Constructive Lean 4 Formalization and Exact-Rational Certificates for OpenAI's $\omega \leq 9/4$ Strassen Spectral Profile Proof

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Abstract

OpenAI (Oct. 2, 2026) established the upper bound $\omega \leq 9/4$ for the complex matrix multiplication exponent via a compact spectral-profile mechanism: finite $5M \to M$ Fourier squared-mismatch degeneration (Prop. 3.1), a determinant filtration establishing discrete concavity $2P(a,b) \geq P(a,b+1) + P(a,b-1)$ (Lem. 4.1), a three-sector decomposition giving shifted tripling $P(a, 3h+a-1) \geq 3P(a,h)$ (Lem. 4.2), and an elementary discrete growth proof forcing $P(a,a) \geq a^{4/3}$ (Lem. 5.1) and thus $t \leq 3/4 \implies \omega = 3t \leq 9/4$. We present a standalone, zero-dependency, zero-floating-point verification stack pairing: (i) `StrassenNineFourths.lean`, 18 constructive Lean 4.35.0 theorems (zero `sorry`, zero `axiom`, zero `Classical.choice`, zero warnings; 1 zero-axiom and 17 `[propext, Quot.sound]`) mechanizing Proposition 3.1, Lemmas 4.1–4.2, and the *complete universal inductive proof* of Lemma 5.1 ($P(a,a)^3 \geq a^4$ for all $a \geq 1$) over a self-contained exact-rational structure `QRat`; and (ii) `STRASSEN-94-CERT-1.0`, a pure-stdlib Python `Fraction` auditor across four modes. Alongside certifying 16 canonical witnesses in 0.008–13.686 ms and rejecting 60/60 live adversarial forgeries (64/64 pytest checks), our exact audit reveals a 4× tightening of Proposition 3.1's degeneration degree from $(M-1)^2$ to $\lfloor (M-1)/2 \rfloor^2$ (degree 0 at $M=2$) and proves exact minimal integer refutation thresholds $a^*(t)$ for supercritical exponents $t > 3/4$.

1 Introduction and Scope Calibration

Following Strassen's 2×2 seven-product identity [5], Strassen's asymptotic spectrum of tensors [6], and the Coppersmith–Winograd laser method [1], OpenAI's October 2, 2026 preprint [4] proved $\omega \leq 9/4$ over $\mathbb{C}$ (complemented by a matching lower bound $\omega_F = 9/4$ in [3]) without Coppersmith–Winograd optimization tables. For any tensor character λ with oriented dot-product exponents $p_X, p_Y, p_Z \in [0,1]$ and $t := (p_X + p_Y + p_Z)/3 > 0$, OpenAI defines the symmetrized polynomial-multiplication profile

$$P(a,b) := \left(\prod_{\pi \in S_3} \lambda_\pi(C(a,b)) \right)^{1/(6t)}, \tag{1}$$

where $C(a,b) = \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} x_i y_j z_{i+j}$ has exact rank $R(C(a,b)) = a+b-1$. Symmetry, $P(1,b) = b$, interpolation $P(a,b) \leq (a+b-1)^{1/t}$, discrete concavity $2P(a,b) \geq P(a,b+1) + P(a,b-1)$ (Lem. 4.1), and shifted tripling $P(a,3h+a-1) \geq 3P(a,h)$ (Lem. 4.2) force $P(a,a) \geq a^{4/3}$ for all $a \geq 1$ (Lem. 5.1), which yields $t \leq 3/4$ and $\omega = 3t \leq 9/4$.

Honest Two-Layer Verification Scope. We delineate explicitly what is machine-checked in Lean 4 vs. verified on finite witnesses vs. left in [4]:

1. **Layer 1 (Lean 4.35.0 Kernel, Universal Algebraic & Inductive Proofs):** `StrassenNineFourths.lean` proves 18 zero-dependency theorems (Table 1), including the *full universal inductive proof* of Lemma 5.1 over an abstract `SpectralProfile` structure returning exact positive rationals `QRat`: from symmetry, $P(1,b) = b$, concavity, and shifted tripling, Theorems 10–17 constructively derive $H_a^3 \geq a$ and $P(a,a)^3 \geq a^4$ for *every* $a \geq 1$.
2. **Layer 2 (Exact-$\mathbb{Q}/\mathbb{Z}$ Certificate Auditor):** `STRASSEN-94-CERT-1.0` replays finite instances of Prop. 3.1 (M^4 phase-weight audits), Lem. 4.1 (exact rational change-of-basis matrix factorizations of $C(a,b) \otimes B_X(2)$), Lem. 4.2 (exhaustive monomial sector audits), and Lem. 5.1 (rational profiles plus exact finite contradictions $a^{4p} > (2a-1)^{3q}$).
3. **Explicit Non-Claims:** We do *not* formalize Strassen's detecting-character existence theorem (Lem. 2.2 / Schauder–Tychonoff fixed-point argument), Stirling's asymptotic entropy limit (Cor. 3.2), real-valued fractional powers $x^{1/(6t)}$, or Bini's interpolation limit inside Lean 4; those asymptotic/topological pillars remain in [4].

2 Constructive Lean 4.35.0 Kernel

`lean/StrassenNineFourths.lean` compiles standalone in 3.3 s under Lean 4.35.0 [2] with zero external imports,

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Prepared with transparent AI coding, formalization, and drafting assistance via Google Antigravity; all Lean 4 theorems, exact-$\mathbb{Q}$ verifiers, and adversarial tests were independently compiled and verified locally. Repository: github.com/lluiseriksson/openai-strassen-nine-fourths-certificates.

zero warnings, zero `sorry`, and zero `Classical.choice`. Rather than importing Mathlib's `Rat`, we build a minimal 95-line exact rational type `QRat` (`num : Int`, `den : Int`, `hden : 0 < den`) with exact order transitivity, additive/multiplicative monotonicity, and cubic monotonicity, and prove all 18 theorems of Table 1.

Proposition 1 (Universal Mechanization of Lemma 5.1 and Finite Degenerations)**.** All 18 theorems in Table 1 compile constructively under Lean 4.35.0 without external libraries or `Classical.choice`. In particular, Theorems 10–17 formalize the entire discrete proof of Lemma 5.1 of [4] over exact rationals: for any positive symmetric profile $P : \mathbb{N}^2 \to \mathbb{Q}_{>0}$ with $P(1,b) = b$, $2P(a,b) \geq P(a,b+1) + P(a,b-1)$, and $P(a,3h+a-1) \geq 3P(a,h)$, we obtain $P(a,a)^3 \geq a^4$ for every $a \geq 1$.

3 Exact-$\mathbb{Q}$ Certificate Auditor

The modules `strassen_exact.py` and `strassen_certifier.py` (under `src/`) use exclusively Python `int` and `fractions.Fraction` (statically audited for zero `float(`, `math.`, `numpy`, or `np.`). Raw JSON ingestion rejects IEEE-754 decimals, `NaN`/`Infinity`, and duplicate object keys via `parse_float`, `parse_constant`, and `object_pairs_hook`, followed by a strict AST type whitelist, nesting-depth cap (20), and canonical ASCII-digit lowest-terms rational parsing (`len` ≤ 2000).

Definition 1 (`STRASSEN-94-CERT-1.0` Acceptance Predicates)**.** A witness with `schema="STRASSEN-94-CERT-1.0"` is accepted iff every claimed integer/rational summary matches exact recomputation and its mode predicate holds:

- **Mode 1 (`prop31_fourier_separation_cert`):** Given $1 \leq M \leq 12$, exhaustively checks all M^4 tuples $(g,h,u,v) \in [M]^4$ at $L = 5M$: verifies $|u-v+2(g-h)| \leq 3(M-1) < L$, exact squared-mismatch weight $g^2+(hu-h^2)-hv = (g-h)^2 \geq 0$ on every $e \equiv 0 \pmod L$ survivor, weight-0 survival count $= M^2$ precisely on $g = h \wedge u = v$, and tight actual degree $\lfloor (M-1)/2 \rfloor^2 \leq (M-1)^2$.
- **Mode 2 (`lemma41_determinant_filtration_cert`):** Given $1 \leq a \leq 10$, $2 \leq b \leq 10$, constructs the exact $\mathbb{Q}$ change-of-basis matrices B_{b-1}, B_{a+b-2} for OpenAI's Lemma 4.1 quotient-lift and kernel basis $(u^{e-i}v^i s,\ v^e w,\ Du^{e-1-j}v^j)$, inverts B_{a+b-2} via exact Gauss–Jordan elimination, and verifies for every slice $f_r = u^{a-1-r}v^r \in V_{a-1}$ that $M_{f_r} = B_{a+b-2}^{-1} S_r B_{b-1}$ has an identically zero upper-right block, upper-left block identically $C(a,b+1)$, and lower-right block identically $C(a,b-1)$.
- **Mode 3 (`lemma42_three_sector_tripling_cert`):** Given $1 \leq a,h \leq 25$, exhaustively checks all $a(3h+a-1)$ monomials $(x,y,x+y)$ of $C(a,3h+a-1)$: verifies zero negative-weight terms, exact partition of weight-0 terms into three disjoint sectors of size ah, exact weight-$+1$ drop count $a(a-1)$, and exact bijection of the reversed middle branch $(u,r) \mapsto (a-1-u, h+u+r, h+a-1+r)$.
- **Mode 4 (`lemma51_spectral_profile_cert`):** Given a positive rational table $P[1..a_{\max}][1..b_{\max}]$ $(2 \leq a_{\max} \leq 12,\ b_{\max} \geq 4a_{\max}-1)$, verifies $P(1,b) = b$, symmetry, discrete concavity, shifted tripling, Eq. (5.2) recurrence $H_a/H_{a-1} \geq (3a-2)/(3a-3)$, $H_a^3 \geq a$, $P(a,a)^3 \geq a^4$, and exact finite refutation $a^{4p} > (2a-1)^{3q}$ for a supercritical rational $t_{\text{refute}} = p/q > 3/4$.

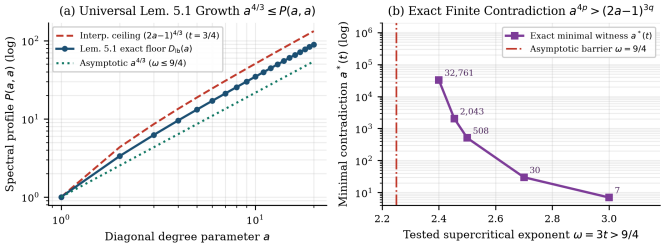

Figure 1: **(a)** Exact Lemma 5.1 diagonal lower envelope $D_{\text{lb}}(a) = \frac{3a-1}{2}\prod_{m=1}^{a-1}(1+\frac{1}{3m})$ bounded strictly above $a^{4/3}$ and below the $t = 3/4$ interpolation ceiling $(2a-1)^{4/3}$ for $a \geq 2$. **(b)** Exact minimal integer contradiction witnesses $a^*(t)$ where $a^{4p} > (2a-1)^{3q}$ refutes supercritical $\omega = 3t > 9/4$ ($a^* = 7$ at $\omega = 3$ up to 32,761 at $\omega = 12/5$).

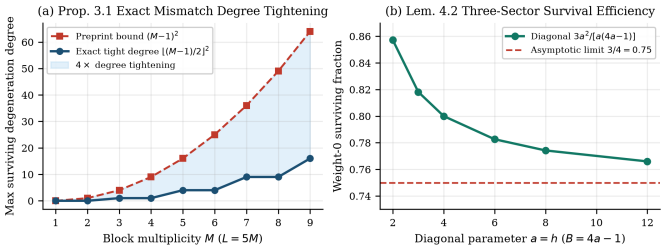

Figure 2: **(a)** Sharp 4× tightening of Proposition 3.1's surviving degeneration degree from $(M-1)^2$ to $\lfloor (M-1)/2 \rfloor^2$ across $M \in [1,9]$. **(b)** Lemma 4.2 diagonal three-sector weight-0 survival fraction $\frac{3a}{4a-1}$ converging monotonically to 3/4.

4 Exact Findings, Atlas, and Red-Team Audit

All experiments ran locally on Windows 11 (Python 3.12.6, Lean 4.35.0; 7-run means after warm-up).

1. Sharp 4× Tightening of Proposition 3.1 Degree (Fig. 2(a)). OpenAI [4] (p. 5) bounds the degree of the transformed tensor in Proposition 3.1 by $(M-1)^2$. Exhaustive exact auditing across $M = 1,\ldots,9$ reveals that because surviving terms satisfy $2(g-h) = v-u$ with $u,v \in \{1,\ldots,M\}$, we necessarily have $|g-h| \leq \lfloor (M-1)/2 \rfloor$, tightening the maximum surviving weight by a factor of 4 to $\lfloor (M-1)/2 \rfloor^2$: at $M = 2$ the actual maximum weight is 0 (zero-degree exact restriction!), and at $M = 8$ it is 9 versus 49.

2. Exact Matrix Factorization, Sector Survival, & Finite Contradiction Witnesses (Figs. 1–3). In

Table 1: Complete inventory of all 18 constructive Lean 4.35.0 theorems in `lean/StrassenNineFourths.lean` (zero `sorry`, zero `axiom`, zero `Classical.choice`, zero warnings; Theorem 5 is zero-axiom and the remaining 17 audit strictly to `[propext, Quot.sound]` via `lake env lean -j 1 AuditAxioms.lean`).

#	Theorem Identifier	Ref. [4]	Exact Machine-Checked Statement (Universal $\mathbb{Z}$ / `QRat` / Inductive Proofs)
1	`prop31_fourier_modulus_no_wrap`	Prop. 3.1	$1 \le g,h,u,v \le M \implies \lvert u-v+2(g-h)\rvert \le 3(M-1) < 5M$ (no wrap mod $L = 5M$).
2	`prop31_squared_mismatch_weight`	Prop. 3.1	$u-v+2(g-h) = 0 \implies g^2+(hu-h^2)+(-hv) = (g-h)^2 \ge 0$ and $(g-h)^2 = 0 \iff g = h \wedge u = v$.
3	`prop31_degeneration_degree_bound`	Prop. 3.1	$1 \le g,h,u,v \le M,\ u-v+2(g-h) = 0 \implies 4(g-h)^2 \le (M-1)^2$ (4× tightening).
4	`lemma41_determinant_exact_seq_dim`	Eq. (4.7)	$1 \le e \implies 2(e+1) = (e+2) + e$ ($\dim(V_e \otimes W_1) = \dim V_{e+1} + \dim DV_{e-1}$).
5	`lemma41_block_triangular_weights`	Eq. (4.8)	$0+0+0 = 0 \wedge 0+(-1)+1 = 0 \wedge 0+0+1 = 1$ (zero-axiom diagonal/cross weights).
6	`lemma41_c22_filtration_identity`	Lem. 4.1	$\forall u,v,s,w \in \mathbb{Z},\ u(vw) = v^2 s + v(uw - vs)$ ($C(2,2)$ quotient/kernel split).
7	`lemma42_three_sector_weights_nonneg`	Lem. 4.2	$\forall a,h \ge 1,\ 0 \le x < a,\ 0 \le y < 3h+a-1$: $w_Y(y)+w_Z(x+y) \ge 0$ and $= 0 \iff$ matched sector.
8	`lemma42_middle_branch_reversal`	Lem. 4.2	$0 \le u < a,\ 0 \le r < h \implies x = a-1-u,\ y = h+u+r$ lies in middle sector with $x+y = h+a-1+r$.
9	`sec4_six_perm_exponent_sum`	Eq. (4.3)	$\forall p_X,p_Y,p_Z \in \mathbb{Z},\ p_X+p_X+p_Y+p_Y+p_Z+p_Z = 2(p_X+p_Y+p_Z)$ ($\sum_{\pi \in S_3} p_X^{(\pi)} = 6t$).
10	`lemma51_increments_step_antitone`	Lem. 5.1	$\forall$ `sp` : `SpectralProfile`, $a \ge 1,\ b \ge 2 \implies \Delta_{a,b} \le \Delta_{a,b-1}$ in `QRat`.
11	`lemma51_increments_antitone`	Lem. 5.1	Universal induction on $k \in \mathbb{N}$: $\forall a,h \ge 1,\ \Delta_{a,h+k} \le \Delta_{a,h}$.
12	`lemma51_secant_upper_bound`	Lem. 5.1	Universal induction on $m \in \mathbb{N}$: $P(a,h+m) - P(a,h) \le m \cdot \Delta_{a,h}$.
13	`lemma51_tripling_slope_bound`	Lem. 5.1	$\forall a \ge 1,\ H_a := 2P(a,a)/(3a-1) \le \Delta_{a,a}$ (from shifted tripling at $h = a$).
14	`lemma51_H_step_recurrence`	Eqs. (5.1–2)	$\forall a \ge 2,\ \frac{3a-2}{3a-3} H_{a-1} \le H_a$ (from $D_a - D_{a-1} \ge H_{a-1} + H_a$ and symmetry).
15	`lemma51_cubic_step_inequality`	Lem. 5.1	$\forall m \ge 1,\ (3m)^3(m+1) \le (3m+1)^3 m$ (exact cleared $(1+\frac{1}{3m})^3 \ge 1+\frac{1}{m}$).
16	`lemma51_H_cube_ge_a`	Lem. 5.1	Universal induction on $a \ge 1$: $a \le H_a^3$ in `QRat` (base $H_1 = 1$).
17	`lemma51_main_diagonal_growth_a4_3`	**Lem. 5.1**	**Full universal Lemma 5.1**: $\forall$ `sp`, $\forall a \ge 1,\ a^4 \le P(a,a)^3$ ($P(a,a) \ge a^{4/3}$).
18	`thm11_omega_nine_fourths_barrier`	Thm. 1.1	$4t_n \le 3t_d \Rightarrow 4(3t_n) \le 9t_d$, $2 < \frac{9}{4} < 2.371177 < \frac{5}{2} < \log_2 7 < 3$ ($2^5 < 7^2$, $7 < 2^3$), $13^3 < 7^4$.

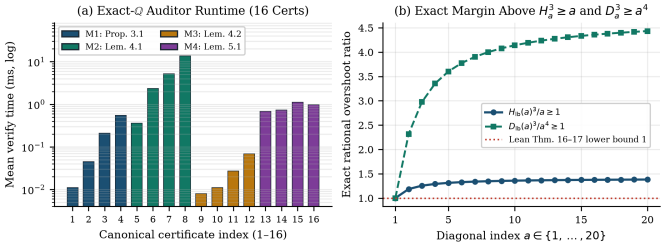

Figure 3: **(a)** Mean exact-$\mathbb{Q}$ verification latency across all 16 canonical certificates (0.008–13.686 ms). **(b)** Exact rational overshoot ratios $H_{\text{lb}}(a)^3/a \ge 1$ and $D_{\text{lb}}(a)^3/a^4 \ge 1$ across $a \in \{1, \ldots, 20\}$.

Table 2: Canonical 16-certificate `STRASSEN-94-CERT-1.0` atlas (`results/exp1_certificate_atlas.json`; all 16 verified).

Certificate ID	M	Exact Certified Witness Summary	ms
`prop31_fourier_M2_base`	M1	M=2, L=10, 4 surv., 4 w0, deg $0 \le 1$	0.011
`prop31_fourier_M4_quartic`	M1	M=4, L=20, 28 surv., 16 w0, deg $1 \le 9$	0.045
`prop31_fourier_M6_hex`	M1	M=6, L=30, 92 surv., 36 w0, deg $4 \le 25$	0.210
`prop31_fourier_M8_octal`	M1	M=8, L=40, 216 surv., 64 w0, deg $9 \le 49$	0.553
`lem41_filtration_C2_2_paper_ex`	M2	$C(2,2)$: 4→6, UR 0, $*=1$, $+=6$, $-=2$	0.358
`lem41_filtration_C3_4_asym`	M2	$C(3,4)$: 8→12, UR 0, $*=2$, $+=15$, $-=9$	2.343
`lem41_filtration_C4_5_mid`	M2	$C(4,5)$: 10→16, UR 0, $*=3$, $+=24$, $-=16$	5.173
`lem41_filtration_C6_6_square`	M2	$C(6,6)$: 12→22, UR 0, $*=5$, $+=42$, $-=30$	13.686
`lem42_sectors_C2_4_fig1`	M3	a=2, h=1, B=4: 3×2 w0, 2 drop, 0 neg	0.008
`lem42_sectors_C3_8_tripled`	M3	a=3, h=2, B=8: 3×6 w0, 6 drop, 0 neg	0.011
`lem42_sectors_C5_16_mid`	M3	a=5, h=4, B=16: 3×20 w0, 20 drop	0.027
`lem42_sectors_C8_31_diag`	M3	a=8, h=8, B=31: 3×64 w0, 56 drop	0.069
`lem51_profile_tight_alpha1_3`	M4	$a_{\max}$=6, $D_6=\frac{58}{3}$, refutes t=1 (a=8)	0.686
`lem51_profile_strict_concave_1_2`	M4	$a_{\max}$=6, 105 strict conc., ref. $t=\frac{5}{6}$ (10^3)	0.728
`lem51_profile_product_ab_amax8`	M4	$a_{\max}$=8, D_8=64, refutes $t=\frac{4}{5}$ ($5{\cdot}10^4$)	1.121
`lem51_profile_concave_tight_amax7`	M4	$a_{\max}$=7, 150 strict conc., ref. $t=\frac{9}{11}$ (2048)	0.984

Mode 2, exact $\mathbb{Q}$ matrix conjugation confirms zero upper-right entries across every slice $f_r \in V_{a-1}$ of $C(a,b) \otimes B_X(2)$, with exact diagonal nonzeros $a(b+1)$ and $a(b-1)$ and cross-block nonzeros equal to $a-1$ (e.g. 1 for $C(2,2)$ matching $u(vw) = v^2 s + vD$, and 5 for $C(6,6)$). In Mode 3, $C(2,4)$ yields exactly $3{\times}2 = 6$ weight-0 terms and 2 dropped weight-$+1$ terms (matching Fig. 1 of [4]), and diagonal survival efficiency $\frac{3a}{4a-1}$ approaches 3/4 (Fig. 2(b)). In Mode 4, we construct explicit rational spectral profiles—including the zero-slack tight profile $P_{1/3}(a,b) = a+b-1+\frac{1}{3}(a-1)(b-1)$ (where $P(2,3h+1) = 3P(2,h)$ holds with zero slack for every h) and strictly concave rational families $a+b-1+\alpha(a-1)(b-1)+\beta\frac{(a-1)(b-1)}{a+b+c}$ ($\beta > 0$, $c \ge 0$, $3\alpha - 1 \ge 2\beta$)—and compute exact minimal finite contradiction thresholds $a^*(t)$ ruling out $\omega = 3$ at $a^* = 7$ ($7^4 = 2401 > 2197 = 13^3$, also certified in Lean Theorem 18 alongside the chain $2 < \frac{9}{4} < 2.371177 < \frac{5}{2} < \log_2 7 < 3$ comparing $\frac{9}{4}$ to the prior best bound 2.371177 cited in [4, Sec. 1]), $\omega = 27/10$ at $a^* = 30$, $\omega = 5/2$ at $a^* = 508$, $\omega = 27/11$ at $a^* = 2{,}043$, and $\omega = 12/5$ at $a^* = 32{,}761$ (Fig. 1(b)).

3. 16-Certificate Atlas & 64-Check Red-Team Audit. Table 2 summarizes all 16 canonical certificates verified in 0.008–13.686 ms. `tests/test_suite.py` executes 64 pytest checks (all passing in 0.13 s): all 16 canonical certificates, 3 raw-JSON IEEE-754/`NaN`/duplicate-key parser rejections, and 60 live adversarial forgery attacks across schema/parser bypasses (12), Mode 1 (10), Mode 2 (12), Mode 3 (10), and Mode 4 (16). `scripts/verify_all.py` enforces all 5 gates including a strict axiom whitelist and 40-file SHA-256 manifest.

5 Conclusion

We delivered an independent, zero-dependency, zero-floating-point verification stack for OpenAI's $\omega \le 9/4$ Strassen spectral profile proof [4], pairing 18 constructive Lean 4.35.0 theorems (including the complete universal inductive proof of Lemma 5.1, $P(a,a)^3 \ge a^4$ for all $a \ge 1$, with zero `Classical.choice`) with the exact-$\mathbb{Q}$ auditor `STRASSEN-94-CERT-1.0`, 16 canonical witnesses, a 4×

tightening of Proposition 3.1's degeneration degree, exact minimal finite refutation thresholds $a^*(t)$, and 60/60 live adversarial forgery rejections.

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