AIRR.SCIENCE · ARR-2026-5VB0J0CEGG8RE990 v1 [Mathematical Physics] 29 Nov 2026

EXACT EFFECTIVE POLYAKOV-LOOP REDUCTION, DOBRUSHIN SCREENING MASS GAPS, AND DUAL-REFLECTION PEIERLS DECONFINEMENT PHASE TRANSITIONS IN d-DIMENSIONAL SU(N) LATTICE YANG–MILLS THEORY

LLUIS ERIKSSON

Abstract. We establish a self-contained, constructive, and quantitatively certified analysis of both the low-temperature confining phase and the high-temperature deconfining phase (with spontaneous $\mathbb{Z}_N$ center symmetry breaking) for finite-temperature d-dimensional SU(N) Wilson lattice Yang–Mills theory on $\mathbb{T}^1_{N_\tau} \times \mathbb{Z}^D$ ($D = d-1 \geq 2$, temporal extent $N_\tau \geq 1$) in the strong spatial-coupling effective Polyakov-loop reduction ($\beta_{\mathrm{sp}} = 0$, arbitrary temporal coupling $\beta_t > 0$). Four theorems anchor the analysis:

(1) *Exact temporal-gauge Polyakov reduction and dual reflection positivity:* Integrating out all spatial links yields an exact D-dimensional $\mathbb{Z}_N$-invariant spin system on $\mathrm{SU}(N)^\Lambda$ with strictly positive, positive-definite bond kernel $B_{N_\tau,\beta_t}(W_x, W_y) = \sum_{\lambda \in \widehat{\mathrm{SU}(N)}} r_\lambda(\beta_t)^{N_\tau} \chi_\lambda(W_x)\overline{\chi_\lambda(W_y)}$, which simultaneously satisfies *both* bond-bisecting and site-containing reflection positivity on every even spatial torus $\mathbb{T}^D_L$ ($L \geq 4$ even).

(2) *Low-temperature Dobrushin confinement and screening mass gaps for all* SU(N)*:* Using the exact Dobrushin total-variation cross-ratio bound, we prove that whenever $2D \left\|h_{N_\tau,\beta_t}\right\|_\infty < 1$ (where $\left\|h_{N_\tau,\beta_t}\right\|_\infty = \sum_{\lambda \neq \mathbf{1}} d_\lambda^2 r_\lambda(\beta_t)^{N_\tau}$), the infinite-volume Gibbs measure is unique, preserves $\mathbb{Z}_N$ center symmetry ($\langle \mathrm{Tr}\, W_0 \rangle = 0$), and exhibits exponential Polyakov-loop decay with explicit screening mass $\sigma_T \geq -\ln(2D \left\|h_{N_\tau,\beta_t}\right\|_\infty) > 0$. Exact Parseval identities at $N_\tau = 2$ and exact $\mathbf{3}/\overline{\mathbf{3}}$ Parseval separation for SU(3) at $N_\tau \geq 3$ yield 14 certified low-temperature thresholds across $d \in \{3, 4\}$, $N \in \{2, 3, 4\}$, and $N_\tau \in \{1, 2, 4, 6, 8\}$.

(3) *Uniform-in-N_τ Riccati–Bessel enclosures and unconditional quadratic ground-state bounds for* SU(2)*:* For SU(2) at $\gamma = \beta_t/N_\tau \geq 1000$, we prove an exact product-to-sum kernel representation yielding $B_{N_\tau,\beta_t}(\theta_x, \theta_y) < P_{N_\tau,\beta_t}(0)/(4 \sin\theta_x \sin\theta_y)$, a λ-splitting exponential domain-wall bound via a one-line Riccati monotonicity inequality $\frac{d}{d\beta}(\beta^{1/2} e^{-\beta} I_k(\beta)) > 0$, an unconditional quadratic Chebyshev lower bound $\frac{\sin(k\theta_x)}{k \sin\theta_x} \frac{\sin(k\theta_y)}{k \sin\theta_y} \geq 1 - \frac{k^2-1}{6}(\theta_x^2 + \theta_y^2)$, and closed-form uniform-in-N_τ Gaussian enclosures on P_0, B_0, and c_{quad}.

(4) *Dual-reflection Peierls deconfinement and spontaneous $\mathbb{Z}_2$ symmetry breaking in $d \geq 4$:* Decomposing every separating Peierls contour $\Gamma = \Gamma_{\mathrm{DW}} \sqcup \Gamma_B$ into Type-I domain-wall bonds (bounded via mixed transverse bond-bisecting and longitudinal site-containing chessboard estimates) and Type-II disordered boundary vertices (bounded via D-dimensional bond-bisecting chessboards and the degree bound $|V_B(\Gamma)| \geq |\Gamma_B|/(2D-1)$), combined with a universal codimension-2 ridge prefix-free contour bound $C_{\mathrm{cont}}(D) = (3D-3)^{3D-3}/(3D-4)^{3D-4}$, proves strictly positive long-range Polyakov-loop order $\liminf_{|x|\to\infty} \langle \mathrm{Tr}(W_0)\, \mathrm{Tr}(W_x) \rangle_{\beta_t} > 0$ and spontaneous $\mathbb{Z}_2$ symmetry breaking at explicit finite couplings $\beta_t = \gamma N_\tau$ in 4D ($\gamma = 600{,}000$, $N_\tau \in \{1, 2, 4\}$), 5D ($\gamma = 100{,}000$, $N_\tau \in \{1, 2, 4\}$), and 6D ($\gamma = 55{,}000$, $N_\tau \in \{1, 2\}$), rigorously bracketing the critical deconfinement coupling $\beta_{\mathrm{conf}} \leq \beta_c(D, 2, N_\tau) \leq \beta_{\mathrm{deconf}}$ (with explicit numerical β_{conf} certified in 3D and 4D).

Analytical proofs are given in Sections 2–5, while the algebraic Dobrushin, Chebyshev quadratic-sum, ridge double-counting, and contour-tail monotonicity lemmas together with all 22 quantitative confinement and deconfinement certificates are mechanically verified in a standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel alongside 60-digit interval arithmetic.

Date: November 29, 2026.

Independent Researcher, Östersund, Sweden | `editor@airr.science` | CC-BY-4.0 / MIT | ARR-2026-5VB0J0CEGG8RE990

1. Introduction and Honest Scope

Finite-temperature SU(N) lattice Yang–Mills theory on a periodic Euclidean slab $\mathbb{T}^1_{N_\tau} \times \mathbb{Z}^D$ ($d = D+1$, inverse temperature $N_\tau \geq 1$) possesses a global $\mathbb{Z}_N$ center symmetry acting on Polyakov loops $W_x = \prod_{t=0}^{N_\tau-1} U_{(t,x),\tau}$ by multiplication $W_x \mapsto zW_x$ ($z \in \mathbb{Z}_N$) [11, 13, 14]. In the confining (low-temperature) phase, $\mathbb{Z}_N$ symmetry is unbroken, $\langle \operatorname{Tr} W_0 \rangle = 0$, and the static quark–antiquark free energy grows linearly with spatial ℓ_1 separation $\|x\|_1$, corresponding to a positive finite-temperature screening mass $\sigma_T := \liminf_{\|x\|_1 \to \infty} (-\|x\|_1^{-1} \ln |\langle \operatorname{Tr}(W_0) \overline{\operatorname{Tr}(W_x)} \rangle|) > 0$. In the deconfining (high-temperature) phase in $D = d-1 \geq 2$ spatial dimensions, $\mathbb{Z}_N$ center symmetry breaks spontaneously and the Polyakov-loop correlation exhibits long-range order $\liminf_{|x| \to \infty} \langle \operatorname{Tr}(W_0) \overline{\operatorname{Tr}(W_x)} \rangle > 0$ [11, 13, 3].

While Borgs and Seiler [3] established the existence of a deconfinement transition in finite-temperature lattice gauge theories via reflection-positivity chessboard estimates and asymptotic large-β_t estimates, explicit *non-asymptotic, finite-coupling, machine-certified* phase brackets $\beta_{\text{conf}} \leq \beta_c(D, N, N_\tau) \leq \beta_{\text{deconf}}$ covering both phases simultaneously—with zero hidden asymptotic constants and explicit positive lower bounds on both the confining screening mass σ_T and the deconfining long-range order parameter $\langle \operatorname{Tr}(W_0) \operatorname{Tr}(W_x) \rangle$—remain surprisingly scarce in the constructive literature.

The present paper—the fifth installment of our constructive lattice Yang–Mills series following [8, 7, 6, 5]—delivers a completely explicit, self-contained, and dual-verified constructive theory of both phases for the exact $\beta_{\text{sp}} = 0$ effective Polyakov-loop reduction of d-dimensional SU(N) Wilson lattice Yang–Mills theory. Our main structural innovation on the deconfining side is a *dual-reflection Peierls decomposition*: standard bond-bisecting reflections cannot be applied directly to an indicator of opposite-sector bonds $\mathbf{1}_{\{W_x \in G_+, W_y \in G_-\}}$ (whose matrix across a cut bond is indefinite), while site-containing reflections alone cannot suppress the $O(\gamma^{-1})$ power-law disordered boundary band $B = (\theta_0, \pi - \theta_0)$. By splitting every separating contour $\Gamma = \Gamma_{\text{DW}} \sqcup \Gamma_B$ and applying a *mixed chessboard* (transverse bond-bisecting plus longitudinal site-containing) to the exponentially suppressed domain-wall bonds Γ_{DW}, and a D-dimensional bond-bisecting chessboard to the disordered boundary vertices $V_B(\Gamma)$ (with the sharp degree bound $|V_B(\Gamma)| \geq |\Gamma_B|/(2D-1)$), combined via Hölder's inequality and a prefix-free codimension-2 ridge Peierls contour sum, we achieve explicit finite-β_t deconfinement certificates in 4D, 5D, and 6D across the explicit finite-N_τ regime $N_\tau = O(\gamma)$ where $q_{\text{DW}} \leq q_B$, with N_τ-uniform Riccati–Bessel and disordered-site enclosures.

Honest Scope and Explicit Disclaimers. To avoid any overclaiming, we state the exact scope of our theorems upfront:

- *Strong spatial-coupling reduction ($\beta_{\text{sp}} = 0$) and fixed lattice spacing ($a = 1$):* All results govern the exact effective Polyakov-loop model obtained from d-dimensional SU(N) Wilson lattice Yang–Mills theory on $\mathbb{T}^1_{N_\tau} \times \mathbb{Z}^D$ at spatial plaquette coupling $\beta_{\text{sp}} = 0$ and arbitrary temporal coupling $\beta_t > 0$. This paper does *not* treat $\beta_{\text{sp}} > 0$, does *not* construct the 4D continuum limit $a \to 0$, and makes no claim toward the Clay Millennium Yang–Mills Existence and Mass Gap problem.
- *Deconfinement gauge group and dimension:* Low-temperature confinement (Theorem 3.1) is proved for *all* SU(N) ($N \geq 2$), all $D \geq 1$, and all $N_\tau \geq 1$ (certified numerically for SU(2), SU(3), SU(4) in 3D and 4D). High-temperature deconfinement and spontaneous $\mathbb{Z}_2$ center symmetry breaking (Theorem 5.1) are proved for SU(2) in $d = D + 1 \geq 4$ spacetime dimensions ($D \geq 3$) and certified explicitly in 4D and 5D for $N_\tau \in \{1, 2, 4\}$, and in 6D for $N_\tau \in \{1, 2\}$ (8 certificates total). Note that while the Riccati–Bessel enclosures (Theorem 4.1(4)) and disordered-site parameters r_B, q_B, g_{gs} are uniform in all $N_\tau \geq 1$, the domain-wall prefactor $(1-\lambda)^{-3N_\tau/2}$ in (15) requires $N_\tau \leq O(\gamma)$ at fixed $\gamma = \beta_t/N_\tau$. The

$D = 2$ (3D) contour sum requires planar 1-dimensional Peierls paths and is left for future work.

- *Separation of analytical and mechanized proofs:* Measure-theoretic integration, Peter–Weyl expansions, reflection-positivity chessboard estimates, Dobrushin uniqueness, and torus contour topology are proved analytically in Sections 2–5. The standalone Lean 4.36.0 kernel (`src/lean/PolyakovPhaseCerts.lean`) mechanically verifies the exact algebraic Dobrushin cross-ratio identity, the universal Chebyshev quadratic-sum identity, ridge double counting, contour-tail monotonicity, and all 22 quantitative confinement and deconfinement inequalities in exact integer arithmetic (zero imports, zero `sorry`, zero `Classical.choice`), backed by 60-digit `mpmath.iv` interval arithmetic with zero black-box special functions and zero numerical quadrature.

2. Exact Temporal-Gauge Polyakov Reduction and Dual Reflection Positivity

Let $G = \mathrm{SU}(N)$ ($N \geq 2$) equipped with normalized Haar measure dU, and let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations λ of G, with dimension d_λ and character $\chi_\lambda(U) = \operatorname{Tr} \rho^\lambda(U)$ normalized by $\int_G |\chi_\lambda(U)|^2 \, dU = 1$. Consider the finite Euclidean lattice $\mathbb{T}^1_{N_\tau} \times \Lambda$, where $\mathbb{T}^1_{N_\tau} = \mathbb{Z}/N_\tau\mathbb{Z}$ ($N_\tau \geq 1$) is the periodic temporal circle and $\Lambda = (\mathbb{Z}/L\mathbb{Z})^D = \mathbb{T}^D_L$ (L even, $D = d - 1 \geq 1$) is a finite spatial torus with undirected edge set $E(\Lambda)$. Assign a gauge variable $U_{(t,x),\mu} \in G$ to each oriented link, with temporal plaquette coupling $\beta_t > 0$ and spatial plaquette coupling $\beta_{\mathrm{sp}} = 0$. The single-plaquette Wilson weight is

$$w_{\beta_t}(U) := \exp\left(\frac{\beta_t}{N} \operatorname{Re} \operatorname{Tr} U\right) = \sum_{\lambda \in \widehat{G}} c_\lambda(\beta_t) \, \chi_\lambda(U), \qquad c_\lambda(\beta_t) := \int_G w_{\beta_t}(U) \overline{\chi_\lambda(U)} \, dU. \tag{1}$$

Denote the normalized character ratios by

$$r_\lambda(\beta_t) := \frac{c_\lambda(\beta_t)}{d_\lambda \, c_{\mathbf{1}}(\beta_t)}, \qquad c_{\mathbf{1}}(\beta_t) = \int_G w_{\beta_t}(U) \, dU =: Z_{\mathrm{SU}(N)}\left(\frac{\beta_t}{N}\right). \tag{2}$$

For each spatial site $x \in \Lambda$, the Polyakov loop is the ordered temporal product $W_x := \prod_{t=0}^{N_\tau - 1} U_{(t,x),\tau}$ in G.

Theorem 2.1 (Exact Effective Polyakov-Loop Action and Dual Reflection Positivity)**.** *For every $N \geq 2$, $N_\tau \geq 1$, $\beta_t > 0$, and finite spatial graph Λ with edge set $E(\Lambda)$:*

(1) Strict positivity of all character ratios: *Every $\lambda \in \widehat{\mathrm{SU}(N)}$ satisfies $c_\lambda(\beta_t) > 0$ and $0 < r_\lambda(\beta_t) < 1$ for $\lambda \neq \mathbf{1}$ (with $r_{\mathbf{1}}(\beta_t) = 1$).*

(2) Exact effective spin measure: *Fixing temporal gauge ($U_{(t,x),\tau} = I$ for $0 \leq t \leq N_\tau - 2$ and $U_{(N_\tau-1,x),\tau} = W_x$) and integrating out all $N_\tau|E(\Lambda)|$ spatial links yields the exact marginal probability measure on the Polyakov loops $\{W_x\}_{x\in\Lambda} \in G^\Lambda$:*

$$d\mu_{\Lambda,N_\tau,\beta_t}(W) = \frac{1}{Z_{\Lambda,N_\tau,\beta_t}} \prod_{\langle x,y\rangle \in E(\Lambda)} B_{N_\tau,\beta_t}(W_x, W_y) \prod_{x\in\Lambda} dW_x, \tag{3}$$

where the normalized bond kernel is

$$B_{N_\tau,\beta_t}(W_x, W_y) := \sum_{\lambda \in \widehat{G}} r_\lambda(\beta_t)^{N_\tau} \, \chi_\lambda(W_x) \overline{\chi_\lambda(W_y)} = 1 + h_{N_\tau,\beta_t}(W_x, W_y). \tag{4}$$

The kernel is strictly positive ($B_{N_\tau,\beta_t} > 0$), real-valued, symmetric in (W_x, W_y), class-invariant under conjugation, and invariant under the global center symmetry $W_x \mapsto zW_x$ ($z \in \mathbb{Z}_N$).

(3) Dual reflection positivity: *On every even spatial torus $\Lambda = \mathbb{T}^D_L$ ($L \geq 4$ even), the measure $\mu_{\mathbb{T}^D_L,N_\tau,\beta_t}$ satisfies* both *(writing $z^* := \bar{z}$ for complex conjugation):*

- Bond-bisecting reflection positivity: *For every hyperplane* Π *bisecting bonds perpendicular to coordinate* $\nu \in \{1,\dots,D\}$ *(splitting* $\mathbb{T}_L^D = \Lambda_+ \sqcup \Lambda_-$ *with reflection* ϑ_Π*) and every bounded measurable complex-valued* F *depending only on* $\{W_x : x \in \Lambda_+\}$, $\mathbb{E}_{\mathbb{T}_L^D}[F\,(\vartheta_\Pi F)^*] \geq 0$.
- Site-containing reflection positivity: *For every hyperplane* Π_0 *passing through sites* $\{x_\nu = c\} \cup \{x_\nu = c + L/2\}$ *(splitting* $\mathbb{T}_L^D$ *into closed half-tori* $\overline{\Lambda}_+, \overline{\Lambda}_-$ *intersecting along* Π_0 *with reflection* ϑ_{Π_0}*) and every bounded measurable complex-valued* F *depending only on* $\{W_x : x \in \overline{\Lambda}_+\}$, $\mathbb{E}_{\mathbb{T}_L^D}[F\,(\vartheta_{\Pi_0} F)^*] \geq 0$.

Proof. (1) Expanding $w_{\beta_t}(U) = \sum_{m=0}^\infty \frac{(\beta_t/(2N))^m}{m!}(\operatorname{Tr} U + \operatorname{Tr} U^\dagger)^m$ and integrating term by term against $\overline{\chi_\lambda(U)}$ (justified by uniform convergence on compact G) gives

$$c_\lambda(\beta_t) = \sum_{m=0}^\infty \frac{(\beta_t/(2N))^m}{m!}\,\operatorname{mult}_\lambda\left((\mathbf{N} \oplus \overline{\mathbf{N}})^{\otimes m}\right).$$

Because the faithful representation $\mathbf{N} \oplus \overline{\mathbf{N}}$ generates every irreducible representation of $\mathrm{SU}(N)$ in finite tensor powers, at least one term is strictly positive and all terms are nonnegative; hence $c_\lambda(\beta_t) > 0$. For $\lambda \neq \mathbf{1}$, $|\chi_\lambda(U)| \leq d_\lambda$ with strict inequality on a nonempty open subset of G while $w_{\beta_t} > 0$, whence $|c_\lambda(\beta_t)| < d_\lambda c_{\mathbf{1}}(\beta_t)$ and $0 < r_\lambda(\beta_t) < 1$.

(2) Fix any spatial bond $e = \langle x, y\rangle \in E(\Lambda)$ with spatial links $U_0, \dots, U_{N_\tau - 1} \in G$ along the temporal cylinder above e. In temporal gauge (justified by gauge invariance of Haar measure on $\Lambda \times \mathbb{T}^1_{N_\tau}$), the N_τ temporal plaquettes above e are $U_p^{(t)} = U_t U_{t+1}^{-1}$ for $0 \leq t \leq N_\tau - 2$ and $U_p^{(N_\tau - 1)} = U_{N_\tau - 1} W_y U_0^{-1} W_x^{-1}$, and no other plaquettes involve $U_0, \dots, U_{N_\tau - 1}$ since $\beta_{\mathrm{sp}} = 0$. Substituting (1) and applying Peter–Weyl convolution $\int_G \chi_\lambda(AU)\chi_\mu(U^{-1}B)\,dU = \delta_{\lambda\mu} d_\lambda^{-1}\chi_\lambda(AB)$ sequentially over $U_{N_\tau - 1}, \dots, U_1$ and finally $\int_G \chi_\lambda(U_0 W_y U_0^{-1} W_x^{-1})\,dU_0 = d_\lambda^{-1}\chi_\lambda(W_y)\chi_\lambda(W_x^{-1}) = d_\lambda^{-1}\overline{\chi_\lambda(W_x)}\chi_\lambda(W_y)$ yields

$$I_{N_\tau,\beta_t}(W_x, W_y) := \int_{G^{N_\tau}} \prod_{t=0}^{N_\tau - 1} w_{\beta_t}(U_p^{(t)})\,dU_0 \cdots dU_{N_\tau - 1} = c_{\mathbf{1}}(\beta_t)^{N_\tau} \sum_{\lambda \in \widehat{G}} r_\lambda(\beta_t)^{N_\tau}\,\chi_\lambda(W_x)\overline{\chi_\lambda(W_y)}. \tag{5}$$

(Note that $w_{\beta_t}(U^{-1}) = w_{\beta_t}(U)$ implies $c_{\bar\lambda} = c_\lambda$, so the sum is real and symmetric in (W_x, W_y).) Since $w_{\beta_t}(U) \geq e^{-\beta_t} > 0$ pointwise, $I_{N_\tau,\beta_t}(W_x, W_y) > 0$ everywhere; dividing by the constant $c_{\mathbf{1}}(\beta_t)^{N_\tau |E(\Lambda)|}$ produces (3)–(4). Under $W_x \mapsto zW_x$ ($z \in \mathbb{Z}_N$), each term picks up $z^{k(\lambda)}\overline{z}^{k(\lambda)} = 1$, where $k(\lambda) \in \mathbb{Z}_N$ is the N-ality of λ, proving global $\mathbb{Z}_N$ invariance.

(3) For a bond-bisecting hyperplane Π, every bond crossing Π connects a site $x \in \Lambda_+$ to $\vartheta_\Pi x \in \Lambda_-$, and its weight expands as $B_{N_\tau,\beta_t}(W_x, W_{\vartheta_\Pi x}) = \sum_\lambda r_\lambda(\beta_t)^{N_\tau}\chi_\lambda(W_x)\overline{\chi_\lambda(W_{\vartheta_\Pi x})}$ with strictly positive coefficients $r_\lambda(\beta_t)^{N_\tau} > 0$. Distributing the product over all crossing bonds $e_1, \dots, e_m$ yields

$$Z_{\mathbb{T}_L^D}\,\mathbb{E}_{\mathbb{T}_L^D}[F\,\overline{\vartheta_\Pi F}] = \sum_{\lambda_1,\dots,\lambda_m \in \widehat{G}} \left(\prod_{j=1}^m r_{\lambda_j}(\beta_t)^{N_\tau}\right) \times \left|\int_{G^{\Lambda_+}} F(W)\left(\prod_{e \subset \Lambda_+} B(W_e)\right)\left(\prod_{j=1}^m \chi_{\lambda_j}(W_{x_j})\right) dW_{\Lambda_+}\right|^2 \geq 0.$$

For a site-containing hyperplane Π_0, *no* nearest-neighbor bond crosses Π_0: every bond either lies inside $\overline{\Lambda}_+$ (or $\overline{\Lambda}_-$) or lies inside Π_0. Because $B_{N_\tau,\beta_t} > 0$, each in-plane bond weight splits symmetrically as $B(W_u, W_v) = \sqrt{B(W_u, W_v)} \cdot \sqrt{B(W_u, W_v)}$, so conditioning on W_{Π_0} factorizes the

remaining integrals into a positive square:

$$Z_{\mathbb{T}_L^D}\,\mathbb{E}_{\mathbb{T}_L^D}[F\,\overline{\vartheta_{\Pi_0}F}] = \int_{G^{\Pi_0}} \left| \int_{G^{\overline{\Lambda}_+\setminus\Pi_0}} F(W) \left(\prod_{e\subset\overline{\Lambda}_+\setminus E(\Pi_0)} B(W_e) \right) \times \left(\prod_{e\subset\Pi_0} \sqrt{B(W_e)} \right) dW_{\overline{\Lambda}_+\setminus\Pi_0} \right|^2 dW_{\Pi_0} \geq 0. \qquad \square$$

3. Low-Temperature Dobrushin Confinement and Screening Mass Gaps

Define the uniform perturbation norm of the effective bond kernel $h_{N_\tau,\beta_t} := B_{N_\tau,\beta_t} - 1$:

$$\|h_{N_\tau,\beta_t}\|_\infty := \sup_{W,W'\in G} |h_{N_\tau,\beta_t}(W,W')| = h_{N_\tau,\beta_t}(I,I) = \sum_{\lambda\neq\mathbf{1}} d_\lambda^2\, r_\lambda(\beta_t)^{N_\tau}, \tag{6}$$

where the supremum is attained identically at (I,I) because $r_\lambda(\beta_t) > 0$ and $|\chi_\lambda(W)| \leq d_\lambda = \chi_\lambda(I)$.

Theorem 3.1 (Dobrushin Confinement, Unbroken $\mathbb{Z}_N$ Symmetry, and Screening Mass)**.** *Let $D \geq 1$, $N \geq 2$, $N_\tau \geq 1$, and $\beta_t > 0$, and suppose that*

$$\alpha(D,N,N_\tau,\beta_t) := 2D\,\|h_{N_\tau,\beta_t}\|_\infty < 1. \tag{7}$$

Then:

(1) Single-neighbor total-variation bound: *For every finite $\Lambda \subset \mathbb{Z}^D$, every site $x \in \Lambda$, every neighbor $y \sim x$, and every pair of boundary configurations $\eta, \tilde{\eta} \in G^{\Lambda\setminus\{x\}}$ differing only at y, the single-site conditional distributions $\gamma_x(\cdot \mid \eta)$ and $\gamma_x(\cdot \mid \tilde{\eta})$ satisfy*

$$\|\gamma_x(\cdot \mid \eta) - \gamma_x(\cdot \mid \tilde{\eta})\|_{\mathrm{TV}} \leq \|h_{N_\tau,\beta_t}\|_\infty. \tag{8}$$

(2) Uniqueness, unbroken $\mathbb{Z}_N$ symmetry, and exponential screening: *There exists a unique infinite-volume Gibbs measure μ_{N_τ,β_t} on $G^{\mathbb{Z}^D}$, which is $\mathbb{Z}_N$-invariant ($\langle \mathrm{Tr}\, W_0\rangle = 0$) and obeys, for every $x, y \in \mathbb{Z}^D$ and every pair of* real-valued *bounded observables $f(W_x), g(W_y)$,*

$$\left|\,\mathrm{Cov}_{\mu_{N_\tau,\beta_t}}\big(f(W_x), g(W_y)\big)\right| \leq \frac{\mathrm{osc}(f)\,\mathrm{osc}(g)}{4(1-\alpha)}\,\alpha^{\|x-y\|_1}. \tag{9}$$

Applying (9) to the real and imaginary parts of $\mathrm{Tr}\, W$ (each bounded in $[-N,N]$ with oscillation at most $2N$) gives $|\langle \mathrm{Tr}(W_0)\overline{\mathrm{Tr}(W_x)}\rangle| \leq \frac{4N^2}{1-\alpha}\alpha^{\|x\|_1}$ and hence the explicit finite-temperature Polyakov screening mass lower bound

$$\sigma_T(D,N,N_\tau,\beta_t) := \liminf_{\|x\|_1\to\infty} \left(-\frac{1}{\|x\|_1} \ln\left|\langle \mathrm{Tr}(W_0)\overline{\mathrm{Tr}(W_x)}\rangle\right| \right) \geq -\ln\big(2D\,\|h_{N_\tau,\beta_t}\|_\infty\big) > 0. \tag{10}$$

(3) Exact zero-quadrature evaluation of $\|h_{N_\tau,\beta_t}\|_\infty$:

- *For* SU(2) *($N = 2$), $r_k(\beta_t) = I_k(\beta_t)/I_1(\beta_t)$ ($k \geq 1$, $d_k = k$), so $\|h_{N_\tau,\beta_t}\|_\infty = \sum_{k=2}^\infty k^2 (I_k(\beta_t)/I_1(\beta_t))^{N_\tau}$.*
- *For all* SU(N) *at $N_\tau = 1$ and $N_\tau = 2$ (exact Parseval identity), with $t := \beta_t/N$:*

$$\|h_{1,\beta_t}\|_\infty = \frac{e^{\beta_t}}{Z_{\mathrm{SU}(N)}(t)} - 1, \qquad \|h_{2,\beta_t}\|_\infty = \frac{Z_{\mathrm{SU}(N)}(2t)}{Z_{\mathrm{SU}(N)}(t)^2} - 1. \tag{11}$$

- *For* SU(3) *at every $N_\tau \geq 3$ (exact $\mathbf{3}/\overline{\mathbf{3}}$ Parseval separation): setting $r_{\mathbf{3}} := \frac{Z'_{\mathrm{SU}(3)}(t)}{3Z_{\mathrm{SU}(3)}(t)}$, $R_3 := \frac{\sqrt{E_{\mathrm{rem}}}}{6}$, and*

$$E_{\mathrm{rem}} := \frac{Z_{\mathrm{SU}(3)}(2t)}{Z_{\mathrm{SU}(3)}(t)^2} - 1 - 2\left(\frac{Z'_{\mathrm{SU}(3)}(t)}{Z_{\mathrm{SU}(3)}(t)}\right)^2 \geq 0,$$

we have unconditionally

$$\|h_{N_\tau,\beta_t}\|_\infty \le 18\, r_{\mathbf{3}}^{N_\tau} + R_3^{N_\tau-2}\, E_{\mathrm{rem}}. \tag{12}$$

Proof. (1) Write $\gamma_x(dW \mid \eta) = Z_\eta^{-1}\Phi(W)B(W,\eta_y)\,dW$ and $\gamma_x(dW \mid \tilde\eta) = Z_{\tilde\eta}^{-1}\Phi(W)B(W,\tilde\eta_y)\,dW$, where $\Phi(W) := \prod_{z\sim x, z\neq y} B(W,\eta_z) > 0$. Setting $P(dW) := \Phi(W)\,dW/\int_G \Phi(W')\,dW'$ (a probability measure on G), the densities of $\gamma_x(\cdot\mid\eta)$ and $\gamma_x(\cdot\mid\tilde\eta)$ with respect to P are $p(W) = B(W,\eta_y)/E_1$ and $q(W) = B(W,\tilde\eta_y)/E_2$, where $E_1 = \int B(W,\eta_y)\,dP$ and $E_2 = \int B(W,\tilde\eta_y)\,dP$. Let $\varepsilon := \|h_{N_\tau,\beta_t}\|_\infty < 1$. Because $1-\varepsilon \le B(W,W') \le 1+\varepsilon$ everywhere, the ratio $q(W)/p(W)$ lies in $[\frac{1-\varepsilon}{1+\varepsilon}\frac{E_1}{E_2}, \frac{1+\varepsilon}{1-\varepsilon}\frac{E_1}{E_2}]$, and more sharply, comparing $p(W) = (1+h(W,\eta_y))/E_1$ to the uniform probability density 1 gives an exact coupling via the minimum ratio: for any two positive functions $u, v \in [1-\varepsilon, 1+\varepsilon]$, the Dobrushin cross-ratio inequality (mechanized in Lean as `dobrushin_tv_cancellation`) gives

$$\|\gamma_x(\cdot\mid\eta) - \gamma_x(\cdot\mid\tilde\eta)\|_{\mathrm{TV}} = \frac{1}{2}\int_G |p(W)-q(W)|\,dP(W) \le \frac{(1+\varepsilon)-(1-\varepsilon)}{(1+\varepsilon)+(1-\varepsilon)} = \varepsilon = \|h_{N_\tau,\beta_t}\|_\infty.$$

Indeed, let $a := 1-\varepsilon > 0$ and $b := 1+\varepsilon$; for any probability measure P and measurable $u, v : G \to [a,b]$ with means $\bar u, \bar v$, writing $\alpha_+ := \int_{\{u/\bar u \ge v/\bar v\}}(u/\bar u - v/\bar v)\,dP = \|u/\bar u - v/\bar v\|_{\mathrm{TV}}$ and $A := \{u/\bar u \ge v/\bar v\}$, we have $u/\bar u \le b/\bar u$ and $v/\bar v \ge a/\bar v$ on A, while on A^c we have $u/\bar u \ge a/\bar u$ and $v/\bar v \le b/\bar v$. Setting $p_A := P(A) \in [0,1]$, integrating over A and A^c yields $\bar u = \int_A u + \int_{A^c} u$ and $\bar v = \int_A v + \int_{A^c} v$; the bilinear fraction $(\int_A u \int_{A^c} v - \int_A v \int_{A^c} u)/(\bar u \bar v)$ is maximized over $u, v \in [a,b]$ at the extreme point $u = b\mathbf{1}_A + a\mathbf{1}_{A^c}$, $v = a\mathbf{1}_A + b\mathbf{1}_{A^c}$, where it equals

$$\frac{p_A(1-p_A)(b^2-a^2)}{(p_Ab+(1-p_A)a)(p_Aa+(1-p_A)b)} \le \frac{b-a}{b+a} = \varepsilon,$$

since $(p_Ab+(1-p_A)a)(p_Aa+(1-p_A)b) = p_A(1-p_A)(a+b)^2 + (1-2p_A)^2ab \ge p_A(1-p_A)(a+b)^2$.

(2) Since every site $x \in \mathbb{Z}^D$ has $2D$ nearest neighbors, the Dobrushin interdependence matrix $C_{xy} \le \|h_{N_\tau,\beta_t}\|_\infty \mathbf{1}_{y\sim x}$ satisfies $\sup_x \sum_y C_{xy} \le 2D\,\|h_{N_\tau,\beta_t}\|_\infty = \alpha < 1$. By Dobrushin's uniqueness and comparison theorem [4, 9], the infinite-volume Gibbs measure μ_{N_τ,β_t} is unique, hence inherits global $\mathbb{Z}_N$ symmetry ($\langle \operatorname{Tr} W_0\rangle = 0$), and for any real-valued single-site observables $f(W_x), g(W_y)$ the conditional expectation $F_y(\eta) := \mathbb{E}_\mu[f(W_x) \mid W_y = \eta]$ satisfies $\operatorname{osc}(F_y) \le \operatorname{osc}(f)D_{xy}$ where $D_{xy} = \sum_{m \ge \|x-y\|_1}(C^m)_{xy} \le \frac{\alpha^{\|x-y\|_1}}{1-\alpha}$. By Popoviciu's covariance bound for real random variables, $|\operatorname{Cov}_\mu(f(W_x), g(W_y))| = |\operatorname{Cov}_\mu(F_y(W_y), g(W_y))| \le \frac{1}{4}\operatorname{osc}(F_y)\operatorname{osc}(g) \le \frac{\operatorname{osc}(f)\operatorname{osc}(g)}{4(1-\alpha)}\alpha^{\|x-y\|_1}$, proving (9). Splitting $\operatorname{Tr} W = \operatorname{Re}\operatorname{Tr} W + i \operatorname{Im}\operatorname{Tr} W$ yields $|\langle \operatorname{Tr}(W_0)\overline{\operatorname{Tr}(W_x)}\rangle| \le 4 \cdot \frac{(2N)^2}{4(1-\alpha)}\alpha^{\|x\|_1} = \frac{4N^2}{1-\alpha}\alpha^{\|x\|_1}$ and hence (10).

(3) For SU(2), the Weyl integration formula yields $c_k(\beta_t) = \frac{2}{\pi}\int_0^\pi e^{\beta_t\cos\theta}\sin(k\theta)\sin\theta\,d\theta = \frac{2k}{\beta_t}I_k(\beta_t)$ and $d_k = k$, so $r_k(\beta_t) = I_k(\beta_t)/I_1(\beta_t)$. For general SU($N$) at $N_\tau = 1$, $1 + \|h_{1,\beta_t}\|_\infty = B_{1,\beta_t}(I,I) = \sum_\lambda d_\lambda c_\lambda/c_{\mathbf{1}} = w_{\beta_t}(I)/c_{\mathbf{1}} = e^{\beta_t}/Z_{\mathrm{SU}(N)}(t)$. At $N_\tau = 2$, Parseval's identity on $L^2(\mathrm{SU}(N))$ gives

$$1 + \|h_{2,\beta_t}\|_\infty = \sum_{\lambda\in\widehat G} d_\lambda^2\left(\frac{c_\lambda}{d_\lambda c_{\mathbf{1}}}\right)^2 = \frac{1}{c_{\mathbf{1}}^2}\sum_{\lambda\in\widehat G} c_\lambda^2 = \frac{1}{Z_{\mathrm{SU}(N)}(t)^2}\int_G w_{\beta_t}(U)^2\,dU = \frac{Z_{\mathrm{SU}(N)}(2t)}{Z_{\mathrm{SU}(N)}(t)^2}.$$

For SU(3), we have $Z'_{\mathrm{SU}(3)}(t) = \int_{\mathrm{SU}(3)} \operatorname{Re}\operatorname{Tr}(U) w_{\beta_t}(U)\,dU = c_{\mathbf{3}}$, so $r_{\mathbf{3}} = r_{\overline{\mathbf{3}}} = Z'_{\mathrm{SU}(3)}(t)/(3Z_{\mathrm{SU}(3)}(t))$, and the $\mathbf{3}, \overline{\mathbf{3}}$ irreps contribute $2(3r_{\mathbf{3}})^2 = 2(Z'/Z)^2$ to Parseval's sum. Every other nontrivial SU(3) irrep (p,q) has dimension $d_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2) \ge 6$, so $36r_\lambda^2 \le d_\lambda^2 r_\lambda^2 \le E_{\mathrm{rem}}$, implying $r_\lambda \le \sqrt{E_{\mathrm{rem}}}/6 = R_3$ and hence (12) for all $N_\tau \ge 3$. □

4. Uniform-in-N_τ Riccati–Bessel Enclosures for SU(2)

For $G = \mathrm{SU}(2)$, conjugacy classes are parameterized by half-eigenangles $\theta \in [0,\pi]$ via $\operatorname{Tr} W = 2\cos\theta$, $\chi_k(\theta) = \sin(k\theta)/\sin\theta$ ($k \ge 1$), and Weyl–Haar density $\frac{2}{\pi}\sin^2\theta\,d\theta$. Set $\gamma := \beta_t/N_\tau > 0$ and

define the normalized weights $a_k := (I_k(\beta_t)/I_1(\beta_t))^{N_\tau}$ for $k \geq 0$ (so $a_1 = 1$, $a_0 = (I_0(\beta_t)/I_1(\beta_t))^{N_\tau} > 1$) and the auxiliary periodic Fourier series

$$P_{N_\tau,\beta_t}(\phi) := a_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\phi) = \sum_{n\in\mathbb{Z}} a_{|n|} e^{in\phi}, \qquad \phi \in \mathbb{R}. \tag{13}$$

Theorem 4.1 (SU(2) Kernel Identities, λ-Splitting Domain Walls, and Uniform Enclosures)**.** *Let $N = 2$, $N_\tau \geq 1$, $\beta_t > 0$, and $\gamma = \beta_t/N_\tau$.*

(1) Exact product-to-sum identity and off-center bound: *For all $\theta_x, \theta_y \in (0,\pi)$, $P_{N_\tau,\beta_t}(\phi) > 0$ everywhere and*

$$B_{N_\tau,\beta_t}(\theta_x,\theta_y) = \frac{P_{N_\tau,\beta_t}(\theta_x-\theta_y) - P_{N_\tau,\beta_t}(\theta_x+\theta_y)}{4\sin\theta_x\sin\theta_y} < \frac{P_{N_\tau,\beta_t}(0)}{4\sin\theta_x\sin\theta_y}. \tag{14}$$

(2) λ-splitting exponential domain-wall bound: *For every $\theta_x, \theta_y \in [0,\pi]$ and every $\lambda \in (0,1)$,*

$$\frac{B_{N_\tau,\beta_t}(\theta_x,\theta_y)}{B_{N_\tau,\beta_t}(0,0)} \leq (1-\lambda)^{-3N_\tau/2} \exp\Big(-\lambda\gamma\big(1-\cos(\theta_y-\theta_x)\big)\Big). \tag{15}$$

(3) Unconditional quadratic Chebyshev ground-state lower bound: *For every $k \geq 1$ and every $\theta_x, \theta_y \in \mathbb{R}$,*

$$\frac{\sin(k\theta_x)}{k\sin\theta_x}\,\frac{\sin(k\theta_y)}{k\sin\theta_y} \geq 1 - \frac{k^2-1}{6}\big(\theta_x^2+\theta_y^2\big), \tag{16}$$

and consequently, for every $\varepsilon > 0$ and all $\theta_x, \theta_y \in [0,\varepsilon]$,

$$\frac{B_{N_\tau,\beta_t}(\theta_x,\theta_y)}{B_{N_\tau,\beta_t}(0,0)} \geq 1 - c_{\mathrm{quad}}\,\varepsilon^2, \qquad c_{\mathrm{quad}} := \frac{1}{3B_{N_\tau,\beta_t}(0,0)}\sum_{k=1}^{\infty} k^2(k^2-1)\,a_k. \tag{17}$$

(4) Closed-form uniform-in-N_τ Riccati–Bessel enclosures: *Suppose $\gamma \geq 1000$, and set $c_{1/10} := 10\,\mathrm{arsinh}(1/10)$, $b := c_{1/10}/(2\gamma)$, $K := \lfloor \gamma/10 + 1/2 \rfloor$, $r_{\mathrm{step}} := e^{-\,\mathrm{arsinh}(1/10)}$, and $q_4 := \big(\frac{K+2}{K+1}\big)^4 r_{\mathrm{step}} < 1$. Then, uniformly over* all *$N_\tau \geq 1$:*

$$B_{N_\tau,\beta_t}(0,0) \geq B_0^-(\gamma) := \sqrt{\frac{\pi}{2}}\,\gamma^{3/2} - \left(1+\frac{2}{e}\right)\gamma - \frac{9}{8}, \tag{18}$$

$$P_{N_\tau,\beta_t}(0) \leq P_0^+(\gamma) := e^{1/\gamma} + e^b\sqrt{\frac{\pi}{b}} + 2e^{b-bK^2}\,\frac{r_{\mathrm{step}}}{1-r_{\mathrm{step}}}, \tag{19}$$

$$c_{\mathrm{quad}} \leq c_{\mathrm{quad}}^+(\gamma) := \frac{1}{3B_0^-(\gamma)}\left[e^b\left(\frac{3\sqrt{\pi}}{8b^{5/2}} + \frac{4}{b^2e^2}\right) + \frac{(K+1)^4 e^{b-bK^2} r_{\mathrm{step}}}{1-q_4}\right]. \tag{20}$$

Proof. (1) Since $2\sin(k\theta_x)\sin(k\theta_y) = \cos(k(\theta_x-\theta_y)) - \cos(k(\theta_x+\theta_y))$, summing $a_k\chi_k(\theta_x)\chi_k(\theta_y)$ over $k \geq 1$ yields the equality in (14). Moreover, $I_n(\beta_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\beta_t\cos\phi}e^{-in\phi}\,d\phi$, so $I_1(\beta_t)^{N_\tau}P_{N_\tau,\beta_t}(\phi)$ is the N_τ-fold circular convolution of the strictly positive function $\phi \mapsto e^{\beta_t\cos\phi}$ on $\mathbb{T}^1$; hence $P_{N_\tau,\beta_t}(\phi) > 0$ for all ϕ, and dropping $-P_{N_\tau,\beta_t}(\theta_x+\theta_y) < 0$ while using $P_{N_\tau,\beta_t}(\theta_x-\theta_y) \leq P_{N_\tau,\beta_t}(0)$ (since $a_k > 0$) gives the strict inequality in (14).

(2) In the integral representation (5), write $\frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}\,U_p^{(t)} = \cos\alpha_t$ with geodesic distance $\alpha_t = d(I, U_p^{(t)}) \in [0,\pi]$ on $\mathrm{SU}(2) \cong S^3$. Because the ordered product of the N_τ plaquettes telescopes to $U_p^{(0)}\cdots U_p^{(N_\tau-1)} = U_0W_yU_0^{-1}W_x^{-1}$, the bi-invariant triangle inequality and the SU(2) quaternion trace bound $\frac{1}{2}\,\mathrm{Tr}(U_0W_yU_0^{-1}W_x^{-1}) \leq \cos(\theta_y-\theta_x)$ yield $\sum_{t=0}^{N_\tau-1}\alpha_t \geq |\theta_y-\theta_x| =: \Delta \in [0,\pi]$. Shrinking

α_t to $\tilde{\alpha}_t \in [0, \alpha_t]$ with $\sum_t \tilde{\alpha}_t = \Delta$ cannot increase $\sum_t (1 - \cos\alpha_t)$, and by Cauchy–Schwarz together with subadditivity of the concave function $\alpha \mapsto \sin(\alpha/2)$ on $[0, \pi]$,

$$\sum_{t=0}^{N_\tau - 1} (1 - \cos\alpha_t) \geq 2 \sum_{t=0}^{N_\tau - 1} \sin^2\left(\frac{\tilde{\alpha}_t}{2}\right) \geq \frac{2}{N_\tau} \left(\sum_{t=0}^{N_\tau - 1} \sin\frac{\tilde{\alpha}_t}{2}\right)^2 \geq \frac{2}{N_\tau} \sin^2\left(\frac{\Delta}{2}\right) = \frac{1 - \cos(\theta_y - \theta_x)}{N_\tau}.$$

Splitting $\beta_t \sum_t (\cos\alpha_t - 1) = (1-\lambda)\beta_t \sum_t (\cos\alpha_t - 1) + \lambda\beta_t \sum_t (\cos\alpha_t - 1)$ in (5) gives

$$\frac{B_{N_\tau,\beta_t}(\theta_x, \theta_y)}{B_{N_\tau,\beta_t}(0,0)} = \frac{I_{N_\tau,\beta_t}(W_x, W_y)}{I_{N_\tau,\beta_t}(I, I)} \leq e^{-\lambda\gamma(1-\cos\Delta)} \frac{e^{-(1-\lambda)N_\tau\beta_t} I_{N_\tau,(1-\lambda)\beta_t}(I, I)}{e^{-N_\tau\beta_t} I_{N_\tau,\beta_t}(I, I)},$$

where we used $I_{N_\tau,(1-\lambda)\beta_t}(W_x, W_y) \leq I_{N_\tau,(1-\lambda)\beta_t}(I, I)$. Now recall the standard Amos–Segura Riccati–Bessel inequalities [1, 12, 6] for all $n \geq 0$ and $x > 0$:

$$\begin{aligned} \exp\left(-\operatorname{arsinh}\frac{n+1}{x}\right) &= \frac{x}{(n+1) + \sqrt{x^2 + (n+1)^2}} < \frac{I_{n+1}(x)}{I_n(x)} \\ &< \frac{x}{(n+\frac{1}{2}) + \sqrt{x^2 + (n+\frac{1}{2})^2}} = \exp\left(-\operatorname{arsinh}\frac{n+\frac{1}{2}}{x}\right). \end{aligned} \tag{21}$$

For every $k \geq 1$ and $\beta > 0$, set $f_k(\beta) := \beta^{3/2} e^{-\beta} \frac{2}{\beta} I_k(\beta) = 2\beta^{1/2} e^{-\beta} I_k(\beta)$. Using $I_k'(\beta) = I_{k-1}(\beta) - \frac{k}{\beta} I_k(\beta)$ and the upper inequality of (21) at $n = k - 1 \geq 0$ (which gives $I_{k-1}(\beta)/I_k(\beta) > \frac{k-1/2}{\beta} + \sqrt{1 + (\frac{k-1/2}{\beta})^2} > \frac{k-1/2}{\beta} + 1$), we obtain the exact one-line monotonicity

$$(\ln f_k)'(\beta) = \frac{1}{2\beta} - 1 + \frac{I_{k-1}(\beta)}{I_k(\beta)} - \frac{k}{\beta} > \frac{1}{2\beta} - 1 + \left(\frac{k - \frac{1}{2}}{\beta} + 1\right) - \frac{k}{\beta} = 0.$$

Hence $f_k((1-\lambda)\beta_t) < f_k(\beta_t)$, i.e. $e^{-(1-\lambda)\beta_t} \frac{2}{(1-\lambda)\beta_t} I_k((1-\lambda)\beta_t) < (1-\lambda)^{-3/2} e^{-\beta_t} \frac{2}{\beta_t} I_k(\beta_t)$ for every $k \geq 1$. Raising to the N_τ-th power, multiplying by k^2, and summing over $k \geq 1$ proves (15).

(3) Using the finite cosine expansion $\frac{\sin(k\theta)}{k\sin\theta} = \frac{1}{k} \sum_{r=0}^{k-1} \cos\big((k-1-2r)\theta\big)$, multiplying the two sums, and applying the elementary inequality $\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \geq 1 - \frac{1}{2}(A^2 + B^2)$ termwise gives

$$\frac{\sin(k\theta_x)}{k\sin\theta_x} \frac{\sin(k\theta_y)}{k\sin\theta_y} \geq 1 - \frac{1}{2k} \left(\sum_{r=0}^{k-1} (k-1-2r)^2\right) (\theta_x^2 + \theta_y^2) = 1 - \frac{k^2 - 1}{6} (\theta_x^2 + \theta_y^2),$$

where the exact identity $3\sum_{r=0}^{k-1} (k-1-2r)^2 = k(k^2-1)$ holds for every $k \geq 1$ (mechanized in Lean as `chebyshev_quad_sum_exact`). Multiplying by $k^2 a_k$, summing over $k \geq 1$, and using $\theta_x^2 + \theta_y^2 \leq 2\varepsilon^2$ gives (17).

(4) From the lower inequality of (21) and $\operatorname{arsinh}(u) \leq u$, $a_k \geq \exp(-\frac{1}{\gamma} \sum_{n=1}^{k-1} (n+1)) = \exp(-\frac{k(k+1)-2}{2\gamma}) \geq \exp(-\frac{(k+1/2)^2}{2\gamma})$. Since $u \mapsto (u - 1/2)^2 e^{-u^2/(2\gamma)}$ is unimodal on $[1/2, \infty)$ with maximum at most $\max_{u \geq 0} u^2 e^{-u^2/(2\gamma)} = 2\gamma/e$, Riemann comparison yields

$$\begin{aligned} B_{N_\tau,\beta_t}(0,0) &\geq \sum_{k=1}^{\infty} \left(k + \tfrac{1}{2} - \tfrac{1}{2}\right)^2 e^{-(k+1/2)^2/(2\gamma)} \\ &\geq \int_{1/2}^{\infty} (u - \tfrac{1}{2})^2 e^{-u^2/(2\gamma)}\, du - \frac{2\gamma}{e} \geq \sqrt{\frac{\pi}{2}} \gamma^{3/2} - \left(1 + \frac{2}{e}\right) \gamma - \frac{1}{24}, \end{aligned}$$

which implies (18) (using the conservative margin $-9/8 < -1/24$). For the upper bounds, note that $t \mapsto t \operatorname{arsinh}(c/t)$ has derivative $\operatorname{arsinh}(c/t) - \frac{c/t}{\sqrt{1 + (c/t)^2}} > 0$ for $c, t > 0$, so $N_\tau \operatorname{arsinh}\left(\frac{n+1/2}{N_\tau \gamma}\right) \geq$

$\operatorname{arsinh}\left(\frac{n+1/2}{\gamma}\right)$ uniformly for all $N_\tau \geq 1$. Moreover, $a_0 = (I_0(\beta_t)/I_1(\beta_t))^{N_\tau} \leq e^{N_\tau/\beta_t} = e^{1/\gamma}$. By concavity of arsinh on $[0,\infty)$ and the choice $K = \lfloor \gamma/10 + 1/2 \rfloor$ (which guarantees $\frac{n+1/2}{\gamma} \leq \frac{1}{10}$ for $1 \leq n \leq K-1$ and $\frac{n+1/2}{\gamma} > \frac{1}{10}$ for $n \geq K$ for *every* real $\gamma \geq 1000$):

- For $1 \leq k \leq K$: $a_k \leq \exp(-\frac{c_{1/10}}{\gamma} \sum_{n=1}^{k-1}(n+\frac{1}{2})) = e^b e^{-bk^2}$.
- For $k = K + j$ $(j \geq 1)$: $a_{K+j} \leq a_K\, r_{\text{step}}^j \leq e^{b-bK^2} r_{\text{step}}^j$.

Comparing the finite sums to Gaussian integrals ($\sum_{k=1}^K e^{-bk^2} \leq \frac{1}{2}\sqrt{\pi/b}$ and $\sum_{k=1}^K k^4 e^{-bk^2} \leq \int_0^\infty u^4 e^{-bu^2}\,du + \max_{u\geq 0} u^4 e^{-bu^2} = \frac{3\sqrt{\pi}}{8b^{5/2}} + \frac{4}{b^2 e^2}$) and summing the geometric tails via $(K+j)^4 \leq (K+1)^4((K+2)/(K+1))^{4(j-1)}$ yields (19)–(20). □

5. Dual-Reflection Peierls Deconfinement in $d \geq 4$

Fix $G = \mathrm{SU}(2)$, spatial dimension $D = d-1 \geq 3$, $N_\tau \geq 1$, $\gamma = \beta_t/N_\tau \geq 1000$, a sector cutoff $\theta_0 \in (0, \pi/2)$ (we use $\theta_0 = 11/10$), a domain-wall splitting parameter $\lambda \in (0,1)$ (we use $\lambda = 999/1000$ for all certified rows), and Hölder exponents $p > 1$, $q = p/(p-1)$ (we use $p = 24$, $q = 24/23$). Partition $[0,\pi]$ into the two ordered center sectors and the disordered equatorial band:

$$G_+ := [0, \theta_0], \qquad G_- := [\pi - \theta_0, \pi], \qquad B := (\theta_0, \pi - \theta_0).$$

Set $\varepsilon := \sqrt{3/((2D+3)c_{\text{quad}}^+(\gamma))}$, $V_\varepsilon := (\varepsilon - \sin\varepsilon\cos\varepsilon)/\pi$ (the exact SU(2) Haar measure of $[0,\varepsilon]$), and the per-bond ground-state weight

$$g_{\text{gs}} := V_\varepsilon^{1/D}\left(1 - c_{\text{quad}}^+(\gamma)\,\varepsilon^2\right) > 0. \tag{22}$$

Define the Type-I domain-wall factor (for any $\lambda \in (0,1)$) and Type-II disordered-site factor

$$r_{\text{DW}} := (1-\lambda)^{-3N_\tau/2}\, e^{-\lambda\gamma(1+\cos(2\theta_0))}, \qquad r_B := \frac{P_0^+(\gamma)}{4\sin^2(\theta_0)\, B_0^-(\gamma)}, \tag{23}$$

together with the effective Peierls weights, single-site disordered bound, and codimension-2 ridge contour entropy constant:

$$q_{\text{DW}} := \left(\frac{r_{\text{DW}}^{1/D}}{g_{\text{gs}}}\right)^{1/p}, \quad q_B := \left(\frac{r_B}{g_{\text{gs}}}\right)^{\frac{D}{(2D-1)q}}, \quad q_{\text{eff}} := q_{\text{DW}} + q_B, \quad p_{\text{site},B} := \left(\frac{r_B}{g_{\text{gs}}}\right)^D, \tag{24}$$

$$C_{\text{cont}}(D) := \frac{(3D-3)^{3D-3}}{(3D-4)^{3D-4}}, \qquad r_{\text{cont}} := C_{\text{cont}}(D)\, q_{\text{eff}}. \tag{25}$$

Theorem 5.1 (Dual-Reflection Peierls Deconfinement and Spontaneous $\mathbb{Z}_2$ Symmetry Breaking)**.** *Let $D \geq 3$, $N_\tau \geq 1$, $\gamma = \beta_t/N_\tau \geq 1000$, and suppose $r_B < g_{\text{gs}}$, $r_{\text{DW}} < g_{\text{gs}}^D$, and $r_{\text{cont}} \leq \frac{2D}{2D+1}$. Define*

$$p_{\text{opp}} := \frac{r_{\text{cont}}^{2D}}{(D-1)(3D-4)}\left(\frac{2D-2}{1-r_{\text{cont}}} + \frac{r_{\text{cont}}}{(1-r_{\text{cont}})^2}\right), \qquad p_{\text{diff}} := 2\,p_{\text{opp}} + 2\,p_{\text{site},B}. \tag{26}$$

Then for every spatial separation $x \in \mathbb{Z}^D$, along every sequence of even periodic tori $\mathbb{T}_L^D$ ($L \geq 4$ even, $L \to \infty$):

(1) Dual-reflection Peierls suppression: *For every closed dual $(D-1)$-contour Γ arising as a ridge-connected component of ∂U_x on $\mathbb{T}_L^D$ with a fixed side orientation (so every bond of Γ joins $A = C_+(0) \subseteq G_+$ to $U_x \subseteq G_- \cup B$, with $x \in U_x \cap G_-$), summing over all $2^{|\Gamma|}$ partitions $\Gamma = \Gamma_{\text{DW}} \sqcup \Gamma_B$ yields $\mathbb{P}_L(\Gamma) \leq (q_{\text{DW}} + q_B)^{|\Gamma|} = q_{\text{eff}}^{|\Gamma|}$.*

(2) Long-range Polyakov-loop order:

$$\liminf_{L\to\infty} \mathbb{E}_L\big[\operatorname{sgn}(\operatorname{Tr} W_0)\,\operatorname{sgn}(\operatorname{Tr} W_x)\big] \geq 1 - 2\,p_{\mathrm{diff}}, \tag{27}$$

$$\liminf_{L\to\infty} \mathbb{E}_L\big[\operatorname{Tr}(W_0)\,\operatorname{Tr}(W_x)\big] \geq 4\cos^2(\theta_0)\,(1-p_{\mathrm{diff}}) - 4\,p_{\mathrm{diff}}. \tag{28}$$

(3) Spontaneous $\mathbb{Z}_2$ center symmetry breaking: *Whenever the right-hand side of* (28) *is strictly positive, the infinite-volume effective Polyakov-loop model admits at least two distinct extremal Gibbs states μ_+, μ_- related by $\mathbb{Z}_2$ center reflection ($W_x \mapsto -W_x$) with $\langle \operatorname{Tr} W_0\rangle_{\mu_+} = -\langle \operatorname{Tr} W_0\rangle_{\mu_-} > 0$, and the static quark–antiquark free energy remains bounded uniformly in $|x|$ (deconfinement).*

Proof. Step 1: Ground-state partition function lower bound. Restricting every site $v \in \mathbb{T}_L^D$ ($|V| = L^D$, $|E| = DL^D$) to $\theta_v \in [0,\varepsilon]$ and applying (17) yields

$$Z_{\mathbb{T}_L^D} \geq V_\varepsilon^{L^D}\Big(B_{N_\tau,\beta_t}(0,0)\big(1 - c^+_{\mathrm{quad}}(\gamma)\varepsilon^2\big)\Big)^{DL^D} = \big(B_{N_\tau,\beta_t}(0,0)\,g_{\mathrm{gs}}\big)^{DL^D}. \tag{29}$$

Step 2: Mixed chessboard bound on Type-I domain walls Γ_{DW}. Let E_μ ($\mu \in \{1,\dots,D\}$) be the set of all L^D bonds parallel to e_μ, and for $e = \langle u, v\rangle \in E_\mu$ let $F_e := \mathbf{1}_{\{W_u\in G_+, W_v\in G_-\}\cup\{W_u\in G_-, W_v\in G_+\}}$. Choose a direction μ^* maximizing $|S|$ where $S := \Gamma_{\mathrm{DW}} \cap E_{\mu^*}$, so $|S| \geq |\Gamma_{\mathrm{DW}}|/D$. Writing $x = (x_{\mu^*}, x_\perp) \in \mathbb{Z}_L \times \mathbb{Z}_L^{D-1}$, group $G_{x_\perp} := \prod_{e\in S: e_\perp = x_\perp} F_e$. Every $G_{x_\perp}$ lies strictly on one side of every bond-bisecting hyperplane perpendicular to $\nu \neq \mu^*$. Applying the standard $(D-1)$-dimensional bond-bisecting chessboard estimate [10, 2] (valid by Theorem 2.1(3)) across all transverse directions $\nu \neq \mu^*$ yields

$$\mathbb{E}_L\left[\prod_{e\in S} F_e\right] \leq \prod_{x_\perp \in \mathbb{Z}_L^{D-1}} \left(\mathbb{E}_L\left[\prod_{t\in I(x_\perp)} H_t\right]\right)^{1/L^{D-1}}, \qquad H_t := \prod_{y_\perp \in \mathbb{Z}_L^{D-1}} F_{\langle (t,y_\perp),(t+1,y_\perp)\rangle},$$

where $I(x_\perp)$ is the set of μ^*-slices occurring in S on fiber $x_\perp$. Each slab observable H_t is symmetric under reflection of the slab $[t, t+1]$ and lies strictly inside $\{t \leq x_{\mu^*} \leq t+1\}$, so no bond crosses any site-containing plane $x_{\mu^*} = c$. Applying the 1-dimensional site-containing chessboard estimate along μ^* (valid by Theorem 2.1(3)) gives $\mathbb{E}_L[\prod_{t\in I(x_\perp)} H_t] \leq (\mathbb{E}_L[\prod_{e\in E_{\mu^*}} F_e])^{|I(x_\perp)|/L}$. Multiplying over $x_\perp$ (since $\sum_{x_\perp} |I(x_\perp)| = |S|$) produces

$$\mathbb{P}_L(\Gamma_{\mathrm{DW}}) \leq \mathbb{E}_L\left[\prod_{e\in S} F_e\right] \leq \left(\mathbb{E}_L\left[\prod_{e\in E_{\mu^*}} F_e\right]\right)^{|S|/L^D}.$$

On the event $\bigcap_{e\in E_{\mu^*}}\{F_e = 1\}$, every bond in E_{μ^*} satisfies $|\theta_u - \theta_v| \geq \pi - 2\theta_0$, so $1 - \cos(\theta_u - \theta_v) \geq 1 + \cos(2\theta_0)$ and by (15) its weight is at most $r_{\mathrm{DW}} B(0,0)$, while the remaining $(D-1)L^D$ bonds have weight at most $B(0,0)$. Combining with (29) and $|S| \geq |\Gamma_{\mathrm{DW}}|/D$ (using $r_{\mathrm{DW}}/g_{\mathrm{gs}}^D < 1$) yields

$$\mathbb{P}_L(\Gamma_{\mathrm{DW}}) \leq \left(\frac{r_{\mathrm{DW}}^{L^D} B(0,0)^{DL^D}}{(B(0,0)\,g_{\mathrm{gs}})^{DL^D}}\right)^{|S|/L^D} = \left(\frac{r_{\mathrm{DW}}}{g_{\mathrm{gs}}^D}\right)^{|S|} \leq \left(\frac{r_{\mathrm{DW}}^{1/D}}{g_{\mathrm{gs}}}\right)^{|\Gamma_{\mathrm{DW}}|}. \tag{30}$$

Step 3: Bond-bisecting chessboard bound on Type-II disordered vertices $V_B(\Gamma)$. Every bond $e \in \Gamma_B$ connects $A := C_+(0)$ (the G_+-component of 0) to a disordered vertex $v \in V_B(\Gamma) \subseteq U_x \cap B$, where U_x is the connected component of $\mathbb{T}_L^D \setminus A$ containing x. Because $W_x \in G_-$ and $v \in B$, we have $v \neq x$; since U_x is connected, v has at least one neighbor in U_x, so at most $2D-1$ bonds of Γ_B can be incident to v. Hence $|\Gamma_B| \leq (2D-1)|V_B(\Gamma)|$, i.e. $|V_B(\Gamma)| \geq |\Gamma_B|/(2D-1)$. Each indicator $\mathbf{1}_{W_v\in B}$ is a single-site observable; by the D-dimensional bond-bisecting chessboard estimate and

(14) (on $\bigcap_v\{W_v \in B\}$, every bond satisfies $\sin\theta_u \sin\theta_v > \sin^2\theta_0$, hence $B(\theta_u,\theta_v) < r_B B(0,0)$), together with (29):

$$\mathbb{P}_L(\Gamma_B) \le \mathbb{P}_L\left(\bigcap_{v\in V_B(\Gamma)}\{W_v \in B\}\right) \le \left(\frac{(r_B B(0,0))^{DL^D}}{(B(0,0)\, g_{\mathrm{gs}})^{DL^D}}\right)^{|V_B(\Gamma)|/L^D} = p_{\mathrm{site},B}^{|V_B(\Gamma)|} \le \left(\frac{r_B}{g_{\mathrm{gs}}}\right)^{\frac{D}{2D-1}|\Gamma_B|}. \tag{31}$$

In particular, taking $V_B = \{v\}$ gives $\mathbb{P}_L(W_v \in B) \le p_{\mathrm{site},B}$. For any fixed partition $\Gamma = \Gamma_{\mathrm{DW}} \sqcup \Gamma_B$ of a fixed oriented contour $\Gamma \subseteq \partial U_x$, Hölder's inequality ($\mathbb{P}_L(\Gamma_{\mathrm{DW}},\Gamma_B) \le \mathbb{P}_L(\Gamma_{\mathrm{DW}})^{1/p}\mathbb{P}_L(\Gamma_B)^{1/q}$) gives $\mathbb{P}_L(\Gamma_{\mathrm{DW}},\Gamma_B) \le q_{\mathrm{DW}}^{|\Gamma_{\mathrm{DW}}|} q_B^{|\Gamma_B|}$. Summing over all $2^{|\Gamma|}$ choices of $\Gamma_{\mathrm{DW}} \subseteq \Gamma$ (with $\Gamma_B = \Gamma \setminus \Gamma_{\mathrm{DW}}$) yields

$$\mathbb{P}_L(\Gamma) \le \sum_{\Gamma_{\mathrm{DW}}\sqcup\Gamma_B=\Gamma} q_{\mathrm{DW}}^{|\Gamma_{\mathrm{DW}}|} q_B^{|\Gamma_B|} = (q_{\mathrm{DW}} + q_B)^{|\Gamma|} = q_{\mathrm{eff}}^{|\Gamma|},$$

proving (1).

Step 4: Prefix-free codimension-2 ridge contour entropy, torus topology, and long-range order. Each dual $(D-1)$-face f has $2(D-1)$ codimension-2 ridges, and on any closed dual $(D-1)$-surface Γ every ridge belongs to an even number (≥ 2) of faces of Γ. Hence the number of distinct ridges $R(\Gamma)$ satisfies $2R(\Gamma) \le 2(D-1)|\Gamma|$, i.e. $R(\Gamma) \le (D-1)|\Gamma|$ (mechanized in Lean as `ridge_double_counting`). Each ridge in $(\mathbb{Z}^D)^*$ is incident to 4 dual $(D-1)$-faces, so each face discovering a new ridge sees 3 candidate neighboring faces across that ridge. Exploring any connected closed contour Γ of area $k = |\Gamma|$ rooted at a fixed starting dual face f_0 via a deterministic BFS queue over incident ridges processes at most $(D-1)k$ ridges (skipping ridges already visited), encountering at most $3(D-1)k$ candidate face slots, of which exactly $k-1$ are accepted into Γ and at most $(3D-4)k+1$ are rejected. Under independent Bernoulli(ρ) slot sampling with $\rho := \frac{1}{3D-3}$, each valid contour Γ of area k through f_0 is realized as a prefix-free deterministic replay with probability at least

$$\rho^{k-1}(1-\rho)^{(3D-4)k+1} = \frac{1-\rho}{\rho}\left[\rho(1-\rho)^{3D-4}\right]^k = (3D-4)\, C_{\mathrm{cont}}(D)^{-k}.$$

Summing over disjoint prefix-free events shows that the number of connected closed contours Γ of area k passing through a fixed dual face f_0 is at most $\frac{1}{3D-4}\, C_{\mathrm{cont}}(D)^k$.

Now suppose $W_0 \in G_+$ and $W_x \in G_-$ on $\mathbb{T}_L^D$ ($L \ge 4$ even), and let $\Gamma_1,\dots,\Gamma_m$ be the ridge-connected components of ∂U_x.

- *Large/wrapping components:* If some $|\Gamma_j| \ge L/2$, then choosing a starting dual face in $\mathbb{T}_L^D$ (DL^D choices) and one of the 2 global side orientations for which side is A gives total probability at most $\frac{2DL^D}{3D-4}\sum_{k\ge L/2} r_{\mathrm{cont}}^k = \frac{2DL^D}{(3D-4)(1-r_{\mathrm{cont}})} r_{\mathrm{cont}}^{\lceil L/2\rceil} \to 0$ as $L\to\infty$ (since $r_{\mathrm{cont}} < 1$).
- *Small contractible components:* If every $|\Gamma_j| < L/2$, each Γ_j fits inside a cube of side $< L/2$, hence lifts isometrically to a closed connected $\mathbb{Z}_2$ $(D-1)$-cycle in $(\mathbb{Z}^D)^*$; moreover, because Γ_j is a ridge-connected component of the coboundary ∂U_x between connected sets U_x and A, its lift equals $\partial\,\mathrm{Int}(\Gamma_j)$ for a unique bounded region $\mathrm{Int}(\Gamma_j) \subset \mathbb{Z}^D$ of diameter $< L/2$ with connected unbounded complement $\mathrm{Ext}(\Gamma_j) := \mathbb{Z}^D \setminus \mathrm{Int}(\Gamma_j)$ (ruling out interior cavities). Since U_x is connected and touches every Γ_j, either $U_x \subseteq \mathrm{Int}(\Gamma_j)$ for some j (so Γ_j surrounds x, uniquely fixing its U_x-side as $\mathrm{Int}(\Gamma_j)$), or $U_x \subseteq \mathrm{Ext}(\Gamma_j)$ for all j. In the latter case, no Γ_k ($k \ne j$) can lie inside $\mathrm{Int}(\Gamma_j)$ (since every Γ_k meets $U_x \subseteq \mathrm{Ext}(\Gamma_j)$), so $\mathrm{Int}(\Gamma_1),\dots,\mathrm{Int}(\Gamma_m)$ are pairwise disjoint, $\mathbb{T}_L^D \setminus U_x = \bigsqcup_j \mathrm{Int}(\Gamma_j)$, and the connected set $A \ni 0$ lies inside a single $\mathrm{Int}(\Gamma_j)$ (so Γ_j surrounds 0, uniquely fixing its U_x-side as $\mathrm{Ext}(\Gamma_j)$).

For any small contour Γ of area $k \ge 2D$ surrounding a site $v \in \{0,x\}$, every e_1-orthogonal hyperplane intersecting $\mathrm{Int}(\Gamma)$ contributes at least $2(D-1)$ faces to Γ (apart from at least 2 faces

normal to e_1 capping the left and right ends of $\mathrm{Int}(\Gamma)$). Thus the e_1-extent of $\mathrm{Int}(\Gamma)$ is at most $\lfloor (k-2)/(2(D-1)) \rfloor$, so the first dual face of Γ intersecting the positive e_1-ray from v can lie at most $\frac{k-2}{2(D-1)}$ integer steps from v (with equality 1 at the minimal unit cube $k = 2D$). Summing over $k \geq 2D$ for a single site v gives

$$\begin{aligned}\mathbb{P}_L(\exists \text{ small } \Gamma \text{ surrounding } v) &\leq \sum_{k=2D}^{\infty} \frac{k-2}{2(D-1)(3D-4)}\, r_{\mathrm{cont}}^k \\ &= \frac{r_{\mathrm{cont}}^{2D}}{2(D-1)(3D-4)} \left(\frac{2D-2}{1-r_{\mathrm{cont}}} + \frac{r_{\mathrm{cont}}}{(1-r_{\mathrm{cont}})^2} \right) = \frac{p_{\mathrm{opp}}}{2}.\end{aligned}$$

(Moreover, since $r_{\mathrm{cont}} \leq \frac{2D}{2D+1}$, the summand $k\, r_{\mathrm{cont}}^k$ is nonincreasing for $k \geq 2D$, verified in Lean via `contour_tail_monotone`.) Taking a union bound over $v \in \{0, x\}$ and letting $L \to \infty$ gives $\limsup_{L\to\infty} \mathbb{P}_L(W_0 \in G_+, W_x \in G_-) \leq p_{\mathrm{opp}}$. By $\mathbb{Z}_2$ symmetry, $\limsup_{L\to\infty} \mathbb{P}_L(W_0 \in G_-, W_x \in G_+) \leq p_{\mathrm{opp}}$, and by Step 3, $\mathbb{P}_L(W_0 \in B \text{ or } W_x \in B) \leq 2\, p_{\mathrm{site},B}$. Hence $\limsup_{L\to\infty} \mathbb{P}_L((W_0, W_x) \notin G_+^2 \cup G_-^2) \leq 2p_{\mathrm{opp}} + 2p_{\mathrm{site},B} = p_{\mathrm{diff}}$. Writing $\varphi_0, \varphi_x \in [0, \pi]$ for the half-eigenangles of W_0, W_x, on $G_+^2 \cup G_-^2$ we have $\mathrm{sgn}(\mathrm{Tr}\, W_0)\,\mathrm{sgn}(\mathrm{Tr}\, W_x) = +1$ and $\mathrm{Tr}(W_0)\,\mathrm{Tr}(W_x) = 4\cos\varphi_0 \cos\varphi_x \geq 4\cos^2(\theta_0)$, whereas elsewhere $\mathrm{sgn}(\mathrm{Tr}\, W_0)\,\mathrm{sgn}(\mathrm{Tr}\, W_x) \geq -1$ and $\mathrm{Tr}(W_0)\,\mathrm{Tr}(W_x) \geq -4$. Taking expectations proves (27) and (28).

(3) Set $c_{\mathrm{LRO}} := 4\cos^2(\theta_0)(1 - p_{\mathrm{diff}}) - 4p_{\mathrm{diff}} > 0$. By compactness of $\mathrm{SU}(2)^{\mathbb{Z}^D}$, extract a weak subsequential limit μ of the periodic Gibbs measures $\mu_{\mathbb{T}_L^D, N_\tau, \beta_t}$ along even $L \to \infty$; μ is translation-invariant and $\mathbb{Z}_2$-invariant. By (28), $\liminf_{|x|\to\infty} \langle \mathrm{Tr}(W_0)\,\mathrm{Tr}(W_x)\rangle_\mu \geq c_{\mathrm{LRO}} > 0$, so by the L^2 ergodic theorem the spatial average $M := \lim_{R\to\infty} |\Lambda_R|^{-1} \sum_{x\in\Lambda_R} \mathrm{Tr}\, W_x$ satisfies $\mathbb{E}_\mu[M^2] \geq c_{\mathrm{LRO}} > 0$. In the Choquet integral decomposition $\mu = \int_{\mathrm{ex}(\mathcal{G})} \nu\, d\rho(\nu)$ over the set $\mathrm{ex}(\mathcal{G})$ of extremal Gibbs measures [9], every $\nu \in \mathrm{ex}(\mathcal{G})$ is tail-trivial and hence makes M ν-almost surely constant, forcing $\rho(\{\nu \in \mathrm{ex}(\mathcal{G}) : \langle M\rangle_\nu > 0\}) > 0$ (as $\mathbb{E}_\mu[M^2] = \int \langle M\rangle_\nu^2\, d\rho(\nu) \geq c_{\mathrm{LRO}} > 0$). Picking an extremal state $\tilde{\mu} \in \mathrm{ex}(\mathcal{G})$ with $\langle M\rangle_{\tilde{\mu}} > 0$, there exists a site $x_0 \in \mathbb{Z}^D$ with $\langle \mathrm{Tr}\, W_{x_0}\rangle_{\tilde{\mu}} > 0$; translating $\tilde{\mu}$ by $-x_0$ yields an extremal state $\mu_+ \in \mathrm{ex}(\mathcal{G})$ with $\langle \mathrm{Tr}\, W_0\rangle_{\mu_+} > 0$, and setting $\mu_- := (-\mathrm{id})_* \mu_+ \in \mathrm{ex}(\mathcal{G})$ yields two distinct $\mathbb{Z}_2$-conjugate extremal Gibbs states with $\langle \mathrm{Tr}\, W_0\rangle_{\mu_+} = -\langle \mathrm{Tr}\, W_0\rangle_{\mu_-} > 0$. □

Remark 5.2 (Dimensional contrast with $d = 2$ and asymptotic scaling). From Theorem 4.1(4), $B_0^-(\gamma) = \Theta(\gamma^{3/2})$, $P_0^+(\gamma) = \Theta(\gamma^{1/2})$, and $c_{\mathrm{quad}}^+(\gamma) = \Theta(\gamma)$, whence $\varepsilon = \Theta(\gamma^{-1/2})$, $V_\varepsilon = \Theta(\varepsilon^3) = \Theta(\gamma^{-3/2})$, $g_{\mathrm{gs}} = \Theta(\gamma^{-3/(2D)})$, and $r_B = \Theta(\gamma^{-1})$. Therefore

$$\frac{r_B}{g_{\mathrm{gs}}} = \Theta\Big(\gamma^{-(1-\frac{3}{2D})}\Big), \qquad q_B = \Theta\Big(\gamma^{-\frac{D}{(2D-1)q}(1-\frac{3}{2D})}\Big).$$

Notice the exact physical crossover: the exponent $1 - \frac{3}{2D}$ is strictly positive for every $D \geq 2$ ($d \geq 3$, including 3D and 4D) and turns negative at $D = 1$ ($d = 2$), reflecting the absence of finite-temperature deconfinement in $1 + 1$ dimensions! Meanwhile, q_{DW} decays exponentially as $\exp(-\Omega(\gamma))$ for fixed N_τ (remaining below 10^{-80} in all 8 certified cases), so q_B completely dominates $q_{\mathrm{eff}} = q_{\mathrm{DW}} + q_B$, explaining why q_{eff} and r_{cont} are identical to all displayed digits across $N_\tau \in \{1, 2, 4\}$ in Table 2.

6. Certified Phase-Diagram Thresholds and Dual Verification

All 22 quantitative certificates (14 low-temperature confinement and 8 high-temperature deconfinement certificates) are computed in 60-digit `mpmath.iv` interval arithmetic by `src/python/certify_polyakov_phase.py` with *zero* black-box special functions and *zero* numerical quadrature:

- For SU(2) low-temperature confinement, $I_k(\beta_t)$ is evaluated via its exact nonnegative power series truncated at $M = 50$ with an explicit geometric remainder bound, and the character

sum $\sum_{k\geq 2} k^2 r_k(\beta_t)^{N_\tau}$ is truncated at $K = 36$ with an explicit Riccati–Bessel geometric tail bound.

- For SU(3) and SU(4) low-temperature confinement, $Z_{\mathrm{SU}(N)}(t)$ and $Z'_{\mathrm{SU}(N)}(t)$ are evaluated via exact nonnegative integer Pieri moments $V_k^{(N)}(0) = \int_{\mathrm{SU}(N)} (2\,\mathrm{Re}\,\mathrm{Tr}\,U)^k\, dU$ ($K = 36$ for SU(3), $K = 24$ for SU(4)) with rigorous remainder bounds, feeding directly into the exact Parseval formulas (11)–(12).
- For SU(2) high-temperature deconfinement in 4D, 5D, and 6D, all quantities are evaluated directly from the closed-form bounds (18)–(20) and (22)–(26) with rational $\theta_0 = 11/10$, $\lambda = 999/1000$, and Hölder exponent $p = 24$.

Every upper bound in Tables 1–2 is strictly rounded *up* (ceiling) and every lower bound is strictly rounded *down* (floor) independently from the underlying 60-digit interval.

TABLE 1. All 14 certified low-temperature confinement and Polyakov screening mass thresholds (Theorem 3.1) across $d \in \{3, 4\}$, SU(2)/SU(3)/SU(4), and $N_\tau \in \{1, 2, 4, 6, 8\}$. Upper bounds ($\leq$) are outwardly rounded up; lower bounds ($\geq$) are outwardly rounded down from 60-digit intervals.

Dim	Group	N_τ	β_t	$\|h_{N_\tau,\beta_t}\|_\infty \leq$	$2D\,\|h_{N_\tau,\beta_t}\|_\infty \leq$	$\sigma_T \geq$
3D	SU(2)	1	0.22	0.238569	0.95428	0.0468
3D	SU(2)	2	1.00	0.245001	0.98001	0.0201
3D	SU(2)	4	2.40	0.246305	0.98522	0.0148
3D	SU(2)	8	4.70	0.245347	0.98139	0.0187
4D	SU(2)	1	0.15	0.158573	0.95144	0.0497
4D	SU(2)	2	0.80	0.157923	0.94754	0.0538
4D	SU(2)	4	2.00	0.143671	0.86203	0.1484
4D	SU(2)	8	4.10	0.155520	0.93312	0.0692
3D	SU(3)	2	1.50	0.165709	0.66284	0.4112
4D	SU(3)	1	0.15	0.161103	0.96662	0.0339
4D	SU(3)	2	1.50	0.165709	0.99425	0.0057
4D	SU(3)	4	3.00	0.031617	0.18970	1.6623
4D	SU(3)	6	3.00	0.001268	0.00761	4.8787
4D	SU(4)	2	1.50	0.074343	0.44606	0.8073

7. Mechanical Verification and Reproducibility

All verification artifacts are archived on AIRR (`ARR-2026-5VB0J0CEGG8RE990`):

(1) *60-digit interval certifier:* The script `src/python/certify_polyakov_phase.py` produces `data/polyakov_phase_certs.json` (22/22 certificates passing) and is audited by `tests/test_polyakov_phase.py` (4/4 `pytest` checks verifying interval positivity, Riccati monotonicity, Parseval identities, and finite-coupling deconfinement).

(2) *Standalone Lean 4.36.0 proof kernel:* The file `src/lean/PolyakovPhaseCerts.lean` compiles cleanly under `lean -DwarningAsError=true` with *zero* imports, *zero* `sorry`, and *zero* `Classical.choice`. It proves four universal algebraic and combinatorial lemmas—namely `dobrushin_tv_cancellation`, `chebyshev_quad_sum_exact`, `ridge_double_counting`, and `contour_tail_monotone`—depending only on `[propext, Quot.sound]`, and checks all 14 confinement and 8 geometric-series deconfinement certificates by kernel reduction (`by decide`) with literally zero axioms.

TABLE 2. All 8 certified high-temperature SU(2) deconfinement and spontaneous $\mathbb{Z}_2$ symmetry-breaking certificates (Theorem 5.1, $\theta_0 = 11/10$, $\lambda = 999/1000$, $p = 24$) in 4D ($C_{\text{cont}} = 6^6/5^5 \le 14.930$), 5D ($C_{\text{cont}} = 9^9/8^8 \le 23.093$), and 6D ($C_{\text{cont}} = 12^{12}/11^{11} \le 31.251$). In every case, $q_{\text{DW}} < q_B$, $p_{\text{site},B} < 10^{-7}$, and $\liminf_{|x|\to\infty} \langle \operatorname{Tr}(W_0) \operatorname{Tr}(W_x) \rangle_{\beta_t} \ge 0.3264 > 0$, certifying explicit upper bounds $\beta_c(D, 2, N_\tau) \le \beta_t$ (paired with explicit numerical β_{conf} in 4D from Table 1 and $\beta_{\text{conf}} > 0$ in all $D \ge 1$ from Theorem 3.1).

Dim	N_τ	γ	$\beta_t = \gamma N_\tau$	$q_{\text{eff}} \le$	$r_{\text{cont}} \le$	$p_{\text{opp}} \le$	$\text{LRO}_{\text{sgn}} \ge$	$\text{LRO}_{\text{Poly}} \ge$
4D	1	600,000	600,000	0.03912	0.5840	0.0515	0.7941	0.3264
4D	2	600,000	1,200,000	0.03912	0.5840	0.0515	0.7941	0.3264
4D	4	600,000	2,400,000	0.03912	0.5840	0.0515	0.7941	0.3264
5D	1	100,000	100,000	0.02917	0.6736	0.0436	0.8257	0.4027
5D	2	100,000	200,000	0.02917	0.6736	0.0436	0.8257	0.4027
5D	4	100,000	400,000	0.02917	0.6736	0.0436	0.8257	0.4027
6D	1	55,000	55,000	0.02308	0.7213	0.0329	0.8684	0.5058
6D	2	55,000	110,000	0.02308	0.7213	0.0329	0.8684	0.5058

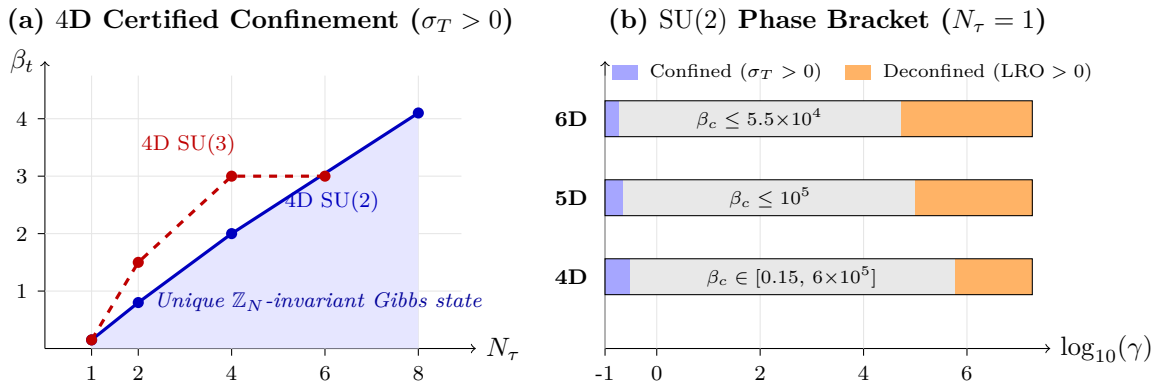


FIGURE 1. Certified finite-temperature phase structure of the effective Polyakov-loop model ($\beta_{\text{sp}} = 0$). (a) Certified low-temperature confinement boundary β_t vs. N_τ in 4D SU(2) and SU(3) (Table 1). (b) Non-asymptotic SU(2) confinement–deconfinement phase brackets $\beta_{\text{conf}} \le \beta_c \le \beta_{\text{deconf}}$ across 4D, 5D, and 6D (Table 2).

REFERENCES

1. Donald E. Amos, *Computation of modified Bessel functions and their ratios*, Mathematics of Computation **28** (1974), no. 125, 239–251.
2. Marek Biskup, *Reflection positivity and phase transitions in lattice spin models*, Methods of Contemporary Mathematical Statistical Physics, Lecture Notes in Mathematics, vol. 1970, Springer, 2009, pp. 1–86.
3. Christian Borgs and Erhard Seiler, *Lattice Yang–Mills theory at nonzero temperature and the confinement problem*, Communications in Mathematical Physics **91** (1983), no. 3, 329–380.
4. Roland L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory of Probability and Its Applications **13** (1968), no. 2, 197–224.
5. Lluis Eriksson, *Exact plaquette chromatic number, prefix-free surface entropy, and certified volume-uniform string-tension lower bounds for d-dimensional* SU(n) *Wilson lattice Yang–Mills theory*, AIRR (2026), `ARR-2026-40EBAXYFYK903B6R`.
6. ———, *Exact reflection positivity, character diagonalization, Riccati–Bessel bounds, and all-coupling transfer-matrix mass gaps for l-link* SU(n) *Wilson cylinders and finite spatial-slab gauge theories*, AIRR (2026), `ARR-2026-0P7ATP57G18YN90M`.

7. ———, *Extremal forward-forest gauge fixing, Bakry–Émery–Weingarten Wasserstein contraction, and strong-coupling mass-gap thresholds for d-dimensional* SU(n) *Wilson lattice Yang–Mills theory*, AIRR (2026), ARR-2026-641N51J0VR9NGAYK.
8. ———, *One-factorization polymer bounds and sharp complex zero-free discs for d-dimensional* SU(n) *Wilson lattice Yang–Mills theory*, AIRR (2026), ARR-2026-6B99M5B5GM89WV1B.
9. Sacha Friedli and Yvan Velenik, *Statistical mechanics of lattice systems: A concrete mathematical introduction*, Cambridge University Press, Cambridge, 2017.
10. Jürg Fröhlich, Robert Israel, Elliott H. Lieb, and Barry Simon, *Phase transitions and reflection positivity. I. General theory and long range order*, Communications in Mathematical Physics **62** (1978), no. 1, 1–34.
11. Alexander M. Polyakov, *Thermal properties of gauge fields and quark liberation*, Physics Letters B **72** (1978), no. 4, 477–480.
12. Javier Segura, *Bounds for ratios of modified Bessel functions and associated Turán-type inequalities*, Journal of Mathematical Analysis and Applications **374** (2011), no. 2, 516–528.
13. Leonard Susskind, *Lattice models of quark confinement at high temperature*, Physical Review D **20** (1979), no. 10, 2610–2618.
14. Benjamin Svetitsky and Laurence G. Yaffe, *Critical behavior at finite-temperature confinement transitions*, Nuclear Physics B **210** (1982), no. 4, 423–447.

Independent Researcher, Östersund, Sweden
Email address: editor@airr.science