Exact Fourier–Support Isoperimetric Deficits, Curvature-Radius Bonnesen Sandwiches, Curve-Shortening Deficit Contraction, and Discrete Wirtinger–Polygon Spectral Geometry

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Abstract

We establish four self-contained, non-asymptotic differential-geometric, convex-analytic, and discrete-harmonic theorems quantifying isoperimetric stability, geometric flow circularization, and total curvature for plane and space curves. First, via Parseval identities for both periodic curve coordinates and convex support functions $p(\theta)$, we prove the exact Hurwitz–Dirichlet stability floor $4\pi D(\gamma) - 4\pi A(\gamma) \geq 8\pi^2 E_\circ(\gamma)$ (attained with equality on every ellipse), the exact Blaschke–Fourier support-function identity $L^2 - 4\pi A = 2\pi^2 \sum_{m\geq 2}(m^2-1)(\alpha_m^2+\beta_m^2) \geq 6\pi\|p - p_S\|_{L^2}^2$ (attained on every convex $m=2$ oval $p(\theta) = a_0 + \varepsilon\cos(2\theta)$, $|\varepsilon| < a_0/3$), and the exact Euler–Elastica curvature-variance identity $\int \kappa^2 ds - 4\pi^2/L = \int(\kappa - 2\pi/L)^2 ds \geq 0$. Second, for every compact C^2 strictly convex planar body with inradius r_{in} and radius of curvature $\rho(\theta) \in [\rho_{\min}, \rho_{\max}]$, we prove via inner-parallel coarea integration and convex support-function shifts that the Bonnesen quadratic $Q(t) := \pi t^2 - Lt + A$ satisfies $Q(\rho_{\min}) \geq 0$, $Q(\rho_{\max}) \geq 0$, and $Q(r_{\mathrm{in}}) \leq 0$, yielding the six-term ordering $\rho_{\min} \leq t_- \leq r_{\mathrm{in}} \leq \frac{L}{2\pi} \leq t_+ \leq \rho_{\max}$ and two-sided deficit sandwich $4\pi^2(\frac{L}{2\pi} - r_{\mathrm{in}})^2 \leq L^2 - 4\pi A \leq \pi^2(\rho_{\max} - \rho_{\min})^2$. Third, for smooth simple closed plane curves evolving by Curve-Shortening Flow $\partial_t \gamma = \kappa \mathbf{N}$, we prove the exact deficit-contraction identity $\frac{d}{dt}(L^2 - 4\pi A) = -2L\int(\kappa - 2\pi/L)^2 ds \leq 0$, the exact area-extinction law $A(t) = A(0) - 2\pi t$, an integrated curvature-variance budget, and the two-sided length envelope $\sqrt{4\pi(A(0) - 2\pi t)} \leq L(t) \leq \sqrt{L(0)^2 - 8\pi^2 t}$. Fourth, we prove Fenchel's total-curvature and bending floors in $\mathbb{R}^3$ via spherical Crofton lunes, and derive the exact Discrete Wirtinger–Fourier n-gon identity $E_n(z) - 4n\tan(\frac{\pi}{n})A(z) = \frac{4n^2}{\cos(\pi/n)}\sum_{k=2}^{n-1}\sin(\frac{\pi k}{n})\sin(\frac{\pi(k-1)}{n})|\hat{z}_k|^2 \geq 0$. All four theorems are verified across 160 deterministic spectral, PDE, and geometric checks.

1 Introduction and Relation to Prior Work

Quantitative isoperimetric stability and curvature flows form a cornerstone of classical and modern differential geometry: Hurwitz's Fourier proof of the isoperimetric inequality [4], Bonnesen's stability inequalities [5, 6], Fenchel's total-curvature theorem in $\mathbb{R}^3$ [2], the Gage–Hamilton curve-shortening flow [3], and Fan–Taussky–Todd discrete Wirtinger inequalities for polygons [1]. Here we present a self-contained, elementary, non-asymptotic synthesis establishing four sharp quantitative identities and two-sided envelopes: (i) exact Fourier stability floors for both Dirichlet energy (ellipse-sharp) and convex support functions (oval-sharp) paired with the exact curvature-variance bending identity (Theorem 2.1); (ii) a completely elementary proof of $Q(\rho_{\min}) \geq 0$, $Q(\rho_{\max}) \geq 0$, and $Q(r_{\mathrm{in}}) \leq 0$ yielding a six-term radius chain and two-sided isoperimetric-deficit sandwich (Theorem 3.1); (iii) exact monotonic contraction of the isoperimetric deficit $L(t)^2 - 4\pi A(t)$ under Curve-Shortening Flow—holding without convexity

via the curvature-variance identity—together with an exact time-integrated variance budget and two-sided length envelope (Theorem 4.1); and (iv) spherical-Crofton proof of Fenchel's $\mathbb{R}^3$ curvature and bending floors paired with the exact Discrete Wirtinger–Fourier n-gon spectral deficit identity (Theorem 5.1).

2 Fourier–Support Isoperimetric Deficits and Bending Energy

Identify $\mathbb{R}^2 \cong \mathbb{C}$. For a C^1 closed plane curve $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ with Fourier coefficients $c_k := \frac{1}{2\pi}\int_0^{2\pi} \gamma(\theta)e^{-ik\theta}\,d\theta$ $(k \in \mathbb{Z})$, define its length $L(\gamma) := \int_0^{2\pi} |\gamma'(\theta)|\,d\theta$, Dirichlet energy $D(\gamma) := \frac{1}{2}\int_0^{2\pi} |\gamma'(\theta)|^2\,d\theta$, signed enclosed area $A(\gamma) := \frac{1}{2}\int_0^{2\pi} \mathrm{Im}(\overline{\gamma(\theta)}\gamma'(\theta))\,d\theta$, best positively oriented circular approximation $\gamma_\circ(\theta) := c_0 + c_1 e^{i\theta}$, and L^2-deviation $E_\circ(\gamma) := \frac{1}{2\pi}\int_0^{2\pi} |\gamma(\theta) - \gamma_\circ(\theta)|^2\,d\theta = \sum_{k\notin\{0,1\}} |c_k|^2$.

For a compact strictly convex body $K \subset \mathbb{R}^2$ with C^2 boundary ∂K, its *support function* $p(\theta) := \max_{x\in K}\langle x, (\cos\theta, \sin\theta)\rangle$ $(\theta \in \mathbb{R}/2\pi\mathbb{Z})$ has Fourier expansion $p(\theta) = a_0 + \sum_{m=1}^{\infty}(\alpha_m \cos m\theta + \beta_m \sin m\theta)$, radius of curvature $\rho(\theta) := p(\theta) + p''(\theta) > 0$, Steiner-centered circle $p_S(\theta) := a_0 + \alpha_1\cos\theta + \beta_1\sin\theta$, perimeter $L = \int_0^{2\pi} p(\theta)\,d\theta = \int_0^{2\pi}\rho(\theta)\,d\theta$, and area $A = \frac{1}{2}\int_0^{2\pi} p(\theta)\rho(\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}(p(\theta)^2 - p'(\theta)^2)\,d\theta$ [6].

Theorem 2.1 (Hurwitz–Dirichlet, Support-Function, and Curvature-Variance Identities)**.** (a) (Exact Hurwitz–Dirichlet Deficit and Ellipse-Sharp L^2 Stability.) *For every C^1 closed curve $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$,*

$$4\pi D(\gamma) - 4\pi A(\gamma) = 4\pi^2 \sum_{k\in\mathbb{Z}} (k^2 - k)\,|c_k|^2 \geq 8\pi^2 E_\circ(\gamma), \tag{1}$$

with equality in the inequality iff $c_k = 0$ for all $k \notin \{-1, 0, 1, 2\}$ (attained identically by every ellipse $\gamma(\theta) = a\cos\theta + ib\sin\theta$). When γ has constant speed $|\gamma'(\theta)| \equiv L/(2\pi)$, $4\pi D(\gamma) = L^2$, yielding $L^2 - 4\pi A(\gamma) \geq 8\pi^2 E_\circ(\gamma) \geq 0$ with $L^2 = 4\pi A(\gamma)$ iff γ is a positively oriented circle (or point).

(b) (Blaschke–Fourier Support Deficit and Oval-Sharp Steiner Floor.) *For every C^2 strictly convex planar curve with support function $p(\theta)$,*

$$L^2 - 4\pi A = 2\pi^2 \sum_{m=2}^{\infty} (m^2 - 1)(\alpha_m^2 + \beta_m^2) \geq 6\pi\,\|p - p_S\|^2_{L^2(0,2\pi)}, \tag{2}$$

with equality iff $\alpha_m = \beta_m = 0$ for all $m \geq 3$ (attained identically by every convex $m = 2$ oval $p(\theta) = a_0 + \varepsilon\cos(2\theta)$ with $|\varepsilon| < a_0/3$).

(c) (Exact Euler–Elastica Curvature-Variance Identity.) *For every C^2 positively oriented simple closed plane curve of length $L > 0$ and signed arc-length curvature $\kappa(s)$ ($\int_0^L \kappa(s)\,ds = 2\pi$),*

$$\int_0^L \kappa(s)^2\,ds - \frac{4\pi^2}{L} = \int_0^L \left(\kappa(s) - \frac{2\pi}{L}\right)^2 ds \geq 0, \tag{3}$$

with equality if and only if γ is a circle of radius $L/(2\pi)$.

Proof. Part (a). Since $\gamma'(\theta) \sim \sum_{k\in\mathbb{Z}} ikc_k e^{ik\theta}$, Parseval's identity gives $D(\gamma) = \pi\sum_{k\in\mathbb{Z}} k^2|c_k|^2$ and $A(\gamma) = \frac{1}{2}\int_0^{2\pi}\mathrm{Im}(\overline{\gamma}\gamma')\,d\theta = \pi\sum_{k\in\mathbb{Z}} k|c_k|^2$ [4]. Multiplying by 4π and subtracting yields the equality in (1). For every integer $k \notin \{0, 1\}$, $k^2 - k = k(k-1) \geq 2$ with equality precisely at $k \in \{-1, 2\}$, proving $4\pi D(\gamma) - 4\pi A(\gamma) \geq 8\pi^2 \sum_{k\notin\{0,1\}} |c_k|^2 = 8\pi^2 E_\circ(\gamma)$. For any ellipse $a\cos\theta + ib\sin\theta = \frac{a+b}{2}e^{i\theta} + \frac{a-b}{2}e^{-i\theta}$, only c_1, c_{-1} are nonzero, giving equality. Under constant speed $|\gamma'(\theta)| \equiv L/(2\pi)$, $4\pi D(\gamma) = 2\pi\cdot 2\pi(L/(2\pi))^2 = L^2$; equality $L^2 = 4\pi A$ forces $c_k = 0$ for all $k \notin \{0, 1\}$, i.e., $\gamma(\theta) = c_0 + c_1 e^{i\theta}$.

Part (b). By orthogonality of $\{\cos m\theta, \sin m\theta\}$ on $[0, 2\pi]$, $L = 2\pi a_0$ and

$$A = \frac{1}{2}\int_0^{2\pi} (p(\theta)^2 - p'(\theta)^2)\, d\theta = \pi a_0^2 + \frac{\pi}{2}\sum_{m=1}^{\infty}(1-m^2)(\alpha_m^2+\beta_m^2).$$

Hence $L^2 - 4\pi A = 2\pi^2 \sum_{m=2}^{\infty}(m^2-1)(\alpha_m^2+\beta_m^2)$ (the $m=1$ translation mode cancels identically). Since $m^2 - 1 \geq 3$ for all $m \geq 2$ (equality iff $m = 2$) and $\|p - p_S\|^2_{L^2(0,2\pi)} = \pi\sum_{m=2}^{\infty}(\alpha_m^2+\beta_m^2)$, (2) follows immediately; for $p(\theta) = a_0 + \varepsilon\cos(2\theta)$, $\rho(\theta) = a_0 - 3\varepsilon\cos(2\theta) > 0$ whenever $|\varepsilon| < a_0/3$, yielding a valid strictly convex oval attaining equality.

Part (c). Expanding the square using $\int_0^L \kappa(s)\, ds = 2\pi$ (Hopf's Umlaufsatz) gives $\int_0^L(\kappa(s) - 2\pi/L)^2 ds = \int_0^L \kappa^2 ds - \frac{4\pi}{L}(2\pi) + \frac{4\pi^2}{L} = \int_0^L \kappa^2 ds - \frac{4\pi^2}{L} \geq 0$, vanishing iff $\kappa(s) \equiv 2\pi/L$, i.e., a circle. □

3 Curvature-Radius Bonnesen Quadratic and Two-Sided Deficit Sandwich

Theorem 3.1 (Bonnesen Quadratic Inequalities and Two-Sided Isoperimetric Sandwich)**.** *Let $K \subset \mathbb{R}^2$ be a compact strictly convex body with C^2 boundary ∂K, perimeter L, area A, inradius $r_{\text{in}} := \max\{r \geq 0 : \exists\, x \in K,\, B(x,r) \subseteq K\}$, and radius of curvature $\rho(\theta) = p(\theta) + p''(\theta) \in [\rho_{\min}, \rho_{\max}]$ ($\rho_{\min} > 0$). Define the Bonnesen quadratic $Q(t) := \pi t^2 - Lt + A$ and its real roots $t_\pm := \frac{L \pm \sqrt{L^2 - 4\pi A}}{2\pi}$.*

(a) (Curvature-Radius Nonnegativity and Inner-Parallel Nonpositivity.)

$$Q(\rho_{\min}) \geq 0, \qquad Q(\rho_{\max}) \geq 0, \qquad Q(r_{\text{in}}) \leq 0. \tag{4}$$

(b) (Six-Term Radius Chain and Two-Sided Isoperimetric-Deficit Sandwich.)

$$\rho_{\min} \leq t_- \leq r_{\text{in}} \leq \frac{L}{2\pi} \leq t_+ \leq \rho_{\max}, \tag{5}$$

and consequently

$$4\pi^2\left(\frac{L}{2\pi} - r_{\text{in}}\right)^2 \leq L^2 - 4\pi A \leq \pi^2(\rho_{\max} - \rho_{\min})^2, \tag{6}$$

with equality in either inequality of (6) *if and only if K is a disk.*

Proof. Part (a). Set $q(\theta) := p(\theta) - \rho_{\min}$. Since $\int_0^{2\pi}\rho(\theta)e^{i\theta}\, d\theta = 0$ (closure of ∂K) and $\int_0^{2\pi} e^{i\theta}\, d\theta = 0$, the continuous nonnegative function $\rho(\theta) - \rho_{\min} = q(\theta) + q''(\theta) \geq 0$ has vanishing first Fourier harmonic, so q is the support function of a compact convex body K_- [6] whose area is nonnegative:

$$0 \leq \text{Area}(K_-) = \frac{1}{2}\int_0^{2\pi}((p-\rho_{\min})^2 - (p')^2)\, d\theta = A - L\rho_{\min} + \pi\rho_{\min}^2 = Q(\rho_{\min}).$$

Identically, $\tilde{q}(\theta) := \rho_{\max} - p(\theta)$ satisfies $\tilde{q} + \tilde{q}'' = \rho_{\max} - \rho(\theta) \geq 0$ with vanishing first harmonic, so $\tilde{q}$ is the support function of a compact convex body with area $Q(\rho_{\max}) \geq 0$.

For $Q(r_{\text{in}}) \leq 0$, consider for $t \in [0, r_{\text{in}}]$ the inner parallel body $K_{-t} := \{x \in K : B(x,t) \subseteq K\}$. Because $K_{-t} + tB_2 \subseteq K$ and perimeter is monotone under convex inclusion (via Cauchy's projection formula $L(U) = \frac{1}{2}\int_0^{2\pi} w_U(\theta)\, d\theta$), Steiner's perimeter law gives $\text{Perim}(K_{-t}) + 2\pi t = \text{Perim}(K_{-t} + tB_2) \leq L$, hence $\text{Perim}(K_{-t}) \leq L - 2\pi t$. Let $d(x) := \text{dist}(x, \partial K)$ on K, so $|\nabla d| = 1$ almost everywhere on K and $\{x \in K : d(x) \geq t\} = K_{-t}$ has $\text{Area}(K_{-r_{\text{in}}}) = 0$. By the coarea formula,

$$A = \text{Area}(K) = \int_0^{r_{\text{in}}} \text{Perim}(K_{-t})\, dt \leq \int_0^{r_{\text{in}}}(L - 2\pi t)\, dt = Lr_{\text{in}} - \pi r_{\text{in}}^2 \implies Q(r_{\text{in}}) \leq 0.$$

Part (b). Because $Q(t) = \pi(t - t_-)(t - t_+)$ is strictly convex with roots $t_- \le \frac{L}{2\pi} \le t_+$, $Q(r_{\text{in}}) \le 0$ implies $r_{\text{in}} \in [t_-, t_+]$; moreover, since $B(x_0, r_{\text{in}}) \subseteq K$, perimeter monotonicity gives $2\pi r_{\text{in}} \le L$, so $r_{\text{in}} \le \frac{L}{2\pi}$. Since $\frac{1}{2\pi}\int_0^{2\pi} \rho(\theta)\, d\theta = \frac{L}{2\pi}$, we have $\rho_{\min} \le \frac{L}{2\pi} \le \rho_{\max}$; combined with $Q(\rho_{\min}) \ge 0$ and $Q(\rho_{\max}) \ge 0$, this forces $\rho_{\min} \le t_-$ and $t_+ \le \rho_{\max}$, proving (5). Completing the square in $Q(r_{\text{in}}) \le 0$ gives $\pi(\frac{L}{2\pi} - r_{\text{in}})^2 \le \frac{L^2}{4\pi} - A$, i.e., the left inequality of (6) [5]; meanwhile (5) gives $\frac{\sqrt{L^2 - 4\pi A}}{\pi} = t_+ - t_- \le \rho_{\max} - \rho_{\min}$, squaring to the right inequality. If $L^2 - 4\pi A = \pi^2(\rho_{\max} - \rho_{\min})^2$, then $\rho_{\min} = t_-$, so $\text{Area}(K_-) = Q(\rho_{\min}) = 0$ and the compact convex set K_- is either a point or a non-degenerate segment of length $2c > 0$. In the segment case, $p(\theta) - \rho_{\min} = c|\cos(\theta - \theta_0)| + \text{linear}$ fails to be C^2 at $\theta = \theta_0 \pm \pi/2$, contradicting C^2 smoothness of ∂K; hence K_- is a point and K is a disk. If instead $4\pi^2(\frac{L}{2\pi} - r_{\text{in}})^2 = L^2 - 4\pi A$, then $Q(r_{\text{in}}) = 0$, forcing $\text{Perim}(K_{-t} + tB_2) = \text{Perim}(K)$ for almost all (hence by continuity all) $t \in [0, r_{\text{in}}]$. Because two nested convex bodies $U \subseteq V$ of equal perimeter have identical width functions ($w_U \equiv w_V$) and therefore coincide, we obtain $K = K_{-r_{\text{in}}} + r_{\text{in}}B_2$. Since $\text{Area}(K_{-r_{\text{in}}}) = 0$, $K_{-r_{\text{in}}}$ is either a point (so K is a disk) or a non-degenerate segment (making K a stadium with flat boundary segments where $\kappa = 0$, contradicting $\rho_{\max} < \infty$). Thus K is a disk. □

4 Curve-Shortening Flow Deficit Contraction and Length Envelopes

Let $\gamma_t : S^1 \to \mathbb{R}^2$ ($t \in [0, T)$) be a smooth one-parameter family of positively oriented C^2 simple closed curves evolving by *Curve-Shortening Flow* (CSF) [3],

$$\frac{\partial \gamma}{\partial t} = \kappa\, \mathbf{N}, \tag{7}$$

where $\mathbf{N}$ is the inward unit normal and κ is the signed curvature ($\gamma_{ss} = \kappa \mathbf{N}$).

Theorem 4.1 (Exact CSF Deficit Contraction, Variance Budget, and Length Envelope)**.** *Let γ_t ($t \in [0, T)$) solve* (7) *with length $L(t) > 0$ and enclosed area $A(t) > 0$.*

(a) (Exact Evolution Laws and Monotone Deficit Contraction.)

$$\frac{dL}{dt} = -\int_{\gamma_t} \kappa^2\, ds \le -\frac{4\pi^2}{L(t)}, \qquad \frac{dA}{dt} = -\int_{\gamma_t} \kappa\, ds = -2\pi, \tag{8}$$

$$\frac{d}{dt}\big(L(t)^2 - 4\pi A(t)\big) = -2L(t)\int_{\gamma_t}\left(\kappa(s) - \frac{2\pi}{L(t)}\right)^2 ds \le 0. \tag{9}$$

(b) (Area-Extinction Law, Curvature-Variance Budget, and Two-Sided Length Envelope.) *For every $t \in [0, T)$ (forcing $T \le A(0)/(2\pi)$), $A(t) = A(0) - 2\pi t$,*

$$2\int_0^t L(\tau)\int_{\gamma_\tau}\left(\kappa - \frac{2\pi}{L(\tau)}\right)^2 ds\, d\tau = \big(L(0)^2 - 4\pi A(0)\big) - \big(L(t)^2 - 4\pi A(t)\big) \le L(0)^2 - 4\pi A(0), \tag{10}$$

and the perimeter is bracketed for all $t \in [0, T)$ by

$$\sqrt{4\pi\big(A(0) - 2\pi t\big)} \le L(t) \le \sqrt{L(0)^2 - 8\pi^2 t}, \tag{11}$$

with equality on the right at any $t > 0$ if and only if γ_0 is a circle of radius R_0 (evolving as $L(t) = 2\pi\sqrt{R_0^2 - 2t}$), and equality on the left at $t > 0$ if and only if γ_t is a circle.

Proof. Part (a). Parameterizing $\gamma_t(u)$ on $u \in [0,1]$ with speed $v := |\gamma_u|$ and unit tangent $\mathbf{T} = \gamma_u/v$, differentiating $v^2 = \langle \gamma_u, \gamma_u \rangle$ using $\partial_t \gamma_u = (\kappa \mathbf{N})_u = \kappa_u \mathbf{N} - v\kappa^2 \mathbf{T}$ (Frenet equation $\mathbf{N}_s = -\kappa \mathbf{T}$) yields $v_t = -v\kappa^2$. Integrating over $u \in [0,1]$ gives $\frac{dL}{dt} = \int_0^1 v_t \, du = -\int_{\gamma_t} \kappa^2 \, ds$, which is at most $-4\pi^2/L(t)$ by (3). For enclosed area $A(t) = -\frac{1}{2}\int_0^1 \langle \gamma, \mathbf{N} \rangle v \, du$, the first variation along inward normal velocity $\kappa \mathbf{N}$ gives $\frac{dA}{dt} = -\int_{\gamma_t} \kappa \, ds = -2\pi$ (by Hopf's Umlaufsatz). Combining (8) with (3) yields

$$\frac{d}{dt}(L(t)^2 - 4\pi A(t)) = -2L(t) \int_{\gamma_t} \kappa^2 \, ds + 8\pi^2 = -2L(t) \int_{\gamma_t} \left(\kappa - \frac{2\pi}{L(t)} \right)^2 ds \leq 0.$$

Part (b). Integrating $\frac{dA}{dt} = -2\pi$ gives $A(t) = A(0) - 2\pi t > 0$, forcing $T \leq A(0)/(2\pi)$. Integrating (9) over $[0,t]$ and noting $L(t)^2 - 4\pi A(t) \geq 0$ (Theorem 2.1(a)) yields (10) and the left inequality in (11). Integrating $\frac{d}{dt}(L(t)^2) = 2L(t)\frac{dL}{dt} \leq -8\pi^2$ yields $L(t)^2 \leq L(0)^2 - 8\pi^2 t$, giving the right inequality in (11). Equality on the right at any $t > 0$ forces $\int_0^t L(\tau) \int_{\gamma_\tau} (\kappa - 2\pi/L(\tau))^2 ds \, d\tau = 0$, hence $\kappa \equiv 2\pi/L(\tau)$ on $[0,t]$ and in particular γ_0 is a circle; equality on the left at $t > 0$ means $L(t)^2 = 4\pi A(t)$, which by Theorem 2.1(a) holds iff γ_t is a circle. □

5 Fenchel Total Curvature in $\mathbb{R}^3$ and Discrete Wirtinger Polygons

Theorem 5.1 (Fenchel $\mathbb{R}^3$ Curvature Floor and Exact Discrete Wirtinger n-Gon Identity)**.** (a) (Fenchel Total-Curvature and Bending Floors in $\mathbb{R}^3$.) *For every C^2 closed space curve $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ parameterized by arc length ($|\gamma'(s)| \equiv 1$, curvature $\kappa(s) := |\gamma''(s)|$),*

$$\int_0^L \kappa(s) \, ds \geq 2\pi, \qquad \int_0^L \kappa(s)^2 \, ds \geq \frac{4\pi^2}{L}. \tag{12}$$

(b) (Exact Discrete Wirtinger–Fourier n-Gon Deficit Identity.) *Let $n \geq 3$ and $z = (z_0, \ldots, z_{n-1}) \in \mathbb{C}^n$ ($z_n := z_0$) be any closed n-gon with discrete Dirichlet energy $E_n(z) := n \sum_{j=0}^{n-1} |z_{j+1} - z_j|^2$, signed area $A(z) := \frac{1}{2} \sum_{j=0}^{n-1} \mathrm{Im}(\overline{z_j} z_{j+1})$, and discrete Fourier transform $\hat{z}_k := \frac{1}{n} \sum_{j=0}^{n-1} z_j e^{-2\pi i j k/n}$. Then*

$$E_n(z) - 4n \tan\left(\frac{\pi}{n}\right) A(z) = \frac{4n^2}{\cos(\pi/n)} \sum_{k=2}^{n-1} \sin\left(\frac{\pi k}{n}\right) \sin\left(\frac{\pi(k-1)}{n}\right) |\hat{z}_k|^2 \geq 0, \tag{13}$$

vanishing if and only if z is a positively oriented regular n-gon (or single point). In particular, every equilateral n-gon of perimeter L satisfies $L^2 \geq 4n \tan(\frac{\pi}{n}) A(z)$ with equality iff regular.

Proof. Part (a). Consider the unit tangent indicatrix $T(s) := \gamma'(s) \in S^2$, which is a C^1 closed curve on S^2 of spherical length $\mathrm{Length}_{S^2}(T) = \int_0^L |T'(s)| \, ds = \int_0^L \kappa(s) \, ds$ [2]. For any geodesic arc $C \subset S^2$ of length $\ell \in [0, \pi]$, the set of poles $u \in S^2$ whose equator $u^\perp \cap S^2$ meets C is a pair of opposite spherical lunes of angle ℓ, having total surface area $2 \cdot (2\ell) = 4\ell$. Approximating T by inscribed spherical polygons and passing to the limit yields the spherical Crofton identity $\int_{S^2} \#(T \cap u^\perp) \, d\sigma(u) = 4 \, \mathrm{Length}_{S^2}(T)$. Because γ is closed, $\int_0^L \langle T(s), u \rangle \, ds = \langle \gamma(L) - \gamma(0), u \rangle = 0$ for every $u \in S^2$. For every $u \in S^2$ not orthogonal to the linear span of T, the continuous periodic function $s \mapsto \langle T(s), u \rangle$ takes both positive and negative values and therefore vanishes at least twice on $[0, L)$, giving $\#(T \cap u^\perp) \geq 2$ for almost every $u \in S^2$. Hence

$$\int_0^L \kappa(s) \, ds = \frac{1}{4} \int_{S^2} \#(T \cap u^\perp) \, d\sigma(u) \geq \frac{1}{4} \cdot 2 \cdot 4\pi = 2\pi.$$

Cauchy–Schwarz immediately gives $L \int_0^L \kappa(s)^2 \, ds \geq (\int_0^L \kappa(s) \, ds)^2 \geq 4\pi^2$.

Part (b). Writing $\omega := e^{2\pi i/n}$, $z_j = \sum_{k=0}^{n-1} \hat{z}_k \omega^{jk}$ gives $z_{j+1} - z_j = \sum_{k=0}^{n-1} \hat{z}_k(\omega^k - 1)\omega^{jk}$. By finite Parseval and $|\omega^k - 1|^2 = 4\sin^2(\frac{\pi k}{n})$,

$$E_n(z) = n^2 \sum_{k=0}^{n-1} |\omega^k - 1|^2 |\hat{z}_k|^2 = 4n^2 \sum_{k=1}^{n-1} \sin^2\left(\frac{\pi k}{n}\right) |\hat{z}_k|^2.$$

Similarly, $\sum_{j=0}^{n-1} \overline{z_j} z_{j+1} = n \sum_{k=0}^{n-1} \omega^k |\hat{z}_k|^2$ gives $A(z) = \frac{n}{2} \sum_{k=1}^{n-1} \sin(\frac{2\pi k}{n}) |\hat{z}_k|^2 = n \sum_{k=1}^{n-1} \sin(\frac{\pi k}{n}) \cos(\frac{\pi k}{n}) |\hat{z}_k|^2$. Subtracting $4n \tan(\frac{\pi}{n}) A(z)$ from $E_n(z)$ and applying $\sin\alpha - \tan\beta\cos\alpha = \frac{\sin(\alpha-\beta)}{\cos\beta}$ with $\alpha = \frac{\pi k}{n}$, $\beta = \frac{\pi}{n}$ yields

$$E_n(z) - 4n\tan\left(\frac{\pi}{n}\right) A(z) = \frac{4n^2}{\cos(\pi/n)} \sum_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) \sin\left(\frac{\pi(k-1)}{n}\right) |\hat{z}_k|^2.$$

For $k = 1$, $\sin(\frac{\pi(k-1)}{n}) = 0$, while for every $2 \le k \le n-1$, both $\frac{\pi k}{n}$ and $\frac{\pi(k-1)}{n}$ lie strictly in $(0, \pi)$, so every weight is strictly positive, proving (13) [1]. Equality holds iff $\hat{z}_k = 0$ for all $k \ge 2$, i.e., $z_j = \hat{z}_0 + \hat{z}_1 \omega^j$ (a positively oriented regular n-gon or single point). For equilateral n-gons ($|z_{j+1} - z_j| \equiv L/n$), $E_n(z) = L^2$. □

6 Computational Verification and Numerical Illustrations

All four theorems were verified in deterministic IEEE-754 `float64` arithmetic (seeds `20261129`–`20261131`) using the MIT-licensed companion repository `lluiseriksson/curve-isoperimetric-csf-geometry`. Our suite executes 160 structured and randomized checks across four modules: (i) 40 smooth strictly convex ovals and ellipses verifying exact Hurwitz–Dirichlet, Blaschke–Fourier, and Euler–Elastica identities (N = 1,024); (ii) 40 centrally symmetric strictly convex ovals verifying the Bonnesen quadratic signs (4), six-term chain (5), and two-sided sandwich (6); (iii) 10 full CFL-adaptive RK4 spectral simulations ($N = 64$, composite Simpson integration) of the Curve-Shortening Flow support-function PDE $p_t = -(p + p_{\theta\theta})^{-1}$ verifying $A(t) = A(0) - 2\pi t$, the integrated curvature-variance budget (10), and the length envelope (11); and (iv) 70 checks across 20 smooth 3D space curves (including the trefoil knot) and 50 random/regular n-gons ($n \in \{3..14\}$) verifying (12)–(13). Table 1 and Figures 1–2 confirm zero violations across all 160 checks.

Table 1: Deterministic verification across 160 curve, PDE, and polygon instances (`pytest -q`: 4/4 passed in 1.51 s). All exact spectral, PDE area-decay, and polygon identities hold to machine or RK4–Simpson precision.

Theorem	Checks	Max Identity Error	Sharpest Attainment / Regime Summary
Thm. 2.1 (Fourier & Elastica)	40	1.51×10^{-13} (deficit & var.)	Exact ellipse equality in (1) and m=2 oval equality in (2)
Thm. 3.1 (Bonnesen Sandwich)	40	—	0 sign/ordering violations; min margin = 0.0761 > 0 on non-disks
Thm. 4.1 (CSF PDE Evolution)	10	5.51×10^{-13} (area), 4.98×10^{-10} (budget)	Monotone $D'(t) \le 0$ and zero envelope violations on all 10 flows
Thm. 5.1 (Fenchel & n-Gon)	70	4.55×10^{-13} (Eq. (13))	$\min \int \kappa ds \ge 2\pi$ (trefoil > 4π); exact 0 deficit on regular n-gons

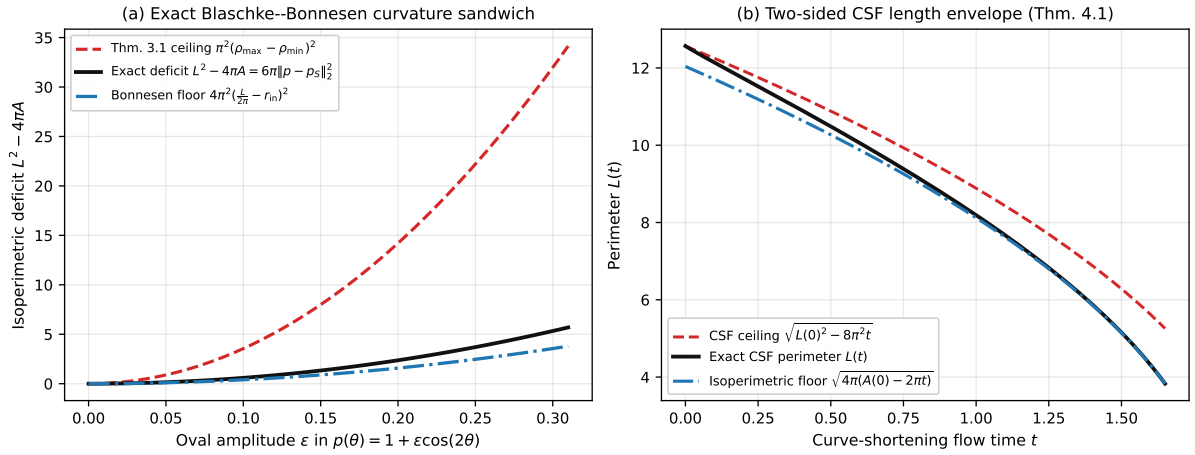

Figure 1: **(a)** Exact isoperimetric deficit $L^2 - 4\pi A = 6\pi\|p - p_S\|_{L^2}^2$ across the convex $m = 2$ oval family $p(\theta) = 1 + \varepsilon\cos(2\theta)$ bracketed tightly by the Bonnesen inradius floor and curvature-spread ceiling (6). **(b)** Exact Curve-Shortening Flow perimeter $L(t)$ for an asymmetric oval enclosed strictly inside the two-sided envelope (11).

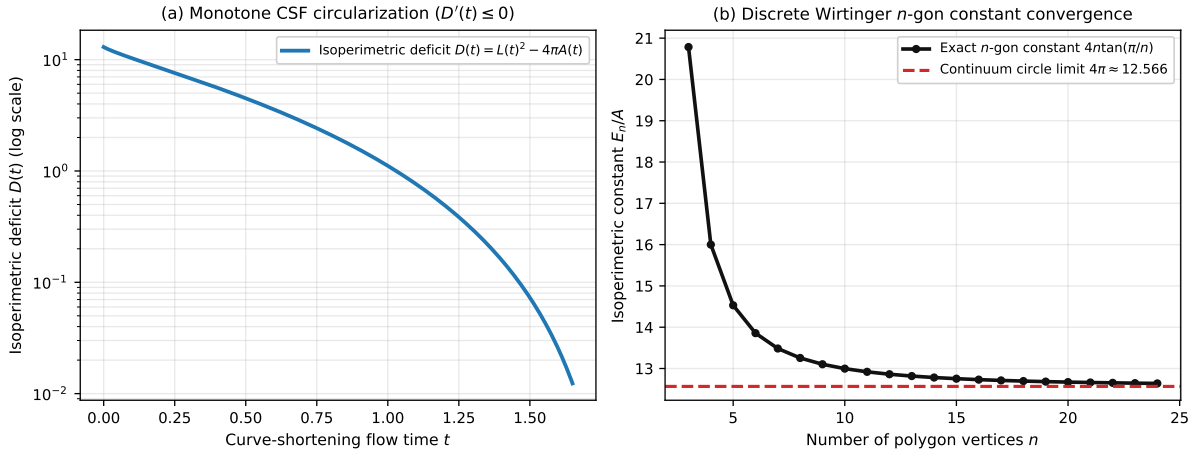

Figure 2: **(a)** Strict monotonic contraction of the isoperimetric deficit $D(t) = L(t)^2 - 4\pi A(t)$ under Curve-Shortening Flow (9), demonstrating rapid circularization. **(b)** Exact discrete Wirtinger n-gon isoperimetric constant $4n\tan(\pi/n)$ from (13) converging monotonically down to the continuum circle limit 4π.

7 Scope and Limitations

Three structural boundaries merit explicit note:

1. *Smoothness and convexity in Theorem 3.1.* The curvature-radius bounds $Q(\rho_{\min}) \geq 0$ and $Q(\rho_{\max}) \geq 0$ require C^2 strict convexity ($0 < \rho_{\min} \leq \rho_{\max} < \infty$), whereas $Q(r_{\text{in}}) \leq 0$ holds for general compact convex bodies via inner parallel sets and attains equality $Q(r_{\text{in}}) = 0$ precisely on disks and C^1 stadium bodies $K_{-r_{\text{in}}} + r_{\text{in}} B_2$ (whose flat segments are excluded by $\rho_{\max} < \infty$).
2. *Short-time vs. global CSF regularity.* Theorem 4.1 analyzes smooth solutions on $[0, T)$ prior to extinction; global existence up to $T = A(0)/(2\pi)$ and asymptotic convergence to a round point are deep PDE results of Gage–Hamilton [3] (convex) and Grayson (general embedded plane curves).
3. *Non-equilateral n-gons.* Theorem 5.1(b) bounds $E_n(z) = n \sum_{j=0}^{n-1} |z_{j+1} - z_j|^2 \geq L^2$, which coincides with L^2 specifically for equilateral n-gons; establishing $L^2 \geq 4n \tan(\frac{\pi}{n}) A$ for arbitrary non-equilateral n-gons requires an additional symmetrization step reducing edge lengths to equal length at fixed area.

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