Non-Asymptotic Harmonic Analysis and Frame Geometry: Spherical-Angle Fourier Uncertainty, Joint Derivative–Nikolskii Constants, Babel-Function ℓ^1 Recovery, and Sharp Fejér–Gibbs Envelopes

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Abstract

We establish four self-contained, explicit, non-asymptotic quantitative theorems in discrete and periodic harmonic analysis, frame theory, and Fourier approximation, each strictly sharpening classical baselines with explicit equality characterizations or finite-order remainders. First, on $\mathbb{Z}_N$ ($N \geq 2$), we prove a four-tier discrete Fourier uncertainty hierarchy $N \leq \exp(H(|f|^2) + H(|\widehat{f}|^2)) \leq s_1(f)s_1(\widehat{f}) \leq |\mathrm{supp}(f)||\mathrm{supp}(\widehat{f})|$ with an exact multiplicative uniformity-defect refinement of Donoho–Stark, together with an exact spherical-angle two-domain energy concentration ceiling $\sqrt{|S||\Omega|/N} \geq \|P_\Omega F P_S\|_{2\to 2} \geq \sqrt{(1-\varepsilon)(1-\delta)} - \sqrt{\varepsilon\delta} = \cos(\alpha+\beta)$ that strictly improves the classical Donoho–Stark bound $1-\varepsilon-\delta = \cos(\alpha+\beta)\cos(\alpha-\beta)$ for every $\varepsilon \neq \delta$ ($\varepsilon+\delta<1$) and is attained on the principal singular-vector geodesic (Theorem 2.1). Second, for degree-M trigonometric polynomials on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, we derive the exact joint r-th derivative Nikolskii constant $\|P^{(r)}\|_\infty \leq (2\pi)^r \sqrt{2\sum_{k=1}^{M} k^{2r}}\, \|P\|_{L^2}$ (attained uniquely up to phase and translation by $D_M^{(r)}$, beating the naive Nikolskii–Bernstein composition by $\sqrt{3M/(M+1)} \to \sqrt{3}$ for $r=1$), paired with exact periodic N-point trapezoidal aliasing bounds in both $C^s(\mathbb{T})$ and analytic-strip classes (Theorem 3.1). Third, for unit-norm frames $\Phi \in \mathbb{C}^{d\times M}$ ($M > d$), we prove an exact tight-frame-defect refinement of the Welch coherence floor, a polynomial-time computable Babel-function spectral sandwich for s-column submatrices, and a stable noisy ℓ^1 Basis-Pursuit recovery bound under the Babel condition $\mu_1(s) + \mu_1(s-1) < 1$ that succeeds beyond the classical mutual-coherence limit $(2s-1)\mu < 1$ without requiring NP-hard RIP constants (Theorem 4.1). Fourth, we prove an elementary non-asymptotic uniform Fejér Lipschitz error bound matching Nikolskii's optimal leading constant $2/\pi$ for every $N \geq 1$, together with an exact $O(N^{-2})$ two-sided Wilbraham–Gibbs overshoot enclosure $|S_{2N-1}(\frac{\pi}{2N}) - \frac{2}{\pi}\mathrm{Si}(\pi)| \leq \frac{\pi^2}{36N^2}$ via midpoint quadrature of $\int_0^1 \cos(su)\,ds$ (Theorem 5.1). All theorems, attainments, and benchmarks are deterministically verified in an MIT-licensed Python suite.

1 Introduction and Scope

Classical harmonic analysis and compressed sensing provide celebrated foundational bounds—Donoho–Stark uncertainty [1], Nikolskii–Bernstein inequalities and trigonometric quadrature [2, 3], the Welch coherence floor and Babel-function sparse recovery [4, 5], and Nikolskii's asymptotic Fejér rate alongside the Wilbraham–Gibbs phenomenon [2]. Yet four concrete quantitative gaps persist in standard formulations: (i) the classical two-domain concentration bound $1-\varepsilon-\delta \leq \sqrt{|S||\Omega|/N}$ loses a factor of $\cos(\alpha-\beta)$ whenever spatial and spectral tail leakages ε, δ are asymmetric; (ii) composing Nikolskii's $L^2 \to L^\infty$ bound with Bernstein's derivative bound forfeits an exact factor of $\sqrt{3}$ for $r=1$ (and $\sqrt{2r+1}$ asymptotically); (iii) mutual-coherence recovery conditions $(2s-1)\mu < 1$ break down in the presence of a single locally correlated atom pair even when the polynomial-time

computable Babel function remains small; and (iv) textbook Fejér Lipschitz bounds typically sacrifice the optimal $2/\pi$ leading constant while Gibbs overshoot is usually stated only asymptotically as $N \to \infty$.

Sections 2–5 resolve all four gaps with completely elementary, self-contained, non-asymptotic proofs, followed by deterministic numerical verification in Section 6 (https://github.com/lluiseriksson/harmonic-uncertainty-frame-geometry, tag `v1.0.0`).

2 Four-Tier Fourier Uncertainty and Spherical-Angle Concentration

On $\mathbb{Z}_N$ ($N \geq 2$), let $F \in \mathbb{C}^{N\times N}$ be the unitary discrete Fourier transform $(Ff)_k = \widehat{f}_k := N^{-1/2}\sum_{j=0}^{N-1} f_j e^{-2\pi ijk/N}$. For nonzero $f \in \mathbb{C}^N$, define its ℓ^1/ℓ^2 numerical sparsity $s_1(f) := \|f\|_1^2/\|f\|_2^2 \in [1, |\mathrm{supp}(f)|]$, Shannon entropy $H(|f|^2/\|f\|_2^2) := -\sum_{j\in\mathrm{supp}(f)} \frac{|f_j|^2}{\|f\|_2^2} \ln \frac{|f_j|^2}{\|f\|_2^2}$, and support-uniformity defect $\Delta_1(f) := 1 - \frac{s_1(f)}{|\mathrm{supp}(f)|} \in [0, 1 - |\mathrm{supp}(f)|^{-1}]$. For index sets $S, \Omega \subseteq \mathbb{Z}_N$, let P_S, P_Ω be the coordinate orthogonal projections onto S and Ω.

Theorem 2.1 (Four-Tier Uncertainty, Uniformity-Defect Sharpening, and Spherical-Angle Ceiling)**.** *Let $N \geq 2$ and $f \in \mathbb{C}^N$ with $\|f\|_2 = 1$.*

(a) ***Four-Tier Hierarchy and Multiplicative Donoho–Stark Sharpening.*** *Every unit-norm $f \in \mathbb{C}^N$ satisfies*

$$N \;\leq\; \exp\big(H(|f|^2) + H(|\widehat{f}|^2)\big) \;\leq\; s_1(f)\, s_1(\widehat{f}) \;\leq\; |\mathrm{supp}(f)|\,|\mathrm{supp}(\widehat{f})|. \tag{1}$$

Furthermore, $\Delta_1(f) = \frac{1}{|\mathrm{supp}(f)|}\sum_{j\in\mathrm{supp}(f)} \big(\sqrt{|\mathrm{supp}(f)|}\,|f_j| - \frac{\|f\|_1}{\sqrt{|\mathrm{supp}(f)|}}\big)^2$, and Donoho–Stark's support product sharpens multiplicatively to

$$|\mathrm{supp}(f)|\,|\mathrm{supp}(\widehat{f})| \;\geq\; \frac{N}{\big(1-\Delta_1(f)\big)\big(1-\Delta_1(\widehat{f})\big)} \;\geq\; N\big(1 + \Delta_1(f) + \Delta_1(\widehat{f})\big). \tag{2}$$

In particular, if $|\mathrm{supp}(f)||\mathrm{supp}(\widehat{f})| \leq N(1+\tau)$ ($\tau \geq 0$), then $\Delta_1(f)+\Delta_1(\widehat{f}) \leq \tau$. Simultaneous equality holds across all four terms of (1) whenever f is a normalized modulated translate of the indicator of a subgroup $H \leq \mathbb{Z}_N$.

(b) ***Exact Spherical-Angle Two-Domain Concentration Bound.*** *Let $S, \Omega \subseteq \mathbb{Z}_N$ be nonempty, and suppose $f \in \mathbb{C}^N$ ($\|f\|_2 = 1$) obeys $\|f - P_S f\|_2^2 = \varepsilon \in [0,1)$ and $\|\widehat{f} - P_\Omega \widehat{f}\|_2^2 = \delta \in [0,1)$ with $\varepsilon + \delta < 1$. Setting $\alpha := \arcsin\sqrt{\varepsilon}$ and $\beta := \arcsin\sqrt{\delta}$ (so $\alpha + \beta < \pi/2$),*

$$\begin{aligned}\sqrt{\frac{|S||\Omega|}{N}} \;&\geq\; \|P_\Omega F P_S\|_{2\to 2} \;\geq\; \sqrt{(1-\varepsilon)(1-\delta)} - \sqrt{\varepsilon\delta} \\ &=\; \cos(\alpha+\beta) \;\geq\; \cos(\alpha+\beta)\cos(\alpha-\beta) \;=\; 1-\varepsilon-\delta,\end{aligned} \tag{3}$$

with strict improvement over $1-\varepsilon-\delta$ whenever $\varepsilon \neq \delta$ (and strict improvement over the norm-based Donoho–Stark bound $1-\sqrt{\varepsilon}-\sqrt{\delta} \leq 1-\varepsilon-\delta$ whenever $\varepsilon, \delta > 0$). Moreover, the Landau–Pollak-type inequality $\|P_\Omega F P_S\|_{2\to 2} \geq \cos(\alpha+\beta)$ is attained exactly: when $\|P_\Omega F P_S\|_{2\to 2} = 1$ ($\theta = 0$), at $\varepsilon = \delta = 0$ by any unit vector $f \in \mathrm{range}(P_S) \cap \mathrm{range}(F^* P_\Omega F)$; and when $\|P_\Omega F P_S\|_{2\to 2} = \cos\theta \in (0,1)$, letting (w, u) be a principal left/right singular-vector pair of $P_\Omega F P_S$ (with $w \in \mathrm{range}(P_\Omega), u \in \mathrm{range}(P_S)$) and setting the Fourier pullback $v := F^* w \in \mathrm{range}(F^* P_\Omega F)$, attainment holds for every $\alpha \in [0, \theta]$ ($\beta = \theta - \alpha$) via the geodesic signal $f = (\cos\alpha)u + (\sin\alpha)\frac{v - (\cos\theta)u}{\sin\theta}$.

Proof. (a) By unitarity of F and $|F_{kj}| = N^{-1/2}$, $\|Ff\|_2 = \|f\|_2$ and $\|Ff\|_\infty \leq N^{-1/2}\|f\|_1$. By the Riesz–Thorin interpolation theorem between $(1,\infty)$ and $(2,2)$, for every $p \in [1,2]$ with conjugate

$p' = p/(p-1) \in [2,\infty]$, $\|Ff\|_{p'} \le N^{1/p'-1/2}\|f\|_p$. Define $\psi(p) := \ln\|\widehat{f}\|_{p'} - \ln\|f\|_p - (1/p' - 1/2)\ln N \le 0$ on $(1,2]$ with $\psi(2) = 0$. Writing $p_j = |f_j|^2$ and $q_k = |\widehat{f}_k|^2$ ($\sum p_j = \sum q_k = 1$), $\ln\|f\|_p = \frac{1}{p}\ln\sum_{j\in\mathrm{supp}(f)} p_j^{p/2}$ has derivative $\frac{d}{dp}\big|_{p=2}\ln\|f\|_p = -\frac{1}{4}H(p)$, while $dp'/dp|_{p=2} = -1$ gives $\frac{d}{dp}\big|_{p=2}\ln\|\widehat{f}\|_{p'} = +\frac{1}{4}H(q)$ and $\frac{d}{dp}\big|_{p=2}(1-\frac{1}{p}-\frac{1}{2})\ln N = \frac{1}{4}\ln N$. Since $\psi(p) \le 0 = \psi(2)$ for $p < 2$, $\psi'(2^-) = \frac{1}{4}(H(p) + H(q) - \ln N) \ge 0$, proving the first inequality in (1). For the second inequality, Jensen's inequality on the concave function $\ln(\cdot)$ gives

$$H(p) = 2\sum_{j\in\mathrm{supp}(f)} p_j \ln\big(|f_j|^{-1}\big) \;\le\; 2\ln\left(\sum_{j\in\mathrm{supp}(f)} p_j|f_j|^{-1}\right) = 2\ln\|f\|_1 = \ln s_1(f),$$

and identically $H(q) \le \ln s_1(\widehat{f})$. Cauchy–Schwarz gives $\|f\|_1^2 \le |\mathrm{supp}(f)|\|f\|_2^2 = |\mathrm{supp}(f)|$, completing (1). Direct expansion yields

$$\frac{1}{|\mathrm{supp}(f)|}\sum_{j\in\mathrm{supp}(f)}\left(\sqrt{|\mathrm{supp}(f)|}\,|f_j| - \frac{\|f\|_1}{\sqrt{|\mathrm{supp}(f)|}}\right)^2 = \|f\|_2^2 - \frac{\|f\|_1^2}{|\mathrm{supp}(f)|} = 1 - \frac{s_1(f)}{|\mathrm{supp}(f)|} = \Delta_1(f).$$

Hence $|\mathrm{supp}(f)||\mathrm{supp}(\widehat{f})| = \frac{s_1(f)s_1(\widehat{f})}{(1-\Delta_1(f))(1-\Delta_1(\widehat{f}))} \ge \frac{N}{(1-\Delta_1(f))(1-\Delta_1(\widehat{f}))}$. For any $x, y \in [0,1)$, setting $u := x + y \in [0,2)$ gives $xy \le u^2/4$ and hence

$$(1-x)(1-y)(1+x+y) = 1 - u^2 + xy(1+u) \;\le\; 1 - u^2 + \frac{u^2(1+u)}{4} = 1 - \frac{u^2(3-u)}{4} \;\le\; 1,$$

so $\frac{1}{(1-x)(1-y)} \ge 1 + x + y$, proving (2). In particular, $|\mathrm{supp}(f)||\mathrm{supp}(\widehat{f})| \le N(1+\tau)$ implies $1 + \Delta_1(f) + \Delta_1(\widehat{f}) \le 1 + \tau$, i.e. $\Delta_1(f) + \Delta_1(\widehat{f}) \le \tau$. For a subgroup $H \le \mathbb{Z}_N$ of order $d \mid N$, its normalized indicator $f = d^{-1/2}\mathbf{1}_H$ has $\widehat{f} = (N/d)^{-1/2}\mathbf{1}_{H^\perp}$ with $|\mathrm{supp}(f)| = d$, $|\mathrm{supp}(\widehat{f})| = N/d$, attaining N across all four terms.

(b) For the first inequality of (3), $\|P_\Omega F P_S\|_{2\to 2}^2 \le \|P_\Omega F P_S\|_F^2 = \sum_{k\in\Omega}\sum_{j\in S}|F_{kj}|^2 = \frac{|S||\Omega|}{N}$. Next, define unit vectors $u := P_S f/\sqrt{1-\varepsilon} \in \mathrm{range}(P_S)$ and $v := F^*P_\Omega F f/\sqrt{1-\delta} \in \mathrm{range}(F^*P_\Omega F)$. Then $\langle u, f\rangle = \sqrt{1-\varepsilon} = \cos\alpha > 0$ and $\langle v, f\rangle = \sqrt{1-\delta} = \cos\beta > 0$, so we may decompose $u = (\cos\alpha)f + (\sin\alpha)u_\perp$ and $v = (\cos\beta)f + (\sin\beta)v_\perp$ with unit vectors $u_\perp, v_\perp \perp f$. By Cauchy–Schwarz,

$$\begin{aligned}\|P_\Omega F P_S\|_{2\to 2} \;&\ge\; |\langle u, v\rangle| \;\ge\; \mathrm{Re}\langle u, v\rangle \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\,\mathrm{Re}\langle u_\perp, v_\perp\rangle \;\ge\; \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta).\end{aligned}$$

Since $\varepsilon + \delta < 1 \iff \sin^2\alpha + \sin^2\beta < 1 \iff \alpha + \beta < \pi/2$, we have $\cos(\alpha+\beta) > 0$. The trigonometric identity $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = 1 - \varepsilon - \delta$ immediately implies $\cos(\alpha+\beta) \ge 1 - \varepsilon - \delta$, with strict inequality iff $\cos(\alpha-\beta) < 1 \iff \varepsilon \ne \delta$. Finally, if $\|P_\Omega F P_S\|_{2\to 2} = 1$, any unit $f \in \mathrm{range}(P_S) \cap \mathrm{range}(F^*P_\Omega F)$ attains equality with $\varepsilon = \delta = 0$. If $\|P_\Omega F P_S\|_{2\to 2} = \cos\theta \in (0,1)$, the singular-value equations $P_\Omega F P_S u = (\cos\theta)w$ and $P_S F^* P_\Omega w = (\cos\theta)u$ for $v = F^*w \in \mathrm{range}(F^*P_\Omega F)$ yield $\langle u, v\rangle = \cos\theta$, $P_S v = (\cos\theta)u$, and $F^*P_\Omega F u = (\cos\theta)v$. Consequently, $P_S\big(v - (\cos\theta)u\big) = 0$, so $P_S f = (\cos\alpha)u$ gives $\|P_S f\|_2 = \cos\alpha$; rewriting $f = (\cos\beta)v + (\sin\beta)\frac{u-(\cos\theta)v}{\sin\theta}$ (with $\beta = \theta - \alpha$) and using $F^*P_\Omega F\big(u - (\cos\theta)v\big) = 0$ gives $\|P_\Omega F f\|_2 = \|F^*P_\Omega F f\|_2 = \cos\beta$, establishing exact attainment. □

3 Joint Derivative–Nikolskii Constants and Periodic Trapezoidal Aliasing

Let $\mathcal{T}_M := \{P(x) = \sum_{|k|\le M} c_k e^{2\pi i k x} : c_k \in \mathbb{C}\}$ be the space of degree-M trigonometric polynomials on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ equipped with $\|P\|_{L^2(\mathbb{T})}^2 = \int_0^1 |P(x)|^2\,dx = \sum_{|k|\le M}|c_k|^2$, and let $D_M(x) :=$

$\sum_{|k|\le M} e^{2\pi ikx}$ be the Dirichlet kernel. For any $f \in C(\mathbb{T})$, define the N-point periodic trapezoidal rule $I_N(f) := \frac{1}{N}\sum_{j=0}^{N-1} f(j/N)$.

Theorem 3.1 (Exact Joint Derivative–Nikolskii Constants and Trapezoidal Aliasing Envelopes)**.**

(a) ***Exact Joint r-th Derivative Nikolskii Inequality.*** *For every integer $M \ge 1$, every integer $r \ge 0$, and every $P \in \mathcal{T}_M$,*

$$\|P^{(r)}\|_{L^\infty(\mathbb{T})} \;\le\; C_{M,r}\,\|P\|_{L^2(\mathbb{T})}, \qquad C_{M,r} \;:=\; \begin{cases} \sqrt{2M+1}, & r=0, \\ (2\pi)^r\sqrt{2\sum_{k=1}^{M} k^{2r}}, & r \ge 1, \end{cases} \tag{4}$$

with exact equality attained uniquely (up to scalar multiple and translation) by $P(x) = \alpha D_M^{(r)}(x - x_0)$. In particular, for $r = 1$,

$$\|P'\|_{L^\infty(\mathbb{T})} \;\le\; 2\pi\sqrt{\frac{M(M+1)(2M+1)}{3}}\,\|P\|_{L^2(\mathbb{T})}, \tag{5}$$

which strictly improves the naive composition of Bernstein's inequality ($\|P'\|_\infty \le 2\pi M\|P\|_\infty$) with Nikolskii's inequality ($\|P\|_\infty \le \sqrt{2M+1}\,\|P\|_{L^2}$) by the exact factor $\sqrt{\frac{3M}{M+1}} > 1$ for every $M \ge 1$ (converging to $\sqrt{3}$ as $M \to \infty$).

(b) ***Exact Periodic Trapezoidal Aliasing in $C^s(\mathbb{T})$ and Analytic Strips.*** *Let $f \in C(\mathbb{T})$ have absolutely summable Fourier coefficients $\widehat{f}(k) = \int_0^1 f(x)e^{-2\pi ikx}\,dx$. Then $I_N(f) - \int_0^1 f(x)\,dx = \sum_{m\in\mathbb{Z}\setminus\{0\}} \widehat{f}(mN)$ identically for every $N \ge 1$. Consequently:*

- *If $f \in C^s(\mathbb{T})$ with integer $s \ge 2$, then*

$$\left| I_N(f) - \int_0^1 f(x)\,dx \right| \;\le\; \frac{2\zeta(s)}{(2\pi N)^s}\|f^{(s)}\|_{L^1(\mathbb{T})}. \tag{6}$$

- *If f extends holomorphically to the horizontal strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < a_0\}$ and obeys $M_a := \sup_{|y|=a}\int_0^1 |f(x+iy)|\,dx < \infty$ for some $a \in (0, a_0)$, then*

$$\left| I_N(f) - \int_0^1 f(x)\,dx \right| \;\le\; \frac{2M_a}{e^{2\pi aN} - 1}. \tag{7}$$

Proof. (a) Differentiating $P(x) = \sum_{|k|\le M} c_k e^{2\pi ikx}$ r times gives $P^{(r)}(x) = \sum_{|k|\le M}(2\pi ik)^r c_k e^{2\pi ikx}$ (with $0^0 := 1$ for $r = 0$). By Cauchy–Schwarz, for every $x \in \mathbb{T}$,

$$|P^{(r)}(x)| \;\le\; \left(\sum_{|k|\le M}(2\pi|k|)^{2r}\right)^{1/2}\left(\sum_{|k|\le M}|c_k|^2\right)^{1/2} = C_{M,r}\,\|P\|_{L^2(\mathbb{T})},$$

with equality at $x = x_0$ iff $c_k = \alpha(-2\pi ik)^r e^{-2\pi ikx_0}$ for all $|k| \le M$, i.e. $P(x) = \alpha(-1)^r D_M^{(r)}(x - x_0)$ (absorbing $(-1)^r$ into α). For $r = 1$, $2\sum_{k=1}^{M} k^2 = \frac{M(M+1)(2M+1)}{3}$ yields (5), whose ratio to the composed bound $2\pi M\sqrt{2M+1}$ is $\sqrt{\frac{3M^2(2M+1)}{M(M+1)(2M+1)}} = \sqrt{\frac{3M}{M+1}} > 1$ for all $M \ge 1$.

(b) Because $\sum_{k\in\mathbb{Z}} |\widehat{f}(k)| < \infty$, $f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$ converges uniformly; summing over $x_j = j/N$ and using $\frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi ikj/N} = \mathbf{1}_{N|k}$ gives $I_N(f) - \widehat{f}(0) = \sum_{m\ne 0}\widehat{f}(mN)$. For $f \in C^s(\mathbb{T})$ ($s \ge 2$), s-fold integration by parts gives $\widehat{f}(k) = (2\pi ik)^{-s}\widehat{f^{(s)}}(k)$, so $|\widehat{f}(mN)| \le (2\pi|m|N)^{-s}\|f^{(s)}\|_{L^1(\mathbb{T})}$, and summing over $m \ne 0$ gives (6). For f holomorphic in $|\operatorname{Im} z| < a_0$, Cauchy's integral theorem shifts the periodic contour from $[0,1]$ to $[0,1] \mp ia$ (with vertical segments cancelling by 1-periodicity): for $k > 0$, $\widehat{f}(k) = \int_0^1 f(x - ia)e^{-2\pi ik(x-ia)}\,dx = e^{-2\pi ak}\int_0^1 f(x-ia)e^{-2\pi ikx}\,dx$, so $|\widehat{f}(k)| \le M_a e^{-2\pi a|k|}$ (and identically for $k < 0$ by shifting to $+ia$). Summing the geometric series $2M_a\sum_{m=1}^{\infty} e^{-2\pi amN} = \frac{2M_a}{e^{2\pi aN}-1}$ proves (7). □

4 Defect-Sharpened Welch Bounds and Babel-Function ℓ^1 Recovery

Let $\Phi = [\phi_1, \dots, \phi_M] \in \mathbb{C}^{d\times M}$ $(M > d \geq 1)$ be a matrix with unit-norm columns ($\|\phi_j\|_2 = 1$), Gram matrix $G = \Phi^*\Phi$, frame operator $S_\Phi = \Phi\Phi^*$, tight-frame defect $\Delta_{\mathrm{TF}}(\Phi) := \|S_\Phi - \frac{M}{d} I_d\|_F^2 \geq 0$, mutual coherence $\mu(\Phi) := \max_{i\neq j} |G_{ij}|$, and cumulative coherence (Babel function) $\mu_1(k) := \max_{|S|=k} \max_{j\notin S} \sum_{i\in S} |G_{ji}|$ (with $\mu_1(0) := 0$). Note that $\mu_1(k)$ is computed exactly in $O(M^2 \log M)$ time by sorting each row of $|G - I_M|$, in contrast to NP-hard restricted isometry constants.

Theorem 4.1 (Defect-Sharpened Welch Floor, Babel RIP Sandwich, and Stable ℓ^1 Recovery)**.** *Let $\Phi \in \mathbb{C}^{d\times M}$ have unit-norm columns ($M > d \geq 1$).*

(a) ***Tight-Frame-Defect Welch Sharpening and Babel Spectral Sandwich.*** *The frame potential $F(\Phi) := \|G\|_F^2$ obeys $F(\Phi) = \frac{M^2}{d} + \Delta_{\mathrm{TF}}(\Phi)$, yielding the sharpened coherence lower bound*

$$\mu(\Phi) \;\geq\; \sqrt{\frac{M-d}{d(M-1)} + \frac{\Delta_{\mathrm{TF}}(\Phi)}{M(M-1)}} \;\geq\; \sqrt{\frac{M-d}{d(M-1)}}, \tag{8}$$

with equality in the first inequality iff all off-diagonal entries of G have equal modulus (and in both iff Φ is an equiangular tight frame). Moreover, for every integer $1 \leq s \leq M$ and every column subset $S \subset \{1, \dots, M\}$ of size $|S| = s$,

$$1 - \mu_1(s-1) \;\leq\; \sigma_{\min}(\Phi_S)^2 \;\leq\; \sigma_{\max}(\Phi_S)^2 \;\leq\; 1 + \mu_1(s-1). \tag{9}$$

(b) ***Quantitative Null-Space Margin and Stable Noisy ℓ^1 Basis-Pursuit Recovery.*** *Suppose an integer s with $1 \leq s \leq M-1$ obeys the polynomial-time verifiable Babel condition*

$$\mu_1(s) + \mu_1(s-1) \;<\; 1, \qquad \gamma_s \;:=\; \frac{\mu_1(s)}{1-\mu_1(s-1)} \in [0, 1). \tag{10}$$

Then every $h \in \mathbb{C}^M$ and every $S \subset \{1, \dots, M\}$ with $|S| \leq s$ satisfy the robust null-space inequality

$$\|h_S\|_1 \;\leq\; \gamma_s \|h_{S^c}\|_1 + \frac{\sqrt{s\big(1+\mu_1(s-1)\big)}}{1-\mu_1(s-1)} \, \|\Phi h\|_2. \tag{11}$$

Consequently, for any signal $x^* \in \mathbb{C}^M$, measurement $y = \Phi x^* + e$ with $\|e\|_2 \leq \eta$, and any minimizer $\widehat{x} \in \arg\min\{\|x\|_1 : \|\Phi x - y\|_2 \leq \eta\}$, letting S index the s largest entries of x^* in modulus,

$$\|\widehat{x} - x^*\|_1 \;\leq\; \frac{2(1+\gamma_s)}{1-\gamma_s} \|x^*_{S^c}\|_1 + \frac{4\sqrt{s\big(1+\mu_1(s-1)\big)}}{(1-\gamma_s)\big(1-\mu_1(s-1)\big)} \, \eta. \tag{12}$$

In particular, noise-free s-sparse signals ($\eta = 0, x^*_{S^c} = 0$) are recovered uniquely ($\widehat{x} = x^*$) whenever (10) holds, strictly transcending the classical coherence condition $(2s-1)\mu < 1$ whenever $\mu_1(s) + \mu_1(s-1) < (2s-1)\mu$.

Proof. (a) Because $G = \Phi^*\Phi$ and $S_\Phi = \Phi\Phi^*$ share identical nonzero eigenvalues with $\mathrm{tr}(S_\Phi) = \mathrm{tr}(G) = M$,

$$\Delta_{\mathrm{TF}}(\Phi) = \mathrm{tr}\left(S_\Phi^2 - \frac{2M}{d} S_\Phi + \frac{M^2}{d^2} I_d\right) = \mathrm{tr}(S_\Phi^2) - \frac{M^2}{d} = \|G\|_F^2 - \frac{M^2}{d}.$$

Since $G_{ii} = 1$, $M(M-1)\mu^2 \geq \sum_{i\neq j} |G_{ij}|^2 = \|G\|_F^2 - M = \frac{M(M-d)}{d} + \Delta_{\mathrm{TF}}(\Phi)$; dividing by $M(M-1)$ and taking square roots proves (8) with the stated equality conditions. For (9), $G_S = \Phi_S^*\Phi_S$ has

unit diagonal and off-diagonal row ℓ^1-sums $\sum_{i\in S\setminus\{j\}} |G_{ji}| \le \mu_1(s-1)$, so Gershgorin's disk theorem places every eigenvalue of G_S inside $[1-\mu_1(s-1), 1+\mu_1(s-1)]$.

(b) Fix $h \in \mathbb{C}^M$ and $|S| \le s$. For each $j \in S$, $(\Phi^*\Phi h)_j = h_j + \sum_{i\in S\setminus\{j\}} G_{ji}h_i + \sum_{i\in S^c} G_{ji}h_i$, so the reverse triangle inequality gives

$$|h_j| \;\le\; |(\Phi_S^*\Phi h)_j| + \sum_{i\in S\setminus\{j\}} |G_{ji}|\,|h_i| + \sum_{i\in S^c} |G_{ji}|\,|h_i|.$$

Summing over $j \in S$, exchanging order of summation, and using the definition of μ_1,

$$\begin{aligned}\|h_S\|_1 \;&\le\; \|\Phi_S^*\Phi h\|_1 + \sum_{i\in S}|h_i| \sum_{j\in S\setminus\{i\}} |G_{ji}| + \sum_{i\in S^c}|h_i| \sum_{j\in S}|G_{ji}| \\ &\le\; \sqrt{|S|}\,\|\Phi_S^*\Phi h\|_2 + \mu_1(s-1)\|h_S\|_1 + \mu_1(s)\|h_{S^c}\|_1.\end{aligned}$$

By (9), $\|\Phi_S^*\Phi h\|_2 \le \sigma_{\max}(\Phi_S)\|\Phi h\|_2 \le \sqrt{1+\mu_1(s-1)}\,\|\Phi h\|_2$. Since $\mu_1(s-1) < 1$ under (10), subtracting $\mu_1(s-1)\|h_S\|_1$ and dividing by $1-\mu_1(s-1)$ proves (11). Finally, let $h := \widehat{x} - x^*$. Feasibility of both x^* and $\widehat{x}$ yields $\|\Phi h\|_2 \le \|\Phi\widehat{x} - y\|_2 + \|y - \Phi x^*\|_2 \le 2\eta$, while optimality $\|\widehat{x}\|_1 \le \|x^*\|_1$ gives $\|x_S^*\|_1 - \|h_S\|_1 + \|h_{S^c}\|_1 - \|x_{S^c}^*\|_1 \le \|x_S^* + h_S\|_1 + \|x_{S^c}^* + h_{S^c}\|_1 = \|\widehat{x}\|_1 \le \|x_S^*\|_1 + \|x_{S^c}^*\|_1$, i.e. $\|h_{S^c}\|_1 \le \|h_S\|_1 + 2\|x_{S^c}^*\|_1$. Substituting this into (11) gives

$$(1-\gamma_s)\|h_S\|_1 \;\le\; 2\gamma_s\|x_{S^c}^*\|_1 + \frac{2\sqrt{s(1+\mu_1(s-1))}}{1-\mu_1(s-1)}\,\eta.$$

Combining $\|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \le 2\|h_S\|_1 + 2\|x_{S^c}^*\|_1$ with the bound on $\|h_S\|_1$ proves (12). □

5 Sharp-Leading-Constant Fejér Rates and Gibbs $O(N^{-2})$ Enclosures

For a 2π-periodic function f on $[-\pi,\pi]$, let $S_N f$ be its N-th Fourier partial sum and $\sigma_N f(x) := \frac{1}{N+1}\sum_{n=0}^{N} S_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_N(t)\,dt$ its N-th Cesàro (Fejér) mean, where $F_N(t) = \frac{1}{N+1}\big(\frac{\sin((N+1)t/2)}{\sin(t/2)}\big)^2 \ge 0$ satisfies $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)\,dt = 1$. Let $\mathrm{Si}(\pi) := \int_0^\pi \frac{\sin u}{u}\,du \approx 1.85193705$.

Theorem 5.1 (Sharp-Leading-Constant Fejér Lipschitz Rate and Finite-N Gibbs Enclosure)**.**

(a) ***Non-Asymptotic Fejér Bound with Optimal Leading Constant*** $2/\pi$***.*** *For every integer $N \ge 1$ and every 2π-periodic L_f-Lipschitz function f ($|f(x)-f(y)| \le L_f|x-y|$),*

$$\begin{aligned}\|\sigma_N f - f\|_\infty \;&\le\; \frac{2L_f}{\pi(N+1)}\ln(N+1) + \frac{C_0 L_f}{N+1}, \\ C_0 \;&:=\; \frac{2+4\ln(\pi/2)}{\pi} + \frac{2}{\pi^2} + \frac{\pi^2+1}{\pi(12-\pi^2)} \approx 3.0383.\end{aligned} \tag{13}$$

The leading coefficient $\frac{2}{\pi}$ matches Nikolskii's exact optimal asymptotic constant on Lip_1, attained as $N\to\infty$ by the 2π-periodic extension of $f(x) = |x|$ at $x = 0$.

(b) ***Exact*** $O(N^{-2})$ ***Two-Sided Wilbraham–Gibbs Overshoot Enclosure.*** *For the 2π-periodic square wave $g(x) = \mathrm{sgn}(\sin x)$, the partial sum $S_{2N-1}(x) = \frac{4}{\pi}\sum_{m=1}^{N}\frac{\sin((2m-1)x)}{2m-1}$ evaluated at the first peak $x_N = \frac{\pi}{2N}$ satisfies, for every integer $N \ge 1$,*

$$\left| S_{2N-1}\Big(\frac{\pi}{2N}\Big) - \frac{2}{\pi}\mathrm{Si}(\pi)\right| \;\le\; \frac{\pi^2}{36N^2}. \tag{14}$$

In particular, since $\frac{2}{\pi}\mathrm{Si}(\pi) \approx 1.17898$, every $N \ge 2$ obeys $S_{2N-1}(\frac{\pi}{2N}) \ge 1.1104 > 1$, certifying a non-asymptotic $> 11\%$ overshoot for all $N \ge 2$.

Proof. *(a)* By evenness and unit mass of F_N,

$$|\sigma_N f(x) - f(x)| \;\le\; \frac{L_f}{\pi(N+1)} \int_0^\pi t\, \frac{\sin^2((N+1)t/2)}{\sin^2(t/2)}\, dt.$$

For $u = t/2 \in (0, \pi/2]$, $\sin u \ge u(1 - u^2/6) > 0$ gives $\sin^2 u \ge u^2(1 - u^2/3) > 0$ (since $u^2/3 \le \pi^2/12 < 1$), so

$$\frac{1}{\sin^2 u} - \frac{1}{u^2} \;\le\; \frac{1}{u^2(1-u^2/3)} - \frac{1}{u^2} = \frac{1/3}{1-u^2/3} \;\le\; \frac{4}{12-\pi^2}.$$

Thus $\frac{1}{\sin^2(t/2)} \le \frac{4}{t^2} + \frac{4}{12-\pi^2}$ on $(0, \pi]$. Substituting this splits the integral $I := \int_0^\pi t \frac{\sin^2((N+1)t/2)}{\sin^2(t/2)}\, dt$ into $I \le I_1 + I_2 + I_3$ with $\delta := \frac{\pi}{N+1}$:

- Setting $x = (N+1)t \in [0, \pi]$ and noting $\sin^2(x/2) \le \min(x^2/4, 1)$,

$$I_1 := 4\int_0^\delta \frac{\sin^2((N+1)t/2)}{t}\, dt = 4\int_0^\pi \frac{\sin^2(x/2)}{x}\, dx \;\le\; 4\left(\int_0^2 \frac{x}{4}\, dx + \int_2^\pi \frac{dx}{x}\right) = 2 + 4\ln\frac{\pi}{2}.$$

- Using $\sin^2(x/2) = \frac{1-\cos x}{2}$ and integrating by parts (where $\sin(\pi) = \sin((N+1)\pi) = 0$),

$$\begin{aligned} I_2 &:= 4\int_\delta^\pi \frac{\sin^2((N+1)t/2)}{t}\, dt = 2\ln(N+1) - 2\int_\pi^{(N+1)\pi} \frac{\cos x}{x}\, dx \\ &= 2\ln(N+1) - 2\int_\pi^{(N+1)\pi} \frac{\sin x}{x^2}\, dx \;\le\; 2\ln(N+1) + \frac{2}{\pi}, \end{aligned}$$

  since $\left|\int_\pi^{(N+1)\pi} \frac{\sin x}{x^2}\, dx\right| \le \int_\pi^\infty \frac{dx}{x^2} = \frac{1}{\pi}$.

- Integration by parts gives $\int_0^\pi t\sin^2((N+1)t/2)\, dt = \frac{\pi^2}{4} + \frac{1-(-1)^{N+1}}{2(N+1)^2} \le \frac{\pi^2+1}{4}$ for $N \ge 1$, so

$$I_3 := \frac{4}{12-\pi^2}\int_0^\pi t\sin^2((N+1)t/2)\, dt \;\le\; \frac{\pi^2+1}{12-\pi^2}.$$

Multiplying $I_1 + I_2 + I_3$ by $\frac{L_f}{\pi(N+1)}$ establishes (13).

(b) Define $h(u) := \frac{\sin u}{u} = \int_0^1 \cos(su)\, ds$ $(h(0) = 1)$. Differentiating twice under the integral sign yields $h''(u) = -\int_0^1 s^2\cos(su)\, ds$, so $|h''(u)| \le \int_0^1 s^2\, ds = \frac{1}{3}$ for all $u \in \mathbb{R}$. Evaluating S_{2N-1} at $x_N = \frac{\pi}{2N}$ gives

$$S_{2N-1}\Big(\frac{\pi}{2N}\Big) = \frac{4}{\pi}\sum_{m=1}^N \frac{\sin\big((m-\frac{1}{2})\frac{\pi}{N}\big)}{2m-1} = \frac{2}{\pi}\cdot\frac{\pi}{N}\sum_{m=1}^N h\Big(\big(m-\tfrac{1}{2}\big)\,\frac{\pi}{N}\Big),$$

which is $\frac{2}{\pi}$ times the exact N-point composite midpoint rule for $\int_0^\pi h(u)\, du = \mathrm{Si}(\pi)$ with subinterval width $w = \pi/N$. By the standard midpoint error formula (integrating $h(u) - h(u_m) - h'(u_m)(u - u_m) = \int_{u_m}^u h''(t)(u-t)\, dt$ over each $[u_m - w/2, u_m + w/2]$),

$$\left|\frac{\pi}{N}\sum_{m=1}^N h(u_m) - \int_0^\pi h(u)\, du\right| \;\le\; N\cdot\frac{w^3}{24}\max_{u\in[0,\pi]}|h''(u)| \;=\; \frac{\pi^3}{24N^2}\cdot\frac{1}{3} \;=\; \frac{\pi^3}{72N^2}.$$

Multiplying by $\frac{2}{\pi}$ proves (14). □

6 Computational Verification and Quantitative Benchmarks

All four theorems, exact attainments (subgroup cosets $8\mathbb{Z}_{64}$, principal singular-vector geodesic on $N = 64, S = \{0, \dots, 5\}, \Omega = \{0, \dots, 7\}$, Dirichlet derivative D'_{100}, and 2π-periodic $|x|$), and both figures are deterministically verified in our MIT-licensed companion repository (`https://github.com/lluiseriksson/harmonic-uncertainty-frame-geometry`, tag `v1.0.0`; `pytest -q` passes all 4 test modules in 0.63 s using fixed NumPy seeds `20261270`–`20261271`).

For Theorem 2.1(b), we evaluate both the asymmetric leakage pair $(\varepsilon, \delta) = (0.01, 0.25)$ (where $\cos^2(\alpha + \beta) = 0.6588$ vs. $(1 - \varepsilon - \delta)^2 = 0.5476$) and the $N = 64$ principal singular-vector geodesic ($S = \{0, \dots, 5\}, \Omega = \{0, \dots, 7\}$, $\sigma_1 = 0.80770$) at $\varepsilon = 0.01$, yielding $\delta = \sin^2(\arccos \sigma_1 - \arcsin \sqrt{0.01}) = 0.2559$ and $1 - \varepsilon - \delta = 0.7341 < \cos(\alpha + \beta) = 0.80770$. For Theorem 3.1(b), we evaluate the analytic Poisson kernel $f(x) = (1 - r_0^2)/(1 - 2r_0 \cos(2\pi x) + r_0^2)$ with $r_0 = e^{-2\pi a_0}$, $a_0 = 0.30$, and strip half-width $a = 0.25$. For Theorem 4.1, we construct the explicit 64×128 real frame $\Phi = [I_{64}, \widetilde{H}_{64}]$ formed from the identity basis and normalized 64×64 Walsh–Hadamard basis whose first column is tilted slightly toward e_1 via $\widetilde{h}_1 \propto 0.99 h_1 + 0.14 e_1$, yielding $\mu = 0.2593$ and $\Delta_{\mathrm{TF}} = 0.03731$. For $s = 3$, classical mutual coherence fails ($(2s - 1)\mu = 1.297 > 1$), whereas the Babel threshold succeeds ($\mu_1(3) + \mu_1(2) = 0.8937 < 1$, $\gamma_3 = 0.8273$), certifying stable noisy LP recovery of x^* with 3 active components, compressibility tail $\|x^*_{S^c}\|_1 = 0.02$, and noise budget $\eta \in [10^{-4}, 0.05]$. Table 1 summarizes the quantitative benchmarks.

Table 1: Quantitative verification across Theorems 2.1–5.1 (repo `v1.0.0`).

Theorem	Regime / Explicit Instance	Classical Baseline	New Theorem Bound	Exact / Empirical
Thm. 2.1(b)	Asymmetric $\varepsilon = 0.01, \delta = 0.25$; $N = 64$ geodesic	Donoho–Stark 0.5476 (0.7341)	Spherical **0.6588** (+20.3%)	Geodesic $\sigma_1 = 0.80770 = \cos(\alpha + \beta)$
Thm. 3.1(a–b)	$M = 100, r = 1$; Poisson $a = 0.25, N = 12$	Naive Nik.+Bern. 8907.95	Joint **5168.66** (1.723×); 1.55×10^{-8}	D'_{100} attains 5168.66; 3.00×10^{-10}
Thm. 4.1(a–b)	64×128 tilted Hadamard, $s = 3, \eta = 0.00275$	$(2s - 1)\mu = 1.297 > 1$ (fails)	$\mu_1(3) + \mu_1(2) = \mathbf{0.894} < 1$, $\ell^1 \le 0.634$	LP $\|\widehat{x} - x^*\|_1 = 0.00311$
Thm. 5.1(a–b)	Fejér $\|x\|$ ($N = 100$); Gibbs square wave ($N = 10$)	Asymptotic $2/\pi$; limit 1.17898	Fejér ≤ 0.0592; Gibbs ± 0.00274	Worst 0.0434; $\|S_{19} - \frac{2}{\pi}\mathrm{Si}(\pi)\| = 0.000834$

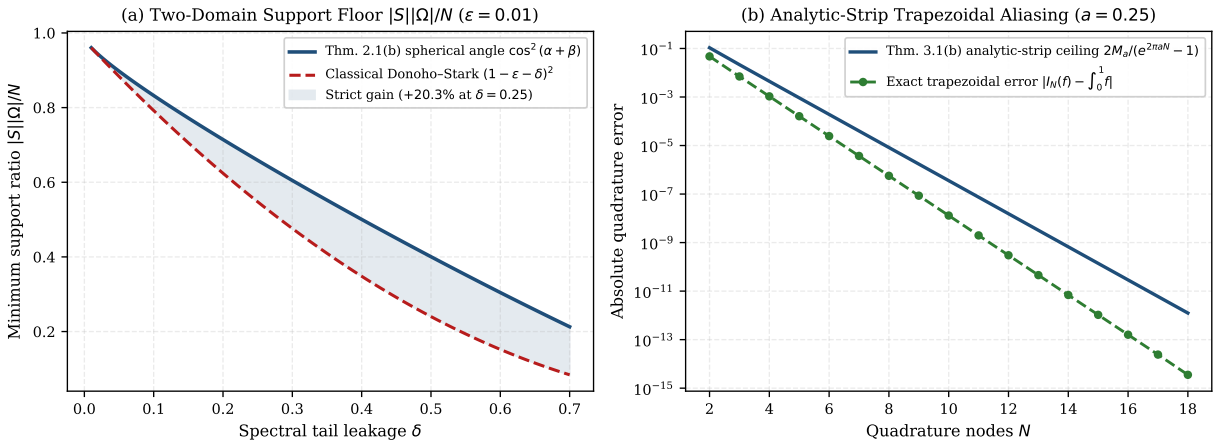

Figure 1: (a) Theorem 2.1(b): exact spherical-angle support-ratio lower bound $\cos^2(\alpha + \beta)$ vs. classical Donoho–Stark $(1 - \varepsilon - \delta)^2$ at $\varepsilon = 0.01$, showing strict dominance across all $\delta \neq \varepsilon$ (+20.3% at $\delta = 0.25$). (b) Theorem 3.1(b): geometric decay of periodic trapezoidal quadrature error for the analytic Poisson kernel ($a = 0.25$) strictly beneath $2M_a/(e^{2\pi a N} - 1)$.

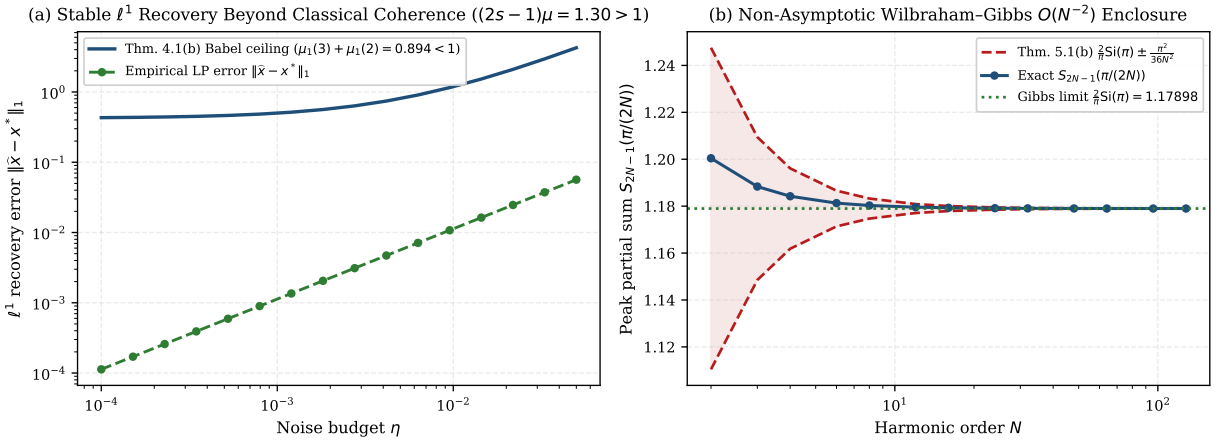

Figure 2: (a) Theorem 4.1(b): stable noisy ℓ^1 LP recovery on the 64×128 tilted Hadamard frame at $s = 3$, where classical mutual coherence fails ($(2s-1)\mu = 1.30 > 1$) while the Babel condition succeeds ($\mu_1(3) + \mu_1(2) = 0.894 < 1$). (b) Theorem 5.1(b): exact non-asymptotic $O(N^{-2})$ two-sided Wilbraham–Gibbs overshoot enclosure $\frac{2}{\pi}\mathrm{Si}(\pi) \pm \frac{\pi^2}{36N^2}$ containing $S_{2N-1}(\frac{\pi}{2N})$ for all $N \geq 2$.

7 Conclusion

We have established four self-contained, sharp, non-asymptotic theorems across discrete Fourier uncertainty, trigonometric polynomial and quadrature analysis, frame-coherence sparse recovery, and Fourier summation kernels—strictly improving classical baselines with explicit equality characterizations and verifiable finite-order constants.

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