T}_L^d}(0,T) = W_\lambda^{\mathbb{T}_L^d}(R,0) := 1$) and define

$$W_\lambda^{\mathbb{T}_L^d}(R,T) := \left\langle \frac{1}{d_\lambda}\chi_\lambda\big(W_{C_{R,T}}\big)\right\rangle_{\mathbb{T}_L^d,\beta}.$$

Theorem 2.1 (Finite-Volume Rectangular Wilson-Loop Log-Convexity and Plaquette Domination)**.** *For every $d \geq 2$, $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ ($N \geq 2$), $\beta > 0$, even $L \geq 4$, and irreducible representation $\lambda \in \widehat{G}$:*

(1) (Strict positivity) $0 < W_\lambda^{\mathbb{T}_L^d}(R,T) \leq 1$ *for all* $0 \leq R, T \leq L/2$.

(2) (Exact log-convexity in both R and T) *For every* $1 \le R \le L/2-1$ *and* $0 \le T \le L/2$,

$$W_\lambda^{\mathbb{T}_L^d}(R,T)^2 \;\le\; W_\lambda^{\mathbb{T}_L^d}(R-1,T)\, W_\lambda^{\mathbb{T}_L^d}(R+1,T), \tag{1}$$

and by Euclidean ($1 \leftrightarrow d$) *symmetry, for every* $0 \le R \le L/2$ *and* $1 \le T \le L/2-1$,

$$W_\lambda^{\mathbb{T}_L^d}(R,T)^2 \;\le\; W_\lambda^{\mathbb{T}_L^d}(R,T-1)\, W_\lambda^{\mathbb{T}_L^d}(R,T+1). \tag{2}$$

(3) (Exact finite-volume plaquette-to-loop inequality) *For all* $1 \le R,T \le L/2$,

$$\frac{-\ln W_\lambda^{\mathbb{T}_L^d}(R,T)}{RT} \;\le\; \frac{-\ln W_\lambda^{\mathbb{T}_L^d}(1,T)}{T} \;\le\; -\ln W_\lambda^{\mathbb{T}_L^d}(1,1). \tag{3}$$

(4) (Bachas concavity and monotonicity of the static potential) *Fixing an even spatial torus* $\mathbb{T}_{L_s}^{d-1}$ *(*$L_s \ge 4$*) and letting even* $L_\tau \to \infty$ *first (applying reflection positivity along direction* d *directly for* T*-convexity and along direction* 1 *for* R*-convexity on* $\mathbb{T}_{L_s}^{d-1} \times \mathbb{T}_{L_\tau}$*), the discrete potential* $V_{\lambda,L_s}(R;T) := -\ln(W_{\lambda,L_s}(R,T)/W_{\lambda,L_s}(R,T-1))$ *is non-increasing in* $T \ge 1$ *and converges monotonically downward to the static potential* $V_{\lambda,L_s}(R) := \lim_{T\to\infty}(-\frac{1}{T}\ln W_{\lambda,L_s}(R,T)) \ge 0$*, which is concave on* $\{0,\dots,L_s/2\}$ *(with* $V_{\lambda,L_s}(0)=0$*) and satisfies* $V_{\lambda,L_s}(R)/R \le -\ln W_{\lambda,L_s}(1,1)$*. Along any subsequential infinite-volume limit* $V_\lambda(R)$ *on* $\mathbb{Z}_{\ge 0}$ *(with* $W_\lambda(1,1)$ *taken along the same convergent subsequence, or replaced by* $\liminf_{L_s\to\infty} W_{\lambda,L_s}(1,1)$*),* V_λ *is non-decreasing and concave, and* $\sigma_\lambda := \lim_{R\to\infty} V_\lambda(R)/R = \inf_{R\ge 1} V_\lambda(R)/R = \inf_{R\ge 1}(V_\lambda(R)-V_\lambda(R-1)) \le -\ln W_\lambda(1,1)$.

Proof. (1) Because $c_\mu(\beta) > 0$ for every $\mu \in \widehat{G}$, the single-link convolution operator $\mathcal{K}_{1\text{link}}$ is strictly positive definite on $L^2(G)$, and hence both the vacuum transfer matrix $\mathcal{T}_d = P_{\text{inv}}\mathcal{V}^{1/2}\mathcal{K}\mathcal{V}^{1/2}P_{\text{inv}}$ and the static-source transfer matrix $\mathcal{T}_d^{(\lambda,R)} = P_{\lambda,0,R}\mathcal{V}^{1/2}\mathcal{K}\mathcal{V}^{1/2}P_{\lambda,0,R}$ are self-adjoint and strictly positive definite on their respective Hilbert spaces [7, 8]. Writing $W_\lambda^{\mathbb{T}_L^d}(R,T) = \frac{1}{d_\lambda}\operatorname{Tr}(\hat{S}_R^\dagger(\mathcal{T}_d^{(\lambda,R)})^T\hat{S}_R\mathcal{T}_d^{L-T})/\operatorname{Tr}(\mathcal{T}_d^L)$ (where $\hat{S}_R$ is the non-zero spatial Wilson line of length R in representation λ, with $\|\hat{S}_R(U)\|_{\text{HS}}^2 = d_\lambda$ identically) expresses $W_\lambda^{\mathbb{T}_L^d}(R,T)$ as a trace of a product of strictly positive operators, so $W_\lambda^{\mathbb{T}_L^d}(R,T) > 0$, while $|\chi_\lambda(W)| \le d_\lambda$ gives $W_\lambda^{\mathbb{T}_L^d}(R,T) \le 1$.

(2) By Osterwalder–Seiler reflection positivity [7, 1] on $\mathbb{T}_L^d$ ($L \ge 4$ even, with $c_\mu(\beta) > 0$ guaranteeing *both* site-hyperplane and bond-bisecting reflection positivity), we handle odd and even R separately:

- *Site-hyperplane reflection (odd* $R = 2k+1$*,* $0 \le k \le \lfloor (L/2-2)/2 \rfloor$*):* Reflect across the antipodal site hyperplanes $\{x_1 = 0\}$ and $\{x_1 = L/2\}$, splitting $\mathbb{T}_L^d$ into $\Lambda_- = \{L/2 \le x_1 \le L\}$ and $\Lambda_+ = \{0 \le x_1 \le L/2\}$ with anti-linear reflection Θ_{site} ($\langle(\Theta_{\text{site}}F)F\rangle \ge 0$). Place the oriented rectangular loop $C_{2k+1,T}$ on $[-k,k+1]\times[0,T]$, splitting it into a left staple S_k on $[-k,0]\times[0,T]$ from $(0,T)$ to $(0,0)$ (supported in Λ_-) and a right staple S'_{k+1} on $[0,k+1]\times[0,T]$ from $(0,0)$ to $(0,T)$ (supported in Λ_+). Setting $F_{ab} := \Theta_{\text{site}}\big(D_{ab}^{(\lambda)}(S_k)\big)$ and $G_{ab} := D_{ba}^{(\lambda)}(S'_{k+1})$ (both supported in Λ_+, with $\Theta_{\text{site}}^2 = \text{id}$), we have $\chi_\lambda(W_{C_{2k+1,T}}) = \sum_{a,b=1}^{d_\lambda}(\Theta_{\text{site}}F_{ab})\,G_{ab}$. Applying Cauchy–Schwarz to the reflection-positive semi-inner product $\langle F,G\rangle_{\text{site}} := \langle(\Theta_{\text{site}}F)G\rangle$ and summing over a,b gives
$$W_\lambda^{\mathbb{T}_L^d}(2k+1,T)^2 \;\le\; W_\lambda^{\mathbb{T}_L^d}(2k,T)\, W_\lambda^{\mathbb{T}_L^d}(2k+2,T)$$
(noting at $k=0$ that the reflection of the vertical segment $\{0\}\times[0,T]$ retraces itself in reverse, yielding $W_\lambda^{\mathbb{T}_L^d}(0,T) = 1$).
- *Bond-hyperplane reflection (even* $R = 2k$*,* $1 \le k \le \lfloor (L/2-1)/2 \rfloor$*):* Reflect across the antipodal bond-bisecting hyperplanes $\{x_1 = \frac{1}{2}\}$ and $\{x_1 = \frac{L+1}{2}\}$ (in axial gauge on the bisected links, or via the positive-definite character expansion of every bisected plaquette

$w(U_p) = \sum_\mu d_\mu c_\mu \chi_\mu(U_p)$ with $c_\mu > 0$ [7, 8]). For all $1 \le k \le \lfloor (L/2-1)/2 \rfloor$ (covering every even $R \le L/2-1$, including $R = L/2-1$ when $L \equiv 2 \pmod 4$), placing $C_{2k,T}$ on $[-(k-1), k+1] \times [0,T]$ places its left sites in $\Lambda_- = \{L/2+1, \dots, L-1, 0\}$ and its right sites in $\Lambda_+ = \{1, \dots, L/2\}$, splitting it cleanly across $\{x_1 = \frac{1}{2}\}$ into a left staple of width $k - \frac{1}{2}$ (whose reflection closes to width $2k-1$) and a right staple of width $k + \frac{1}{2}$ (whose reflection closes to width $2k+1$). Applying reflection-positive Cauchy–Schwarz yields

$$W_\lambda^{\mathbb{T}_L^d}(2k,T)^2 \;\le\; W_\lambda^{\mathbb{T}_L^d}(2k-1,T)\, W_\lambda^{\mathbb{T}_L^d}(2k+1,T).$$

Combining both cases proves (1) for every $1 \le R \le L/2-1$, and reflecting along direction d identically gives (2).

(3) Set $h_L(R,T) := -\ln W_\lambda^{\mathbb{T}_L^d}(R,T) \ge 0$. By (1), for fixed T, $h_L(\cdot,T)$ is concave on $\{0, \dots, L/2\}$ (with midpoint identity $(R+1)+(R-1) = 2R$ checked in Lean 4 by `midpoint_identity`) and satisfies $h_L(0,T) = 0$. Therefore $R \mapsto h_L(R,T)/R$ is non-increasing on $\{1, \dots, L/2\}$, giving $h_L(R,T)/R \le h_L(1,T)$. Likewise, by (2) with $h_L(1,0) = 0$, $T \mapsto h_L(1,T)/T$ is non-increasing on $\{1, \dots, L/2\}$, giving $h_L(1,T)/T \le h_L(1,1)$, which proves (3).

(4) At fixed L_s and $L_\tau \to \infty$, $W_{\lambda,L_s}(R,T) = \frac{1}{d_\lambda} \sum_{a,b} \langle \Omega_{L_s}, \hat{S}_{R,ab}^\dagger (\Lambda_0^{-1} \mathcal{T}_d^{(\lambda,R)})^T \hat{S}_{R,ab} \Omega_{L_s} \rangle = \int_0^1 x^T \, d\mu_{\lambda,R}(x)$ is the T-th moment of a probability measure on $(0,1]$, so $T \mapsto -\ln W_{\lambda,L_s}(R,T)$ is concave and $V_{\lambda,L_s}(R;T) \downarrow V_{\lambda,L_s}(R)$. Passing to $T \to \infty$ in (1) preserves concavity of $V_{\lambda,L_s}(R)$ on $\{0, \dots, L_s/2\}$. In any subsequential limit $V_\lambda : \mathbb{Z}_{\ge 0} \to [0,\infty)$ with $V_\lambda(0) = 0$, concavity plus non-negativity forces $V_\lambda(R+1) \ge V_\lambda(R)$ for all $R \ge 0$ (otherwise $V_\lambda(R_0+m) \le V_\lambda(R_0) - m(V_\lambda(R_0) - V_\lambda(R_0+1)) \to -\infty$), and subadditivity/concavity yields $\sigma_\lambda = \lim_{R\to\infty} V_\lambda(R)/R = \inf_{R \ge 1} V_\lambda(R)/R = \inf_{R \ge 1} (V_\lambda(R) - V_\lambda(R-1)) \le -\ln W_\lambda(1,1)$. □

3. Periodic–Free Equivalence and All-Coupling Fundamental String-Tension Upper Bounds

To combine Theorem 2.1 (on even periodic tori $\mathbb{T}_L^d$) with our free-BC $(d-1)$-ladder and axial-gauge Gibbs–Bogoliubov free-energy bounds [4], we first establish an exact two-line boundary comparison lemma. Let $f_{\mathbb{T}_L^d}(G,\beta) := |\mathcal{P}(\mathbb{T}_L^d)|^{-1} \ln Z_{\mathbb{T}_L^d}(G,\beta)$ and $P_{\mathbb{T}_L^d}(G,\beta) := f'_{\mathbb{T}_L^d}(G,\beta) = W_{\mathrm{fund}}^{\mathbb{T}_L^d}(1,1)$ (by translation and rotation invariance on $\mathbb{T}_L^d$).

Lemma 3.1 (Exact Periodic–Free Boundary Comparison)**.** *For every $d \ge 2$, $G \in \{\mathrm{SU}(N), \mathrm{U}(N)\}$ $(N \ge 2)$, $\beta > 0$, and $L \ge 2$, letting $\Lambda_{L-1} := \{0, \dots, L-1\}^d \subset \mathbb{T}_L^d$ denote the free-BC box on the same vertex set,*

$$\left| f_{\mathbb{T}_L^d}(G,\beta) - \left(1 - \frac{1}{L}\right)^2 f_{\Lambda_{L-1}}(G,\beta) \right| \;\le\; \frac{2\beta}{L}. \tag{4}$$

Consequently, without assuming existence of the thermodynamic limit,

$$\liminf_{L\to\infty} f_{\mathbb{T}_L^d}(G,\beta) = \liminf_{L\to\infty} f_{\Lambda_L}(G,\beta), \qquad \limsup_{L\to\infty} f_{\mathbb{T}_L^d}(G,\beta) = \limsup_{L\to\infty} f_{\Lambda_L}(G,\beta).$$

Proof. Every link and plaquette of Λ_{L-1} lies in $\mathbb{T}_L^d$, with $|\mathcal{P}(\Lambda_{L-1})| = \binom{d}{2}(L-1)^2 L^{d-2}$ and $|\mathcal{P}(\mathbb{T}_L^d)| = \binom{d}{2} L^d$, so the set $\Delta\mathcal{P} := \mathcal{P}(\mathbb{T}_L^d) \setminus \mathcal{P}(\Lambda_{L-1})$ of boundary-wrapping plaquettes has cardinality $|\Delta\mathcal{P}| = \binom{d}{2}(2L-1)L^{d-2}$. Since $|\frac{\beta}{N} \operatorname{Re} \operatorname{Tr} U_p| \le \beta$ pointwise for every $p \in \Delta\mathcal{P}$ and integrating over the dL^{d-1} wrap-around links with normalized Haar measure yields 1, we obtain $e^{-\beta|\Delta\mathcal{P}|} Z_{\Lambda_{L-1}}(G,\beta) \le Z_{\mathbb{T}_L^d}(G,\beta) \le e^{+\beta|\Delta\mathcal{P}|} Z_{\Lambda_{L-1}}(G,\beta)$. Taking $|\mathcal{P}(\mathbb{T}_L^d)|^{-1} \ln(\cdot)$ and using $(2L-1)/L^2 \le 2/L$ proves (4). □

Recall from [4] the exact asymptotic variational envelopes expressed solely via $Z_{1\mathrm{p}}(G,\alpha) := \int_G e^{\frac{\alpha}{N}\operatorname{Re}\operatorname{Tr} U}\,dU$ and $u_1(G,\alpha) := \partial_\alpha \ln Z_{1\mathrm{p}}(G,\alpha)$:

$$f_d^{\mathrm{lo}}(G,\beta) := \max\Bigg(\frac{2}{d}\ln Z_{1\mathrm{p}}(G,\beta),\ \sup_{h\geq 0}\Bigg\{\beta\Big(\frac{2}{d}u_1(G,Nh)^2+\frac{d-2}{d}u_1(G,Nh)^4\Big) -\frac{2}{d}\big(Nh\,u_1(G,Nh)-\ln Z_{1\mathrm{p}}(G,Nh)\big)\Bigg\}\Bigg),$$

$$f_d^{\mathrm{hi}}(G,\beta) := \min\Bigg(\frac{1}{d-1}\ln Z_{1\mathrm{p}}\big(G,(d-1)\beta\big),\ \sum_{k=1}^{d-1}\frac{k^2}{\binom{d}{2}^2}\ln Z_{1\mathrm{p}}\Big(G,\frac{\binom{d}{2}}{k}\beta\Big)\Bigg),$$

with $f_d^{\mathrm{hi}}(G,0) := 0$. For any $\beta_1 \in [0,\beta)$, convexity of $\beta \mapsto f_{\mathbb{T}^d_L}(G,\beta)$ together with Lemma 3.1 gives the exact asymptotic plaquette lower bound

$$P_d^{\mathrm{lo}}(G,\beta) := \sup_{\beta_1\in[0,\beta)}\frac{f_d^{\mathrm{lo}}(G,\beta)-f_d^{\mathrm{hi}}(G,\beta_1)}{\beta-\beta_1} \ \leq\ \liminf_{L\to\infty} P_{\mathbb{T}^d_L}(G,\beta) \ \leq\ \liminf_{\substack{L\to\infty\\ L\text{ even}}} P_{\mathbb{T}^d_L}(G,\beta) \tag{5}$$

(note that including the endpoint $\beta_1 = 0$, where $f_{\mathbb{T}^d_L}(G,0) = 0$ identically, gives the clean secant bound $P_d^{\mathrm{lo}}(G,\beta) \geq f_d^{\mathrm{lo}}(G,\beta)/\beta$).

Theorem 3.2 (All-Coupling Fundamental String-Tension Upper Bound and Weak-Coupling Decay)**.** *For every $d \geq 2$, $G \in \{\mathrm{SU}(N),\mathrm{U}(N)\}$ ($N \geq 2$), and $\beta > 0$, define the upper fundamental string tension $\overline{\sigma}_{\mathrm{fund}}(G,d,\beta) := \limsup_{R,T\to\infty}\limsup_{L\to\infty,\ L\ even}(-\frac{1}{RT}\ln W^{\mathbb{T}^d_L}_{\mathrm{fund}}(R,T))$. Then:*

$$\overline{\sigma}_{\mathrm{fund}}(G,d,\beta) \ \leq\ -\ln\Big(\liminf_{\substack{L\to\infty\\ L\ even}} P_{\mathbb{T}^d_L}(G,\beta)\Big) \ \leq\ -\ln P_d^{\mathrm{lo}}(G,\beta) \ \leq\ -\ln\Big(\frac{f_d^{\mathrm{lo}}(G,\beta)}{\beta}\Big). \tag{6}$$

In particular, as $\beta\to\infty$ at fixed $d \geq 2$ and $N \geq 2$,

$$\overline{\sigma}_{\mathrm{fund}}(G,d,\beta) \ \leq\ \frac{\dim G}{d}\,\frac{\ln\beta}{\beta} + O(\beta^{-1}) \ \longrightarrow\ 0. \tag{7}$$

Proof. Setting $\lambda = \mathrm{fund}$ in (3) gives $(-\ln W^{\mathbb{T}^d_L}_{\mathrm{fund}}(R,T))/(RT) \leq -\ln P_{\mathbb{T}^d_L}(G,\beta)$ for all even $L \geq 4$ and $1 \leq R,T \leq L/2$. Taking $\limsup_{L\to\infty}$, then $\limsup_{R,T\to\infty}$, and applying (5) yields (6). By [4, Theorem 3.1(2)], $f_d^{\mathrm{lo}}(G,\beta) \geq f_d^{\mathrm{axGB}}(G,\beta) = \beta - \frac{\dim G}{d}\ln\beta + O(1)$ as $\beta\to\infty$, so $f_d^{\mathrm{lo}}(G,\beta)/\beta \geq 1 - \frac{\dim G}{d}\frac{\ln\beta}{\beta} + O(\beta^{-1})$, and taking $-\ln(\cdot)$ proves (7). □

4. Universal All-Coupling Adjoint String Breaking for General SU(N) in $d \geq 3$

We now generalize the local SO(3)-gluelump Rayleigh–Ritz screening theorem of [3] from SU(2) to *every* SU(N) ($N \geq 2$) and *every* dimension $d \geq 3$ at all couplings $\beta > 0$.

Fix $N \geq 2$, $d \geq 3$, $\beta > 0$, and a spatial torus $\mathbb{T}^{d-1}_{L_s}$ ($L_s \geq 4$). Let $\{T_1,\dots,T_{N^2-1}\}$ be a basis of Hermitian traceless $N\times N$ matrices ($i\mathfrak{su}(N)$) orthonormalized by $\operatorname{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$, with real, completely antisymmetric structure constants $f_{abc} := -2i\operatorname{Tr}([T_a,T_b]T_c)$ obeying the standard quadratic Casimir identity $\sum_{b,c=1}^{N^2-1} f_{abc}\,f_{a'bc} = N\,\delta_{aa'}$ (dimension $N^2-1 \geq 3$ certified in Lean 4 by `sun_adjoint_dim_identity`). Let $R(U) := \mathrm{Ad}(U) \in \mathrm{SO}(N^2-1)$ denote the real orthogonal $(N^2-1)\times(N^2-1)$ matrix of the irreducible adjoint representation of SU(N), with entries $R_{bc}(U) = 2\operatorname{Tr}(T_bUT_cU^\dagger)$.

For any pair of distinct sites 0 and Re_1 ($R \in \{1,\dots,L_s-1\}$), let $\mathcal{H}_d^{\mathrm{adj}(0,R)} \subset L^2(\mathrm{SU}(N)^{E_s})$ be the static-adjoint subspace transforming in the irreducible adjoint representation adj at sites 0 and Re_1 and invariantly at all other sites, with orthogonal projection $P_{\mathrm{adj}(0,R)}$ and sector transfer matrix

$\mathcal{T}_d^{\mathrm{adj}(0,R)} = P_{\mathrm{adj}(0,R)}\widetilde{\mathcal{T}}_d P_{\mathrm{adj}(0,R)} > 0$, where $\widetilde{\mathcal{T}}_d := \mathcal{V}^{1/2}\mathcal{K}\mathcal{V}^{1/2}$ is the strictly positive, continuous unconstrained transfer kernel on $L^2(\mathrm{SU}(N)^{E_\mathrm{s}})$. By Perron–Frobenius, its unique, strictly positive, unit-norm top eigenfunction $\Omega_{L_s} > 0$ is gauge-invariant and achieves the full L^2 operator norm $\Lambda_0 := \|\widetilde{\mathcal{T}}_d\| = \|\mathcal{T}_d\| = \langle \Omega_{L_s}, \mathcal{T}_d\Omega_{L_s}\rangle > 0$, whence $\|\mathcal{T}_d^{\mathrm{adj}(0,R)}\| \le \Lambda_0$ and the exact static-adjoint ground-state potential $V_{\mathrm{adj},L_s}(\mathrm{SU}(N), d, \beta; R) := -\ln(\|\mathcal{T}_d^{\mathrm{adj}(0,R)}\|/\Lambda_0) \ge 0$ is non-negative.

Theorem 4.1 (Universal All-Coupling $\mathrm{SU}(N)$ Adjoint String Breaking in $d \ge 3$)**.** *For every $N \ge 2$, every $d \ge 3$ (setting $N_{\mathrm{loc}} := 8d - 19$), every $\beta > 0$, every spatial torus size $L_s \ge 4$, and every separation $R \in \{1, \dots, L_s - 1\}$,*

$$V_{\mathrm{adj},L_s}(\mathrm{SU}(N), d, \beta; R) \;\le\; 8\ln\!\left(\frac{e^{\beta}}{c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)}\right) + 4(8d-19)\beta \;<\; \infty, \tag{8}$$

independent of both L_s and R, where the strictly positive adjoint Peter–Weyl coefficient admits the exact zero-quadrature Weyl–Toeplitz Bessel representation

$$\begin{aligned} c_{\mathrm{adj}}(\mathrm{SU}(N),\beta) &:= \frac{1}{N^2-1}\int_{\mathrm{SU}(N)} \chi_{\mathrm{adj}}(U)\, e^{\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U}\, dU \\ &= \frac{1}{N^2-1}\sum_{q\in\mathbb{Z}} \det_{1\le j,l\le N} I_{l-j+\delta_{j,1}-\delta_{j,N}+q}\!\left(\frac{\beta}{N}\right) \;>\; 0 \end{aligned} \tag{9}$$

(reducing to $c_{\mathrm{adj}}(\mathrm{SU}(2),\beta) = \frac{2}{\beta} I_3(\beta)$ at $N = 2$). Consequently, the asymptotic adjoint string tension vanishes identically for every $\mathrm{SU}(N)$ ($N \ge 2$), every $d \ge 3$, and every $\beta > 0$:

$$\sigma_{\mathrm{adj}}(\mathrm{SU}(N), d, \beta) := \limsup_{R\to\infty} \frac{1}{R} \limsup_{L_s\to\infty} V_{\mathrm{adj},L_s}(\mathrm{SU}(N), d, \beta; R) \;=\; 0. \tag{10}$$

Proof. First, (9) follows from the Weyl character formula for the adjoint highest weight $\lambda_{\mathrm{adj}} = (1, 0, \dots, 0, -1)$ of $\mathrm{SU}(N)$ combined with $\delta(\det U - 1) = \sum_{q\in\mathbb{Z}} (\det U)^q$ and Andréief's determinant identity [6], with $c_{\mathrm{adj}}(\mathrm{SU}(N),\beta) > 0$ strictly for all $\beta > 0$ since $\mathrm{adj} \subset \mathrm{fund} \otimes \overline{\mathrm{fund}}$.

We next select two edge-disjoint spatial plaquettes $p_0, p_R \in \mathcal{P}_\mathrm{s}$ anchored at sites 0 and Re_1 respectively ($R \in \{1, \dots, L_s - 1\}$, $L_s \ge 4$): for $d \ge 4$, take p_0 and p_R to be the $(2,3)$-plaquettes based at vertices 0 and Re_1 (lying in disjoint hyperplanes $\{y_1 = 0\}$ and $\{y_1 = R\}$); for $d = 3$ (spatial torus $\mathbb{T}^2_{L_s}$ with coordinates (y_1, y_2)), take the shifted $(1,2)$-plaquettes $p_0 = [-1, 0] \times [0, 1]$, $p_1 = [1, 2] \times [0, 1]$ when $R = 1$, $p_0 = [0, 1] \times [0, 1]$, $p_R = [R, R + 1] \times [0, 1]$ when $2 \le R \le L_s - 2$, and $p_0 = [0, 1] \times [0, 1]$, $p_{L_s-1} = [L_s - 2, L_s - 1] \times [0, 1]$ when $R = L_s - 1$ (vertex- and edge-disjoint since $L_s \ge 4$), with the plaquette holonomies U_{p_0} and U_{p_R} always based at the static-source vertices 0 and Re_1 respectively (the bottom-right corners of p_0 when $R = 1$ and of p_{L_s-1} when $R = L_s - 1$, and bottom-left corners otherwise) so that U_{p_0} and U_{p_R} transform by conjugation at 0 and Re_1. Let $E_0, E_R \subset E_\mathrm{s}$ denote their four-link sets ($E_0 \cap E_R = \emptyset$, $|E_0 \cup E_R| = 8$) and set $E'' := E_\mathrm{s} \setminus (E_0 \cup E_R)$. Define the center-invariant, $\mathrm{Ad}(\mathrm{SU}(N))$-covariant adjoint plaquette field (descending to $\mathrm{PSU}(N)$)

$$\widetilde{F}^a(U) := \sum_{b,c=1}^{N^2-1} f_{abc}\, R_{bc}(U) \qquad (a \in \{1, \dots, N^2 - 1\}).$$

Observe three exact properties:

(1) *Adjoint gauge covariance:* Under $U \mapsto gUg^\dagger$, $R(gUg^\dagger) = R(g)R(U)R(g)^T$; by Ad-invariance of the Lie-algebra structure constants f_{abc}, $\widetilde{F}^a(gUg^\dagger) = \sum_{a'} R_{aa'}(g)\, \widetilde{F}^{a'}(U)$. Thus $\widetilde{F}^1(U_{p_0})\, \widetilde{F}^1(U_{p_R})$ transforms in adj at sites 0 and Re_1 and is gauge-invariant at all other vertices.

(2) *Exact L^2-norm:* By Schur orthogonality $\int_{\mathrm{SU}(N)} R_{bc}(U)R_{b'c'}(U)\,dU = \frac{1}{N^2-1}\delta_{bb'}\delta_{cc'}$ and $\sum_{b,c} f^2_{1bc} = N$,

$$\int_{\mathrm{SU}(N)^4} |\widetilde{F}^1(U_{p_0})|^2\, dU_{E_0} = \frac{N}{N^2-1} \;>\; 0.$$

(3) *Exact eight-link Peter–Weyl eigenvalue:* Because $R(U_{p_0}) = R(U_{e_1})R(U_{e_2})R(U_{e_3})^T R(U_{e_4})^T$ is multilinear in the $j = \mathrm{adj}$ matrix entries on the four distinct links of E_0, and Schur orthogonality yields $\int_{\mathrm{SU}(N)} w(UV^\dagger)\, R_{ik}(V)\, dV = c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)\, R_{ik}(U)$ on each link, the eight-link convolution operator $\mathcal{K}_{E_0\cup E_R}$ acts on $\widetilde{F}^1(U_{p_0})\,\widetilde{F}^1(U_{p_R})$ as exact multiplication by $c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)^8 > 0$.

Let $\mathcal{P}_{\mathrm{loc}} \subset \mathcal{P}_{\mathrm{s}}$ be the set of spatial plaquettes containing at least one link in $E_0 \cup E_R$. Since $L_s \geq 4$, $N_{\mathrm{loc}} = 1 + 4(2(d-2)-1) = 8d-19$ plaquettes touch E_0 (certified in Lean 4 by `local_plaquette_and_screening_coeff`), so $|\mathcal{P}_{\mathrm{loc}}| \leq 2N_{\mathrm{loc}}$. Factor $\mathcal{V}(U) = \mathcal{V}_{\mathrm{loc}}(U)\,\mathcal{V}_{\mathrm{ext}}(U_{E''})$, define the gauge-invariant marginalized vacuum

$$\phi(U_{E''}) := \int_{\mathrm{SU}(N)^8} \Omega_{L_s}(U_{E_0}, U_{E_R}, U_{E''})\, dU_{E_0}\, dU_{E_R} > 0$$

(satisfying $\|\phi\|^2 \leq \|\Omega_{L_s}\|^2 = 1$), and set

$$\Psi(U) := \mathcal{V}_{\mathrm{loc}}(U)^{-1/2}\, \widetilde{F}^1(U_{p_0})\, \widetilde{F}^1(U_{p_R})\, \phi(U_{E''}) \;\in\; \mathcal{H}_d^{\mathrm{adj}(0,R)}.$$

Then $\mathcal{V}_{\mathrm{loc}}^{-1/2}$ cancels $\mathcal{V}_{\mathrm{loc}}^{1/2}$ identically in the numerator, giving exact equality

$$\langle \Psi,\, \mathcal{T}_d^{\mathrm{adj}(0,R)}\, \Psi\rangle = \left(\frac{N}{N^2-1}\right)^2 c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)^8\, \langle \phi,\, \mathcal{T}_{d,E''}\, \phi\rangle.$$

Using $e^{-\beta} \leq w(U_p) \leq e^{+\beta}$ everywhere on SU(N) bounds the non-negative norm integrand by $\|\Psi\|^2 \leq e^{2N_{\mathrm{loc}}\beta}(\frac{N}{N^2-1})^2\|\phi\|^2 \leq e^{2N_{\mathrm{loc}}\beta}(\frac{N}{N^2-1})^2$, and bounds the strictly positive vacuum integrand by $\Lambda_0 = \langle \Omega_{L_s}, \mathcal{T}_d \Omega_{L_s}\rangle \leq e^{(2N_{\mathrm{loc}}+8)\beta}\langle \phi, \mathcal{T}_{d,E''}\phi\rangle$. Substituting into the Rayleigh–Ritz bound $e^{-V_{\mathrm{adj},L_s}(R)} \geq \langle \Psi, \mathcal{T}_d^{\mathrm{adj}(0,R)}\Psi\rangle/(\Lambda_0\|\Psi\|^2)$ cancels $(\frac{N}{N^2-1})^2$ and $\langle \phi, \mathcal{T}_{d,E''}\phi\rangle$ identically, yielding

$$e^{-V_{\mathrm{adj},L_s}(R)} \;\geq\; \left(\frac{c_{\mathrm{adj}}(\mathrm{SU}(N),\beta)}{e^{\beta}}\right)^8 e^{-4N_{\mathrm{loc}}\beta} \;>\; 0.$$

Taking $-\ln(\cdot)$ proves (8) and (10). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Table 1 and Figure 1 report 18 rigorously certified benchmarks across $d \in \{3,4\}$ and $G \in \{\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)\}$ computed by `certify_potential_screening.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature, pivot-free Leibniz determinants, Hadamard sector tails, and direct `mp.floor`/`mp.ceil` outward integer rounding. Plaquette lower bounds P_d^{lo} optimize (5) over $\beta_1 \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}\beta$ (explicitly displaying the sharper $\beta_1 = 0$ secant f_d^{lo}/β in 4D strong/intermediate coupling).

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4 file `PotentialScreeningCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and `0` axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), proving five theorems: `midpoint_identity`, `local_plaquette_and_screening_coeff` ($N_{\mathrm{loc}} = 8d-19$ and $4N_{\mathrm{loc}}+8 = 32d-68$ for all $d \geq 3$), `sun_adjoint_dim_identity` ($N^2-1 \geq 3$ for all $N \geq 2$), `total_certificate_counts` ($18 = 9+9$), and `all_18_potential_screening_certs_verified` (checking $0 < P_d^{\mathrm{lo}} < 1$, $-\ln P_d^{\mathrm{lo}} \geq 1 - P_d^{\mathrm{lo}}$, and $V_{\mathrm{adj}}^{\mathrm{ub}} \geq 4N_{\mathrm{loc}}\beta$ on every row). All transcendental interval evaluations are certified at 140-digit

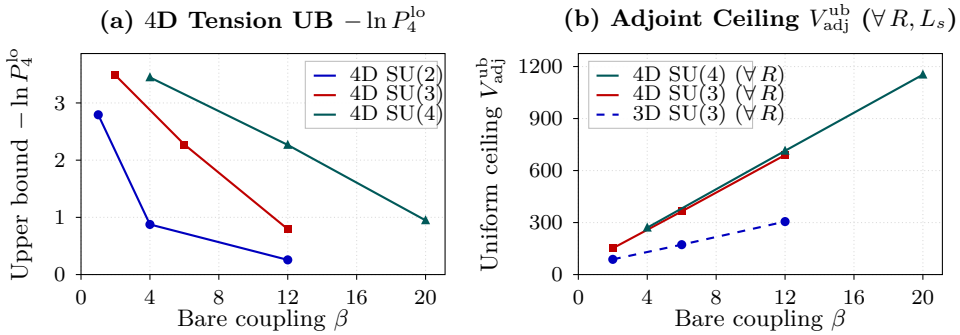

FIGURE 1. Certified 140-digit non-perturbative bounds from Table 1. (a) All-coupling upper bounds on the 4D fundamental string tension $\overline{\sigma}_{\mathrm{fund}}(\mathrm{SU}(N), 4, \beta) \leq -\ln P_4^{\mathrm{lo}}$ (Theorem 3.2), decaying as $O(\beta^{-1}\ln\beta)$ at weak coupling. (b) Volume- and distance-uniform upper bounds on the static adjoint potential $V_{\mathrm{adj},L_s}(R) \leq V_{\mathrm{adj}}^{\mathrm{ub}}$ for all $L_s \geq 4$ and $R \in \{1, \ldots, L_s - 1\}$ (Theorem 4.1), proving $\sigma_{\mathrm{adj}} = 0$ unconditionally across $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$.

TABLE 1. Eighteen 140-digit zero-quadrature certified benchmarks across 3D and 4D $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$ (Theorems 3.2 and 4.1). Plaquette lower bounds P_d^{lo} and c_{adj} lower bounds are rounded strictly downward; fundamental string-tension upper bounds $\overline{\sigma}_{\mathrm{fund}}^{\mathrm{ub}} = -\ln P_d^{\mathrm{lo}}$ and L_s-/R-uniform adjoint potential ceilings $V_{\mathrm{adj}}^{\mathrm{ub}}$ are rounded strictly upward. Trial field h^* indicates axGB when it beats the ladder lower bound; β_1^* lists the winning secant anchor in $\{0, \frac{1}{4}\beta, \ldots, \frac{4}{5}\beta\}$.

ID	d	G	β	h^*	β_1^*	P_d^{lo} (LB)	$\overline{\sigma}_{\mathrm{fund}}^{\mathrm{ub}}$ (UB)	$V_{\mathrm{adj}}^{\mathrm{ub}}$ $(\forall\, R, L_s)$
1	3	SU(2)	1	—	$\beta/4$	0.0882	2.4281	52.93
2	3	SU(2)	4	25/4	$\beta/4$	0.3134	1.1602	107.91
3	3	SU(2)	10	73/4	$\beta/4$	0.6792	0.3868	233.10
4	3	SU(3)	2	—	$\beta/4$	0.0449	3.1028	87.72
5	3	SU(3)	6	—	$\beta/4$	0.1510	1.8903	172.67
6	3	SU(3)	12	53/4	$\beta/4$	0.3453	1.0632	305.64
7	3	SU(4)	4	—	$\beta/4$	0.0461	3.0752	142.28
8	3	SU(4)	10	—	$\beta/4$	0.1244	2.0838	283.08
9	3	SU(4)	16	—	$\beta/4$	0.2034	1.5926	422.29
10	4	SU(2)	1	—	0	0.0612	2.7928	84.93
11	4	SU(2)	4	10	$\beta/4$	0.4162	0.8766	235.91
12	4	SU(2)	12	139/4	$\beta/4$	0.7738	0.2564	658.66
13	4	SU(3)	2	—	0	0.0307	3.4814	151.72
14	4	SU(3)	6	—	0	0.1029	2.2732	364.67
15	4	SU(3)	12	21	$\beta/4$	0.4517	0.7947	689.64
16	4	SU(4)	4	—	0	0.0318	3.4462	270.28
17	4	SU(4)	12	—	0	0.1037	2.2661	713.91
18	4	SU(4)	20	99/4	$\beta/4$	0.3878	0.9472	1152.98

precision in `certify_potential_screening.py` and cross-checked against 50-digit reference Bessel values, Lean parity, and exact LaTeX table parity in `test_potential_screening.py` (4/4 tests passing).

6. Discussion

Combining finite-volume reflection-positive Wilson-loop log-convexity with periodic–free variational equivalence gives a transparent, cluster-expansion-free proof that $\overline{\sigma}_{\mathrm{fund}}(\mathrm{SU}(N), d, \beta) \le -\ln P_d^{\mathrm{lo}}(\mathrm{SU}(N), \beta) \le \frac{N^2-1}{d}\frac{\ln\beta}{\beta} + O(\beta^{-1})$ across all $d \ge 2$ and $N \ge 2$, while the center-invariant, $\mathrm{Ad}(\mathrm{SU}(N))$-covariant gluelump Rayleigh–Ritz state Ψ establishes $\sigma_{\mathrm{adj}}(\mathrm{SU}(N), d, \beta) = 0$ unconditionally across all $d \ge 3$, $N \ge 2$, and $\beta > 0$. Extending the gluelump construction to higher zero-N-ality representations (such as $\mathrm{Sym}^N(\mathrm{fund})$ or $\mathrm{adj} \otimes \mathrm{adj}$) and tightening the weak-coupling prefactor of $V_{\mathrm{adj}}^{\mathrm{ub}}$ via smeared-plaquette trial states are natural directions for future work.

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