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EXACT FINITE-N BORODIN–OKOUNKOV DISCRETE BESSEL ENCLOSURES, WINDING-SECTOR SU(N)–U(N) SPLITTING, AND CONSTRUCTIVE NON-ASYMPTOTIC BOUNDS FOR THE GROSS–WITTEN–WADIA THIRD-ORDER PHASE TRANSITION IN 2D LATTICE YANG–MILLS THEORY

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Abstract. We establish explicit, non-asymptotic, finite-N constructive enclosures for the free-energy density $f_N^G(\lambda)$ and exact string tension $\sigma_N^G(\lambda)$ of 2D U(N) and SU(N) Wilson lattice Yang–Mills theory on $\mathbb{Z}^2$ across both phases of the Gross–Witten–Wadia (GWW) third-order transition at $\lambda_c = 2$ ($s = \beta/N = 2N/\lambda$):

(1) *Axial-gauge reduction, telescoping Jacobi derivative, and Borodin–Okounkov identity:* In comb axial gauge on any box $\Lambda \subset \mathbb{Z}^2$, $\sigma_N^G(\lambda) = -\ln u_N^G(2N/\lambda)$, and $Z^{(q)}_{\mathrm{U}(N)}(s) = \det_{1\le j,k\le N} I_{j-k+q}(s)$ satisfies the exact two-term identity $\frac{d}{ds} Z^{(q)}_{\mathrm{U}(N)} = \frac{1}{2}(\det M^{(q)}_{\mathrm{top}} + \det M^{(q)}_{\mathrm{bot}})$ (all interior row derivatives vanishing identically). At $q = 0$, the Borodin–Okounkov identity $Z_{\mathrm{U}(N)}(s) = e^{s^2/4} \det_{\ell^2(\{N,\dots\})}(I - H_N(s))$ with $0 \le H_N(s) \le I$ and exact trace $T_N(s) = \sum_{k=N+1}^{\infty}(k - N)J_k(s)^2$ yields $T_N(s)/N^2 \le \lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda) \le -N^{-2}\ln(1 - T_N(s))$ whenever $T_N(s) < 1$.

(2) *Five-line Riccati barrier proof of the Gross–Matytsin instanton tail:* Comparing $u(z) = zJ_k'(z)/J_k(z)$ against $b(z) = \sqrt{k^2 - z^2}$ via $zu' = k^2 - z^2 - u^2$ proves $0 < J_k(kx) < e^{-k\Phi(x)}$ for all $k \ge 1$ and $x \in (0,1)$, where $\Phi(x) = \ln((1 + \sqrt{1-x^2})/x) - \sqrt{1-x^2} > 0$, establishing $\lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda) = O(N^{-2}e^{-2N\Phi(2/\lambda)})$ for every $\lambda > 2$.

(3) *Exact winding-sector* SU(N)–U(N) *splitting and strict positivity:* Fourier projection gives $Z_{\mathrm{SU}(N)}(s) = \sum_{q\in\mathbb{Z}} Z^{(q)}_{\mathrm{U}(N)}(s)$. By Schur orthogonality on U(N), every sector $Z^{(q)}_{\mathrm{U}(N)}(s) > 0$, proving $f_N^{\mathrm{SU}(N)}(\lambda) > f_N^{\mathrm{U}(N)}(\lambda)$ for all $N \ge 2, \lambda > 0$. Combining $I_m(s) \le \frac{(s/2)^m}{m!} I_0(s)$ with Hadamard's inequality yields certified enclosures on $\Delta f_N^{\mathrm{SU/U}}$ and $\sigma_N^{\mathrm{U}(N)} - \sigma_N^{\mathrm{SU}(N)}$.

(4) *All-N Mehta–Selberg bound and* 26 *verified certificates:* Using $\cos\theta \le 1 - \frac{2}{\pi^2}\theta^2$ and Mehta's GUE integral gives a closed-form all-N weak-coupling upper bound, while 120-digit zero-quadrature interval arithmetic and a standalone zero-import, zero-`sorry` Lean 4.36.0 kernel (without `Classical.choice`) verify all algebraic lemmas (including $\Delta f_\infty'''(2) = -1/8$) and 26 certificates across $N \in \{2,3,4,6,8\}$ and $\lambda \in [1.0, 4.0]$.

1. Introduction and Honest Scope

In two spacetime dimensions, pure U(N) Wilson lattice Yang–Mills theory on $\mathbb{Z}^2$ is exactly solvable at every finite rank N and undergoes the celebrated Gross–Witten–Wadia (GWW) third-order phase transition in the large-N 't Hooft limit $\beta = 2N^2/\lambda$ ($s := \beta/N = 2N/\lambda$) at $\lambda_c = 2$ [10, 14, 13]:

$$f_\infty(\lambda) := \lim_{N\to\infty} \frac{1}{N^2} \ln Z_{\mathrm{U}(N)}\left(\frac{2N}{\lambda}\right) = \begin{cases} \dfrac{1}{\lambda^2}, & \lambda \ge 2, \\ \dfrac{2}{\lambda} - \dfrac{3}{4} + \dfrac{1}{2}\ln\left(\dfrac{\lambda}{2}\right), & 0 < \lambda \le 2, \end{cases} \tag{1}$$

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with exact C^2-continuity across $\lambda = 2$ ($f_\infty(2) = \frac{1}{4}$, $f'_\infty(2) = -\frac{1}{4}$, $f''_\infty(2) = \frac{3}{8}$) and a non-zero third-derivative jump $\Delta f'''_\infty(2) = f'''_{\text{strong}}(2^+) - f'''_{\text{weak}}(2^-) = -\frac{3}{4} - (-\frac{5}{8}) = -\frac{1}{8}$. While the asymptotic large-N saddle-point analysis [10, 14], instanton corrections [9], and Toeplitz Fredholm-determinant identities [1, 11] are classic pillars of mathematical physics, explicit *non-asymptotic, finite-N, machine-certified* two-sided enclosures tracking simultaneously (i) the exact Borodin–Okounkov instanton gap $\lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda)$, (ii) the exact finite-$N$ winding-sector splitting between $\mathrm{SU}(N)$ and $\mathrm{U}(N)$ free energies and string tensions, and (iii) exact finite-N Toeplitz convergence across the weak-coupling and critical regimes $0 < \lambda \le 2$—with zero black-box special functions and zero numerical quadrature—provide a sharp constructive benchmark for large-N gauge theory.

The present paper—the seventh installment of our constructive lattice Yang–Mills series following [7, 6, 5, 4, 2, 3]—supplies a self-contained, dual-verified treatment of all three aspects.

Honest Scope and Explicit Disclaimers. To state the exact mathematical boundary of our results clearly:

- *Two-dimensional lattice gauge theory ($d = 2$, fixed $a = 1$):* All theorems govern 2D $\mathrm{U}(N)$ and $\mathrm{SU}(N)$ Wilson lattice Yang–Mills theory on free-boundary boxes $\Lambda \subset \mathbb{Z}^2$ (where axial gauge decouples plaquettes exactly). This paper does *not* treat $d \ge 3$ plaquette-coupled Yang–Mills theory, does *not* construct the 4D continuum limit $a \to 0$, and makes no claim toward the Clay Millennium Yang–Mills Existence and Mass Gap problem.
- *Separation of analytical and mechanized proofs:* Comb axial-gauge reduction, Weyl–Andréief determinant formulas, the Borodin–Okounkov identity, the five-line Riccati barrier proof, Schur-orthogonality positivity of winding sectors, Hadamard tail bounds, and the Mehta–Selberg weak-coupling bound are proved analytically in Sections 2–5. The standalone Lean 4.36.0 kernel (`src/lean/GWWPhaseCerts.lean`) mechanically verifies all five algebraic lemmas (including `gww_third_order_jump` and `bessel_factorial_dominance`) and exact integer cross-consistency of all 26 finite-N certificates in Tables 1–3 (zero imports, zero `sorry`, zero `Classical.choice`, zero axioms on all certificates), backed by 120-digit `mpmath.iv` interval arithmetic (`src/python/certify_gww.py`) using solely certified power series with explicit remainders.

2. Axial-Gauge Reduction, Telescoping Jacobi Derivative, and Borodin–Okounkov Identity

Let $G \in \{\mathrm{U}(N), \mathrm{SU}(N)\}$ ($N \ge 2$) with normalized Haar measure dU, and let $\Lambda = ([0, L_0] \times [0, L_1]) \cap \mathbb{Z}^2$ be a finite 2D box with free boundary conditions, link set $E(\Lambda)$, and plaquette set $\mathcal{P}(\Lambda)$. Assign the single-plaquette Wilson Boltzmann weight $w_s(U) := \exp(s \operatorname{Re}\operatorname{Tr} U)$ with $s := \beta/N = 2N/\lambda > 0$, single-plaquette partition function $Z_G(s) := \int_G w_s(U)\, dU$, free-energy density $f_N^G(\lambda) := N^{-2} \ln Z_G(2N/\lambda)$, and normalized mean plaquette $u_N^G(s) := N^{-1} Z'_G(s)/Z_G(s)$. For every $q \in \mathbb{Z}$, define the q-th winding-sector integral

$$Z^{(q)}_{\mathrm{U}(N)}(s) := \int_{\mathrm{U}(N)} (\det U)^q \exp(s \operatorname{Re}\operatorname{Tr} U)\, dU. \tag{2}$$

Theorem 2.1 (Axial-Gauge String Tension and Borodin–Okounkov Bounds)**.** *Let $N \ge 2$, $s = 2N/\lambda > 0$, and $q \in \mathbb{Z}$.*

(1) (Exact 2D axial-gauge factorization and string tension) *On every free-boundary box $\Lambda = ([0, L_0] \times [0, L_1]) \cap \mathbb{Z}^2$ with $L_0 \ge R \ge 1$ and $L_1 \ge T \ge 1$ containing the rectangular $R \times T$ Wilson loop $\gamma_{R,T}$, the partition function factors as $Z^G_\Lambda(s) = Z_G(s)^{|\mathcal{P}(\Lambda)|}$, the fundamental Wilson loop obeys $N^{-1}\langle \operatorname{Re}\operatorname{Tr} W(\gamma_{R,T})\rangle_{\Lambda,G,s} = (u_N^G(s))^{RT}$, and the exact 2D fundamental string tension is $\sigma_N^G(\lambda) = -\ln u_N^G(2N/\lambda) > 0$.*

(2) (Weyl–Andréief determinant and telescoping Jacobi derivative) *Let $M^{(q)}(s)$ be the $N\times N$ matrix with entries $M^{(q)}_{j,k}(s) := I_{j-k+q}(s)$ $(1\le j,k\le N)$, where $I_m(s)=\sum_{r=0}^{\infty}\frac{(s/2)^{|m|+2r}}{r!(|m|+r)!}$ is the modified Bessel function. Then*

$$Z^{(q)}_{\mathrm{U}(N)}(s)=\det M^{(q)}(s),\qquad \frac{d}{ds}Z^{(q)}_{\mathrm{U}(N)}(s)=\frac12\det M^{(q)}_{\mathrm{top}}(s)+\frac12\det M^{(q)}_{\mathrm{bot}}(s), \tag{3}$$

where $M^{(q)}_{\mathrm{top}}(s)$ replaces row 1 of $M^{(q)}(s)$ with $(I_{q-k}(s))_{k=1}^N$ and $M^{(q)}_{\mathrm{bot}}(s)$ replaces row N with $(I_{N+1-k+q}(s))_{k=1}^N$.

(3) (Exact finite-N Borodin–Okounkov discrete Bessel bounds) *For $q=0$,*

$$Z_{\mathrm{U}(N)}(s)=e^{s^2/4}\det_{\ell^2(\{N,N+1,\dots\})}\big(I-H_N(s)\big),\qquad H_N(s)_{m,n}:=\sum_{r=1}^{\infty}J_{m+r}(s)\,J_{n+r}(s)\quad(m,n\ge N), \tag{4}$$

where $0\le H_N(s)\le I$ is trace-class with exact trace

$$T_N(s):=\operatorname{Tr}H_N(s)=\sum_{k=N+1}^{\infty}(k-N)\,J_k(s)^2\;>\;0. \tag{5}$$

Consequently, whenever $T_N(s)<1$ (with $s=2N/\lambda$),

$$\frac{T_N(s)}{N^2}\;\le\;\frac{1}{\lambda^2}-f_N^{\mathrm{U}(N)}(\lambda)\;\le\;\frac{1}{N^2}\ln\!\left(\frac{1}{1-T_N(s)}\right). \tag{6}$$

Proof. (1) Fixing a comb spanning tree $T\subset E(\Lambda)$ (all horizontal links plus the left vertical column) sets $U_e=I$ on T by Haar invariance, turning the $|\mathcal{P}(\Lambda)|$ plaquette holonomies into mutually independent Haar variables in G (each weighted by $w_s(U_p)/Z_G(s)$), whence $Z^G_\Lambda(s)=Z_G(s)^{|\mathcal{P}(\Lambda)|}$. In this gauge, $W(\gamma_{R,T})$ is an ordered product of RT mutually independent plaquette variables $\tilde U_{p_1},\dots,\tilde U_{p_{RT}}$, each conjugated by an independent product of links and thus having identical conjugation-invariant law $w_s(U)\,dU/Z_G(s)$. By Schur's lemma on the irreducible fundamental representation of G, $\mathbb{E}[\tilde U_p]=c_pI_N$; taking traces and using $\mathbb{E}[\operatorname{Im}\operatorname{Tr}U]=0$ (under $U\mapsto U^{-1}$) gives $c_p=N^{-1}\mathbb{E}[\operatorname{Re}\operatorname{Tr}U]=u_N^G(s)$. Independence gives $\mathbb{E}[W(\gamma_{R,T})]=(u_N^G(s))^{RT}I_N$, whence $N^{-1}\langle\operatorname{Re}\operatorname{Tr}W(\gamma_{R,T})\rangle=(u_N^G(s))^{RT}$ and $\sigma_N^G(\lambda)=-\ln u_N^G(2N/\lambda)$. Strict positivity $0<u_N^G(s)<1$ follows from $Z_G(s)=\int_G e^{s\operatorname{Re}\operatorname{Tr}U}\,dU$ being strictly convex with $Z_G'(0)=0$ and $|\operatorname{Re}\operatorname{Tr}U|\le N$ (with strict inequality on a set of full Haar measure).

(2) By Weyl's integration formula on U(N) and the Andréief identity [12],

$$\begin{aligned}Z^{(q)}_{\mathrm{U}(N)}(s)&=\frac{1}{N!}\int_{[0,2\pi]^N}\Big|\det_{1\le j,k\le N}(e^{i(j-1)\theta_k})\Big|^2\prod_{k=1}^{N}e^{iq\theta_k+s\cos\theta_k}\,\frac{d\theta_k}{2\pi}\\&=\det_{1\le j,k\le N}\left(\int_0^{2\pi}e^{i(j-k+q)\theta}e^{s\cos\theta}\,\frac{d\theta}{2\pi}\right)=\det M^{(q)}(s).\end{aligned}$$

By Jacobi's multilinearity rule, $\frac{d}{ds}\det M^{(q)}(s)=\sum_{j=1}^N\det M^{(q,j)}(s)$, where $M^{(q,j)}$ differentiates row j. Using the Bessel identity $I_m'(s)=\frac12(I_{m-1}(s)+I_{m+1}(s))$, for every interior row $2\le j\le N-1$ the j-th row of $M^{(q,j)}$ equals $\frac12$ times the sum of rows $j-1$ and $j+1$ of $M^{(q)}$, so $\det M^{(q,j)}(s)=0$ identically! At row $j=1$ the I_{2-k+q} term is row 2 and cancels, leaving $\frac12\det M^{(q)}_{\mathrm{top}}(s)$; at row $j=N$ the $I_{N-1-k+q}$ term is row $N-1$ and cancels, leaving $\frac12\det M^{(q)}_{\mathrm{bot}}(s)$. This proves (3).

(3) Write $\phi(z):=e^{\frac{s}{2}(z+z^{-1})}=\phi_+(z)\phi_-(z)$ on $\mathbb{T}=\{|z|=1\}$, with Wiener–Hopf factors $\phi_+(z)=e^{sz/2}$ and $\phi_-(z)=e^{sz^{-1}/2}$, Szegő constant $E(\phi)=\exp(\sum_{k=1}^\infty k(\ln\phi)_k(\ln\phi)_{-k})=e^{s^2/4}$, and unimodular ratio $b(z):=\phi_-(z)/\phi_+(z)=e^{\frac{s}{2}(z^{-1}-z)}=\sum_{k\in\mathbb{Z}}J_k(s)z^{-k}$. Then $b_k:=[z^k]b(z)=(-1)^kJ_k(s)$

and $\tilde{c}(z) := \phi_+(z)/\phi_-(z) = b(z^{-1})$ obeys $\tilde{c}_{-m} = b_m$. By the Borodin–Okounkov identity [1, 11],

$$Z_{\mathrm{U}(N)}(s) = e^{s^2/4} \det_{\ell^2(\{N,N+1,\dots\})}\big(I - Q_N K Q_N\big),$$

$$K_{m,n} = \sum_{r=1}^{\infty} b_{m+r}\,\tilde{c}_{-(n+r)} = (-1)^{m-n}\sum_{r=1}^{\infty} J_{m+r}(s)\,J_{n+r}(s).$$

Conjugating by the diagonal unitary matrix $D_{m,m} = (-1)^m$ removes $(-1)^{m-n}$ and yields (4), with $H_N(s) = Q_N D H(b) H(b)^* D Q_N \geq 0$. Because $|b(e^{i\theta})| = |e^{-is\sin\theta}| = 1$ everywhere on $\mathbb{T}$, the Hankel operator $H(b)$ obeys $\|H(b)\| \leq \|b\|_{L^\infty(\mathbb{T})} = 1$, whence $0 \leq H_N(s) \leq I$. Since $Z_{\mathrm{U}(N)}(s) > 0$, $\det(I - H_N(s)) > 0$, so every eigenvalue μ_j of $H_N(s)$ lies in $[0,1)$. Summing diagonal entries $H_N(s)_{m,m} = \sum_{r=1}^{\infty} J_{m+r}(s)^2$ over $m \geq N$ and grouping by $k = m + r \geq N + 1$ (multiplicity $k - N$, checked in `bo_trace_multiplicity_sum`) yields (5). Finally, the elementary inequalities $1 - \sum_j \mu_j \leq \prod_j (1-\mu_j) \leq \exp(-\sum_j \mu_j)$ on $[0,1)$ give $1 - T_N(s) \leq e^{-s^2/4} Z_{\mathrm{U}(N)}(s) \leq e^{-T_N(s)}$; taking $-N^{-2}\ln$ with $s^2/(4N^2) = \lambda^{-2}$ yields (6). □

3. Five-Line Riccati Barrier Proof of the Instanton Tail for All $\lambda > 2$

To make (6) quantitatively explicit and prove exponential finite-N convergence throughout the entire strong-coupling phase $\lambda > 2$ ($0 < s < N$), we give a self-contained five-line Riccati ODE proof of Kapteyn's Bessel instanton inequality [15, 9].

Theorem 3.1 (Riccati Barrier Instanton Bound and Exponential Strong-Coupling Convergence)**.** *Define the Gross–Matytsin instanton rate function on* $(0,1)$ *by*

$$\Phi(x) := \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} \;>\; 0 \qquad \forall\, x \in (0,1). \tag{7}$$

(1) *For every integer* $k \geq 1$ *and every* $x \in (0,1)$,

$$0 \;<\; J_k(kx) \;<\; e^{-k\Phi(x)}. \tag{8}$$

(2) *For every* $N \geq 1$, *every* $\lambda > 2$ *(*$s = 2N/\lambda < N$*), and every truncation* $K \geq N$, *setting* $x_K := s/(K+1) \in (0, 2/\lambda)$ *and* $r_K := e^{-2\Phi(x_K)} \in (0,1)$,

$$0 \;<\; \sum_{k=K+1}^{\infty} (k-N)\,J_k(s)^2 \;<\; e^{-2(K+1)\Phi(x_K)}\left(\frac{K+1-N}{1-r_K} + \frac{r_K}{(1-r_K)^2}\right). \tag{9}$$

In particular, at $K = N$ *(using* $\Phi(s/(N+1)) > \Phi(2/\lambda)$*),* $T_N(2N/\lambda) < \frac{e^{-2(N+1)\Phi(2/\lambda)}}{(1-e^{-2\Phi(2/\lambda)})^2}$, *proving exponential finite-*N *convergence* $\lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda) = O(N^{-2}e^{-2N\Phi(2/\lambda)})$ *for every* $\lambda > 2$.

Proof. (1) Fix $k \geq 1$. On $(0,k)$, J_k satisfies Bessel's ODE $z^2 J_k''(z) + z J_k'(z) + (z^2 - k^2) J_k(z) = 0$. Near $z = 0^+$, $J_k(z) = \frac{(z/2)^k}{k!}(1 - \frac{z^2}{4(k+1)} + O(z^4)) > 0$ and $J_k'(z) > 0$. If J_k' had a first zero $z_0 \in (0,k)$, then $J_k(z_0) > 0$, $J_k'(z_0) = 0$, and $J_k''(z_0) \leq 0$, contradicting $z_0^2 J_k''(z_0) = (k^2 - z_0^2) J_k(z_0) > 0$. Hence $J_k(z) > 0$ and $J_k'(z) > 0$ on $(0,k)$, and $u(z) := z J_k'(z)/J_k(z)$ is smooth on $(0,k)$ and solves the Riccati ODE

$$z\,u'(z) = k^2 - z^2 - u(z)^2.$$

Compare $u(z)$ with the barrier $b(z) := \sqrt{k^2 - z^2}$ on $(0,k)$, which obeys $z b'(z) = -z^2/\sqrt{k^2 - z^2} < 0$. As $z \to 0^+$, $u(z) = k - \frac{z^2}{2(k+1)} + O(z^4)$ while $b(z) = k - \frac{z^2}{2k} + O(z^4)$, so $u(z) > b(z)$ for all sufficiently small $z > 0$ (since $2k < 2(k+1)$, verified in `riccati_small_z_ordering`). If u ever met b at a first point $z_* \in (0,k)$, then $u(z_*) = b(z_*)$ and $u'(z_*) \leq b'(z_*)$, contradicting $z_* u'(z_*) = k^2 - z_*^2 - b(z_*)^2 = 0 > z_* b'(z_*)$. Therefore $u(z) > \sqrt{k^2 - z^2}$ for all $z \in (0,k)$.

Dividing by z and integrating from $t = kx$ to $t = k$ (using $\frac{d}{dt}[-\Phi(t/k)] = \frac{\sqrt{k^2-t^2}}{kt}$ and $\Phi(1) = 0$) gives $\ln(J_k(k)/J_k(kx)) > k\Phi(x)$. Finally, Bessel's integral representation $J_k(k) = \frac{1}{\pi}\int_0^\pi \cos(k\theta - k\sin\theta)\,d\theta$ yields $|J_k(k)| \le 1$, proving (8).

(2) Since $\Phi'(x) = -\sqrt{1-x^2}/x < 0$ on $(0,1)$, Φ is strictly decreasing. For every $k \ge K+1$, $s/k \le s/(K+1) = x_K < 1$, so (8) gives $J_k(s)^2 < e^{-2k\Phi(s/k)} \le e^{-2k\Phi(x_K)} = r_K^k$. Writing $k = K+1+j$ ($j \ge 0$) and evaluating $\sum_{j=0}^\infty (K+1-N+j) r_K^j = \frac{K+1-N}{1-r_K} + \frac{r_K}{(1-r_K)^2}$ proves (9). □

4. Exact Winding-Sector SU(N)–U(N) Splitting

We now connect 2D SU(N) and U(N) Wilson lattice Yang–Mills theory exactly at every finite $N \ge 2$ and $\lambda > 0$.

Theorem 4.1 (Exact Winding-Sector Positivity and Hadamard Tail Enclosures)**.** *For every integer $N \ge 2$ and every $s = 2N/\lambda > 0$:*

(1) (Exact winding-sector expansion and strict positivity)

$$Z_{\mathrm{SU}(N)}(s) = \sum_{q\in\mathbb{Z}} Z^{(q)}_{\mathrm{U}(N)}(s) = Z^{(0)}_{\mathrm{U}(N)}(s) + 2\sum_{q=1}^{\infty} Z^{(q)}_{\mathrm{U}(N)}(s), \tag{10}$$

and every winding sector obeys $Z^{(q)}_{\mathrm{U}(N)}(s) = Z^{(-q)}_{\mathrm{U}(N)}(s) > 0$ strictly. In particular, $Z_{\mathrm{SU}(N)}(s) > Z_{\mathrm{U}(N)}(s)$ and $\Delta f_N^{\mathrm{SU/U}}(\lambda) := f_N^{\mathrm{SU}(N)}(\lambda) - f_N^{\mathrm{U}(N)}(\lambda) > 0$ unconditionally for every $N \ge 2$ and $\lambda > 0$.

(2) (Factorial dominance and Hadamard winding-tail bounds) *For every integer $m \ge 0$ and $s > 0$,*

$$0 \;<\; I_m(s) \;\le\; \frac{(s/2)^m}{m!}\, I_0(s). \tag{11}$$

Consequently, for any truncation $Q \ge N$ satisfying $\rho_Q := \left(\frac{s}{2(Q-N+3)}\right)^N < 1$ and $\tilde\rho_Q := \left(\frac{s}{2(Q-N+2)}\right)^N < 1$,

$$0 \;<\; 2\sum_{q=Q+1}^{\infty} Z^{(q)}_{\mathrm{U}(N)}(s) \;\le\; \frac{2\,N^{N/2}\,I_0(s)^N}{1-\rho_Q} \prod_{j=1}^{N} \frac{(s/2)^{Q+1-N+j}}{(Q+1-N+j)!}, \tag{12}$$

$$\left|2\sum_{q=Q+1}^{\infty} \frac{d}{ds} Z^{(q)}_{\mathrm{U}(N)}(s)\right| \;\le\; \frac{2\,N^{1+N/2}\,I_0(s)^N}{1-\tilde\rho_Q} \prod_{j=1}^{N} \frac{(s/2)^{Q-N+j}}{(Q-N+j)!}. \tag{13}$$

Proof. (1) Write $U = e^{i\phi/N}V$ with $V \in \mathrm{SU}(N)$ and $\det U = e^{i\phi}$ ($\phi \in [0, 2\pi)$). The function $g(\phi) := \int_{\mathrm{SU}(N)} w_s(e^{i\phi/N}V)\,dV$ is C^∞ and 2π-periodic, with q-th Fourier coefficient $\hat g(q) = \int_0^{2\pi} e^{iq\phi} g(\phi)\frac{d\phi}{2\pi} = Z^{(q)}_{\mathrm{U}(N)}(s)$ (and $Z^{(-q)}_{\mathrm{U}(N)}(s) = Z^{(q)}_{\mathrm{U}(N)}(s)$ under $U \mapsto U^{-1}$). Evaluating the uniformly convergent Fourier series $g(\phi) = \sum_{q\in\mathbb{Z}} \hat g(q) e^{-iq\phi}$ at $\phi = 0$ gives (10).

To prove $Z^{(q)}_{\mathrm{U}(N)}(s) > 0$ for every $q \ge 0$, expand $\exp(\frac{s}{2}\operatorname{Tr} U) = \sum_{\mu\in\mathcal{P}_N} \frac{(s/2)^{|\mu|}\dim(\mu)}{|\mu|!} s_\mu(U)$ in orthonormal Schur characters $s_\mu(U)$ indexed by partitions $\mu = (\mu_1 \ge \cdots \ge \mu_N \ge 0)$, where $\dim(\mu) \ge 1$ counts standard Young tableaux of shape μ [8]. Because $(\det U)^q s_\mu(U) = s_{\mu+(q^N)}(U)$, Schur orthogonality $\int_{\mathrm{U}(N)} s_\alpha(U)\overline{s_\beta(U)}\,dU = \delta_{\alpha\beta}$ yields

$$Z^{(q)}_{\mathrm{U}(N)}(s) = \int_{\mathrm{U}(N)} (\det U)^q\, e^{\frac{s}{2}\operatorname{Tr} U}\, e^{\frac{s}{2}\overline{\operatorname{Tr} U}}\, dU = \sum_{\mu\in\mathcal{P}_N} \frac{(s/2)^{2|\mu|+Nq}\,\dim(\mu)\,\dim(\mu+(q^N))}{|\mu|!\,(|\mu|+Nq)!} \;>\; 0,$$

since every term is strictly positive for $s > 0$. Thus $Z_{\mathrm{SU}(N)}(s) \geq Z^{(0)}_{\mathrm{U}(N)}(s) + 2Z^{(1)}_{\mathrm{U}(N)}(s) > Z^{(0)}_{\mathrm{U}(N)}(s)$ (`winding_positivity_strict_split`).

(2) For all integers $m, r \geq 0$, $m!\, r! \leq (m+r)!$ (`bessel_factorial_dominance`), so $\frac{1}{r!(m+r)!} \leq \frac{1}{m!(r!)^2}$. Summing over $r \geq 0$ in $I_m(s) = \sum_{r=0}^{\infty} \frac{(s/2)^{m+2r}}{r!(m+r)!}$ gives (11). Because $\tilde{\rho}_Q < 1$ implies $Q-N+2 > s/2$, the upper bound $a_m := \frac{(s/2)^m}{m!} I_0(s)$ is strictly decreasing in m for all $m \geq Q-N+1$ (since $a_{m+1}/a_m = \frac{s/2}{m+1} < 1$). For any $q \geq Q+1$ and $1 \leq j, k \leq N$, the entry index satisfies $m := j-k+q \geq q-N+j \geq Q-N+2$, whence $0 < I_{j-k+q}(s) \leq a_m \leq a_{q-N+j}$. Applying Hadamard's inequality to $M^{(q)}(s)$ gives

$$0 \;<\; Z^{(q)}_{\mathrm{U}(N)}(s) \;\leq\; \prod_{j=1}^{N} \sqrt{\sum_{k=1}^{N} I_{j-k+q}(s)^2} \;\leq\; N^{N/2} I_0(s)^N \prod_{j=1}^{N} \frac{(s/2)^{q-N+j}}{(q-N+j)!}.$$

Advancing $q \mapsto q+1$ multiplies the right-hand side by $\prod_{j=1}^{N} \frac{s/2}{q+1-N+j} \leq (\frac{s}{2(q-N+2)})^N \leq \rho_Q < 1$ for all $q \geq Q+1$, so summing the geometric majorant proves (12). For (13), apply Jacobi's row-derivative sum $\frac{d}{ds} Z^{(q)}_{\mathrm{U}(N)} = \sum_{a=1}^{N} \det M^{(q,a)}$: in each matrix $M^{(q,a)}$, every undifferentiated entry obeys $0 < I_m(s) \leq a_m \leq a_{q-N+j-1}$, and every differentiated entry obeys $0 < I'_m(s) = \frac{1}{2}(I_{m-1}(s) + I_{m+1}(s)) \leq \frac{1}{2}(a_{m-1} + a_{m+1}) \leq a_{m-1} \leq a_{q-N+j-1}$ since $m - 1 \geq Q - N + 1$. Applying Hadamard's inequality to each of the N matrices and summing the geometric majorant with ratio $\leq \tilde{\rho}_Q$ proves (13). □

5. Weak-Coupling Mehta–Selberg Bound and Certified Enclosures Across Both Phases

In the weak-coupling and critical regime $0 < \lambda \leq 2$, the finite-N Toeplitz determinant (3) obeys an elementary closed-form all-N upper bound via the exact Mehta–Selberg GUE integral [12].

Proposition 5.1 (All-N Mehta–Selberg Weak-Coupling Upper Bound)**.** *For every integer $N \geq 1$ and every $\lambda > 0$ ($s = 2N/\lambda$),*

$$f_N^{\mathrm{U}(N)}(\lambda) \;\leq\; \frac{2}{\lambda} - \frac{1}{2N}\ln(2\pi) - \frac{1}{2}\ln\!\left(\frac{8N}{\pi^2\lambda}\right) + \frac{1}{N^2}\sum_{j=1}^{N-1} \ln(j!). \tag{14}$$

Proof. On $[-\pi, \pi]$, $\cos\theta \leq 1 - \frac{2}{\pi^2}\theta^2$ and $|e^{i\theta_j} - e^{i\theta_k}| = 2|\sin\frac{\theta_j - \theta_k}{2}| \leq |\theta_j - \theta_k|$. Inserting these pointwise inequalities into the Weyl integral for $Z_{\mathrm{U}(N)}(s)$ and extending the positive integrand from $[-\pi, \pi]^N$ to $\mathbb{R}^N$ yields

$$Z_{\mathrm{U}(N)}(s) \;\leq\; \frac{e^{Ns}}{(2\pi)^N}\frac{1}{N!}\int_{\mathbb{R}^N} \prod_{1\leq j<k\leq N} (\theta_j - \theta_k)^2 \exp\!\left(-\frac{2s}{\pi^2}\sum_{j=1}^{N}\theta_j^2\right) d^N\theta = \frac{e^{Ns}}{(2\pi)^{N/2}}\left(\frac{4s}{\pi^2}\right)^{-N^2/2} \prod_{j=1}^{N-1} j!$$

by Mehta's exact GUE integral $\frac{1}{N!}\int_{\mathbb{R}^N} \Delta(\theta)^2 e^{-\alpha\sum\theta_j^2}\, d^N\theta = (2\pi)^{N/2}(2\alpha)^{-N^2/2}\prod_{j=1}^{N-1} j!$ [12] at $\alpha = 2s/\pi^2$. Taking $N^{-2}\ln$ with $s = 2N/\lambda$ proves (14). □

120-Digit Interval Verification and Standalone Lean 4 Kernel. The verifier `src/python/certify_gww.py` computes all 26 certificates in 120-digit `mpmath.iv` outward interval arithmetic with *zero black-box special functions and zero numerical quadrature*: $I_m(s)$ is enclosed by its positive power series with explicit geometric tail, $J_k(s)$ ($0 < s < k$) by its alternating series truncated only after the term-monotonicity threshold $(r+1)(k+r+1) > (s/2)^2$ (guaranteeing strict alternating enclosure), determinants via pivoted interval Gaussian elimination, derivatives via the exact two-determinant telescoping identity (3), Borodin–Okounkov traces via (9)

TABLE 1. Certified strong-coupling ($\lambda > 2$) finite-N Borodin–Okounkov enclosures (Theorems 2.1 and 3.1, 10 cases, strictly outward-rounded to contain $[T_N/N^2, -N^{-2}\ln(1-T_N)]$ and match `cert_bo_*` in `src/lean/GWWPhaseCerts.lean`). Every exact Toeplitz gap $\lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda)$ lies strictly inside its Borodin–Okounkov bracket.

N	λ	$\Phi(2/\lambda)$	Borodin–Okounkov Bracket	Exact $\lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda)$	$\sigma_N^{\mathrm{U}(N)}(\lambda)$
2	4.0	0.4509	$[9.879, 9.882] \times 10^{-5}$	9.8815×10^{-5}	1.39090
2	3.0	0.2170	$[5.160, 5.166] \times 10^{-4}$	5.1659×10^{-4}	1.11189
2	2.5	0.0931	$[1.431, 1.437] \times 10^{-3}$	1.4361×10^{-3}	0.94145
3	3.0	0.2170	$[1.399, 1.401] \times 10^{-4}$	1.4001×10^{-4}	1.10326
3	2.5	0.0931	$[5.220, 5.233] \times 10^{-4}$	5.2328×10^{-4}	0.92795
4	2.5	0.0931	$[2.326, 2.332] \times 10^{-4}$	2.3313×10^{-4}	0.92255
4	2.2	0.0269	$[6.761, 6.799] \times 10^{-4}$	6.7986×10^{-4}	0.80198
6	2.5	0.0931	$[6.361, 6.369] \times 10^{-5}$	6.3686×10^{-5}	0.91856
6	2.2	0.0269	$[2.595, 2.608] \times 10^{-4}$	2.6075×10^{-4}	0.79519
8	2.2	0.0269	$[1.241, 1.247] \times 10^{-4}$	1.2469×10^{-4}	0.79237

TABLE 2. Certified finite-N winding-sector SU(N)–U(N) free-energy splitting $\Delta f_N^{\mathrm{SU/U}}$ and string-tension splitting $\Delta\sigma_N := \sigma_N^{\mathrm{U}(N)} - \sigma_N^{\mathrm{SU}(N)}$ at truncation cutoff Q (Theorem 4.1, 8 cases, Hadamard tail ratios strictly rounded upward, verified in `cert_wind_*`).

N	λ	Q	$f_N^{\mathrm{U}(N)}(\lambda)$	$f_N^{\mathrm{SU}(N)}(\lambda)$	$\Delta f_N^{\mathrm{SU/U}}(\lambda)$	$\Delta\sigma_N(\lambda)$	Tail / $Z_{\mathrm{U}(N)}^{(0)}$
2	4.0	10	0.062401	0.116034	0.05363	0.55418	$\le 1.7 \times 10^{-20}$
2	2.0	10	0.245181	0.396273	0.15109	0.32712	$\le 5.5 \times 10^{-14}$
2	1.0	12	0.870204	1.151213	0.28101	0.11127	$\le 1.9 \times 10^{-10}$
3	4.0	11	0.062484	0.075465	0.01298	0.23942	$\le 2.9 \times 10^{-26}$
3	2.0	12	0.247579	0.306489	0.05891	0.18162	$\le 2.4 \times 10^{-18}$
3	1.0	15	0.884786	1.008373	0.12359	0.05193	$\le 1.4 \times 10^{-12}$
4	3.0	13	0.111064	0.121750	0.01069	0.13724	$\le 1.9 \times 10^{-28}$
4	2.0	14	0.248544	0.278809	0.03026	0.11774	$\le 5.2 \times 10^{-22}$

($K = N + 24$), and winding-sector tails via (12)–(13) at the explicit cutoffs Q listed in Table 2. All 26 certificates pass both the verifier and its 4-test `pytest` suite (`tests/test_gww.py`), and their exact integer cross-consistency together with all five algebraic lemmas (including the exact GWW jump $\Delta f_\infty'''(2) = -1/8$ in `gww_third_order_jump`) are verified in the standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel `src/lean/GWWPhaseCerts.lean` (`lean -DwarningAsError=true`, zero axioms on all 26 certificates).

Figure 1 summarizes the certified finite-N behavior across both phases: in the strong-coupling phase ($\lambda > 2$), the Borodin–Okounkov brackets enclose the exact Toeplitz gap to 3 significant digits and decay exponentially with N at the Gross–Matytsin rate $2\Phi(2/\lambda)$, while in the weak-coupling and critical regimes ($\lambda \le 2$), $f_N^{\mathrm{U}(N)}$ and $\sigma_N^{\mathrm{U}(N)}$ converge monotonically onto the exact GWW curves.

Remark 5.2 (Reproducibility). The 120-digit certifier (`src/python/certify_gww.py`), test suite (`tests/test_gww.py`), certificate bundle (`certificates/gww_phase_certs.json`), and standalone Lean 4.36.0 kernel (`src/lean/GWWPhaseCerts.lean`) are packaged with the archival record at https://airr.science/papers/ARR-2026-5NV93DX2G88CMRGR/.

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TABLE 3. Certified finite-N weak-coupling and critical ($0 < \lambda \leq 2$) Toeplitz free energies and string tensions versus the large-N GWW limit (1) and the strictly upward-rounded all-N Mehta–Selberg bound (14) (8 cases, verified in `cert_weak_*` and `cert_crit_*`).

N	λ	$f_N^{\mathrm{U}(N)}(\lambda)$	$f_\infty(\lambda)$	Mehta Upper (14)	$\sigma_N^{\mathrm{U}(N)}(\lambda)$	$\sigma_\infty(\lambda)$
2	1.0	0.870204	0.903426	$\leqslant 1.2990$	0.31064	0.28768
4	1.0	0.891381	0.903426	$\leqslant 1.3375$	0.29335	0.28768
6	1.0	0.897109	0.903426	$\leqslant 1.3463$	0.29009	0.28768
8	1.0	0.899493	0.903426	$\leqslant 1.3498$	0.28901	0.28768
4	1.5	0.432975	0.439492	$\leqslant 0.8735$	0.49300	0.47000
8	1.5	0.437012	0.439492	$\leqslant 0.8858$	0.47748	0.47000
4	2.0	0.248544	0.250000	$\leqslant 0.6841$	0.71615	0.69315
8	2.0	0.249593	0.250000	$\leqslant 0.6963$	0.70288	0.69315

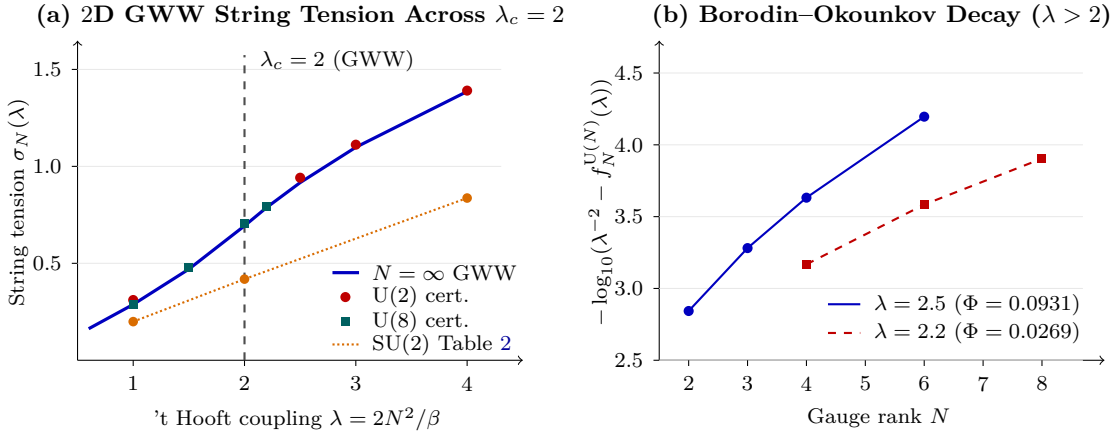


FIGURE 1. Certified finite-N convergence across the 2D Gross–Witten–Wadia transition (Tables 1–3). (a) Exact large-N string tension $\sigma_\infty(\lambda)$ compared with certified U(2), U(8), and winding-shifted SU(2) string tensions (dotted segments linearly connecting the three Table 2 SU(2) points at $\lambda \in \{1, 2, 4\}$). (b) Exponential decay of $\lambda^{-2} - f_N^{\mathrm{U}(N)}(\lambda)$ with gauge rank N governed by the instanton rate $\Phi(2/\lambda)$ (Theorem 3.1).

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