A Wronskian Identity for Antisymmetrized Sine Sums:
unconditional denominator positivity via circular convolution,
sharp monotonicity distinctions, and counterexamples to naive sine-series principles

Lluis Eriksson

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Abstract

For real arrays $(a_m)_{m\ge1}$ and $(b_m)_{m\ge1}$ that are finitely supported, or summable with $\sum_{m\ge1} m(|a_m|+|b_m|)<\infty$, we prove the exact Wronskian identity

$$\sum_{1\le m<n} (a_m b_n - a_n b_m)\big[(m-n)\sin((m+n)t)+(m+n)\sin((n-m)t)\big] \;=\; 2\,\big[F_a'(t)\,F_b(t) - F_a(t)\,F_b'(t)\big],$$

where $F_x(t)=\sum_{m\ge1} x_m \sin(mt)$, with explicit Fubini justification of the rearrangement in the infinite-series case. Applying this identity to the modified-Bessel coefficient arrays (A_m, B_m) from a companion two-dimensional lattice gauge theory surface-positivity problem, we first prove *unconditionally* for every $\beta>0$ that $F_B(t) = -(f^{*4})'(t)/(2I_1(\beta)^4) > 0$ on $(0,\pi)$, where $f(t)=e^{\beta\cos t}$ and f^{*4} is its fourfold normalized circular convolution (Theorem 5), closing part (i) of v1's Conjecture 3 without numerics. We then distinguish precisely between: (a) strict decrease of the real-analytic ratio F_A/F_B on $(0,\pi)$ (equivalent to $W\le0$ and $W\not\equiv0$), (b) strict pointwise negativity $W(t)<0$ on $(0,\pi)$ (strictly stronger, as shown by the finite example $F_b=\sin t$, $F_a=\frac14\sin2t+\frac18\sin4t$ with $F_a/F_b=\cos^3 t$ and $W(\pi/2)=0$), and (c) local negativity near $t=\pi$ (which is strictly weaker under generic side conditions, as shown by the finite counterexample $a=(6,10,24)$, $b=(6,5,2)$). We also retain the hand-checkable counterexample $a=(3,4,3)$, $b=(3,2,1)$ refuting any unrestricted sine-series analogue of the Biernacki–Krzyż principle even when a_m/b_m is strictly increasing on the support and $F_b>0$ on $(0,\pi)$. The finite Wronskian identity (arbitrary K) is machine-checked in Lean 4/Mathlib (`WronskianIdentity.lean`, zero `sorry`, standard axioms only), and every counterexample is an exact algebraic evaluation. The global Bessel sign conjecture $W(t)<0$ on $(0,\pi)$ remains open.

1 The Wronskian identity

Theorem 1 (Wronskian identity)**.** *Let $(a_m)_{m\ge1}$ and $(b_m)_{m\ge1}$ be real sequences satisfying $\sum_{m\ge1} m(|a_m|+|b_m|)<\infty$ (in particular, any finitely supported sequences). Set $F_a(t)=\sum_{m\ge1} a_m\sin(mt)$ and $F_b(t)=\sum_{m\ge1} b_m\sin(mt)$. Then $F_a, F_b\in C^1(\mathbb{R})$ with termwise derivatives $F_a'(t)=\sum_{m\ge1} m a_m\cos(mt)$ and $F_b'(t)=\sum_{m\ge1} m b_m\cos(mt)$, the double series below converges absolutely and uniformly on $\mathbb{R}$, and for all $t\in\mathbb{R}$,*

$$\sum_{1\le m<n} (a_m b_n - a_n b_m)\big[(m-n)\sin((m+n)t)+(m+n)\sin((n-m)t)\big] = 2\,\big[F_a'(t)F_b(t) - F_a(t)F_b'(t)\big]. \tag{1}$$

Proof. Since $|a_m \sin(mt)| \le |a_m| \le m|a_m|$ and $|m a_m \cos(mt)| \le m|a_m|$, the Weierstrass M-test yields uniform convergence of F_a, F_b, F_a', F_b' on $\mathbb{R}$ and validates termwise differentiation. For any $m, n \ge 1$, the product-to-sum identities give

$$\begin{aligned}(m-n)\sin((m+n)t) + (m+n)\sin((n-m)t) \\ = 2\left[m\cos(mt)\sin(nt) - n\sin(mt)\cos(nt)\right] =: 2\,G(m,n).\end{aligned}$$

Note that $|G(m,n)| \le m+n$, $G(n,m) = -G(m,n)$, and $G(n,n) = 0$. Moreover,

$$\sum_{m,n\ge1} |a_m b_n|\,(m+n) = \Big(\sum_{m\ge1} m|a_m|\Big)\Big(\sum_{n\ge1}|b_n|\Big) + \Big(\sum_{m\ge1}|a_m|\Big)\Big(\sum_{n\ge1} n|b_n|\Big) < \infty,$$

so every double sum below is absolutely convergent and may be rearranged freely by Fubini's theorem. Using the joint antisymmetry of $(a_m b_n - a_n b_m)$ and $G(m,n)$ together with $G(n,n) = 0$,

$$\begin{aligned}\sum_{1\le m<n} (a_m b_n - a_n b_m)\,2G(m,n) &= \sum_{m,n\ge1} (a_m b_n - a_n b_m)\,G(m,n) = 2\sum_{m,n\ge1} a_m b_n\,G(m,n) \\ &= 2\Big[\Big(\sum_{m\ge1} m a_m\cos(mt)\Big)\Big(\sum_{n\ge1} b_n\sin(nt)\Big) \\ &\qquad - \Big(\sum_{m\ge1} a_m\sin(mt)\Big)\Big(\sum_{n\ge1} n b_n\cos(nt)\Big)\Big] \\ &= 2\left[F_a'(t)F_b(t) - F_a(t)F_b'(t)\right].\end{aligned}$$

□

Remark 2. Wherever $F_b(t) \ne 0$, the quotient rule gives

$$W(t) := 2\big[F_a'(t)F_b(t) - F_a(t)F_b'(t)\big] = 2\,F_b(t)^2\,\frac{d}{dt}\left(\frac{F_a(t)}{F_b(t)}\right).$$

Thus on any open interval I where $F_b \ne 0$, pointwise strict negativity $W(t) < 0$ on I is equivalent to $(F_a/F_b)'(t) < 0$ on I, which is *strictly stronger* than strict decrease of F_a/F_b on I (see Proposition 6).

2 The Bessel arrays and unconditional denominator positivity

We recall the modified-Bessel coefficient arrays from the companion note [4] (AIRR record `ARR-2026-72G1SSCNNG8YDVC2`; programme context in [5]). For $\beta > 0$, let $I_\mu(\beta) = \sum_{j=0}^{\infty} \frac{(\beta/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)}$ be the modified Bessel function of the first kind, set $r_k = I_{k+1}(\beta)/I_1(\beta)$ for $k \ge 0$, and define for $k \ge 1$,

$$A_k = r_{k-1}^2 \sum_{j\in\{k-2,\,k\},\,j\ge0} (j+1) r_j^2, \qquad B_k = k\,r_{k-1}^4 = k\,\frac{I_k(\beta)^4}{I_1(\beta)^4}, \qquad c_{mn} = A_m B_n - A_n B_m.$$

Lemma 3 (Non-asymptotic super-exponential decay)**.** *For every $\beta > 0$ and every integer $k \ge 0$,*

$$0 < I_k(\beta) \le \frac{(\beta/2)^k}{k!}\,I_0(\beta).$$

Consequently $r_{k-1} \le \frac{I_0(\beta)}{I_1(\beta)}\frac{(\beta/2)^k}{k!}$, both A_m and B_m decay at least as $O(((\beta/2)^m/m!)^4)$ (up to a polynomial factor in m), and $\sum_{m\ge1} m(A_m + B_m) < \infty$ for every $\beta > 0$. Moreover, F_A and F_B extend to entire (hence real-analytic on $\mathbb{R}$) 2π-periodic odd functions.

Proof. Since $\binom{k+j}{j} = \frac{(k+j)!}{k!\,j!} \geq 1$ for all $k, j \geq 0$, we have $(k+j)! \geq k!\,j!$, whence

$$I_k(\beta) = \sum_{j=0}^{\infty} \frac{(\beta/2)^{k+2j}}{j!\,(k+j)!} \leq \frac{(\beta/2)^k}{k!} \sum_{j=0}^{\infty} \frac{(\beta/2)^{2j}}{(j!)^2} = \frac{(\beta/2)^k}{k!}\, I_0(\beta).$$

Positivity follows termwise for $\beta > 0$. Substituting into A_m and B_m gives the stated factorial-fourth-power bound, which implies $\sum_{m\geq 1} m^r(A_m + B_m) < \infty$ for every $r \geq 0$ and absolute convergence of the complex trigonometric series on all of $\mathbb{C}$ (since $|\sin(mz)| \leq e^{m|\operatorname{Im} z|}$). □

We now prove unconditionally that $F_B(t) > 0$ on $(0, \pi)$ for every $\beta > 0$, closing part (i) of v1's Conjecture 3 analytically.

Lemma 4 (Circular convolution preserves even strict decrease)**.** *Let $g, h \in C^1(\mathbb{R})$ be 2π-periodic and even, and suppose $g'(s) < 0$ and $h'(s) < 0$ for all $s \in (0, \pi)$. Define their normalized circular convolution*

$$(g * h)(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s)\, h(t-s)\, ds.$$

Then $g * h \in C^1(\mathbb{R})$ is 2π-periodic, even, and satisfies $(g * h)'(t) < 0$ for all $t \in (0, \pi)$.

Proof. Periodicity and evenness follow by the substitution $s \mapsto -s$. By periodicity we may write $(g*h)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t-u)\, h(u)\, du$, differentiate under the integral sign, and change variables back to $s = t - u$:

$$(g * h)'(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g'(s)\, h(t-s)\, ds = \frac{1}{2\pi} \int_{0}^{\pi} g'(s)\, \big[h(t-s) - h(t+s)\big]\, ds,$$

using $g'(-s) = -g'(s)$ because g is even. Because h is 2π-periodic, even, and strictly decreasing on $[0, \pi]$, there is a strictly increasing function $\tilde{h} : [-1, 1] \to \mathbb{R}$ such that $h(\theta) = \tilde{h}(\cos\theta)$ for all $\theta \in \mathbb{R}$. For any $t, s \in (0, \pi)$,

$$\cos(t-s) - \cos(t+s) = 2 \sin t\, \sin s > 0,$$

so $h(t-s) - h(t+s) = \tilde{h}(\cos(t-s)) - \tilde{h}(\cos(t+s)) > 0$. Together with $g'(s) < 0$ on $(0, \pi)$, the integrand is strictly negative on all of $(0, \pi)$, giving $(g * h)'(t) < 0$. □

Theorem 5 (Unconditional denominator positivity)**.** *For every $\beta > 0$ and every $t \in (0, \pi)$,*

$$F_B(t) = \sum_{k=1}^{\infty} k\, \frac{I_k(\beta)^4}{I_1(\beta)^4}\, \sin(kt) = -\frac{(f^{*4})'(t)}{2\, I_1(\beta)^4} \;>\; 0,$$

where $f(t) := e^{\beta \cos t}$ and $f^{*4} := (f * f) * (f * f)$.

Proof. By the Jacobi–Anger expansion [3, 10.35.2], $f(t) = e^{\beta\cos t} = \sum_{k\in\mathbb{Z}} I_k(\beta) e^{ikt}$ with $I_{-k}(\beta) = I_k(\beta)$, converging uniformly on $\mathbb{R}$. Under normalized circular convolution, Fourier coefficients multiply, so the k-th Fourier coefficient of f^{*4} is $I_k(\beta)^4$:

$$f^{*4}(t) = I_0(\beta)^4 + 2 \sum_{k=1}^{\infty} I_k(\beta)^4 \cos(kt).$$

By Lemma 3, termwise differentiation is justified uniformly on $\mathbb{R}$ and yields

$$-(f^{*4})'(t) = 2 \sum_{k=1}^{\infty} k\, I_k(\beta)^4 \sin(kt) = 2\, I_1(\beta)^4\, F_B(t).$$

For $\beta > 0$, $f \in C^\infty(\mathbb{R})$ is 2π-periodic, even, and satisfies $f'(t) = -\beta \sin t\, e^{\beta \cos t} < 0$ on $(0, \pi)$. Applying Lemma 4 to (f, f) yields f^{*2} even, 2π-periodic, C^∞, and strictly decreasing on $(0, \pi)$; applying Lemma 4 once more to (f^{*2}, f^{*2}) gives $(f^{*4})'(t) < 0$ on $(0, \pi)$. Dividing by $2I_1(\beta)^4 > 0$ gives $F_B(t) > 0$ on $(0, \pi)$. □

3 Sharp monotonicity distinctions and the global conjecture

Let $W(t) := 2[F_A'(t)F_B(t) - F_A(t)F_B'(t)]$ be the Wronskian of Theorem 1 at $(a, b) = (A, B)$. In the companion note [4], the surface-positivity problem asks for $W(t) < 0$ in a left neighborhood of $t = \pi$, and proves that the coefficient ratio $\varphi_m = A_m/B_m$ is strictly increasing ($c_{mn} < 0$ for all $m < n$). By Theorem 5, F_A/F_B is real-analytic and well-defined on all of $(0, \pi)$, with $(F_A/F_B)' = W/(2F_B^2)$.

We clarify two logical distinctions that were blurred in v1. First, strict decrease of F_a/F_b is *not* equivalent to pointwise strict negativity $W < 0$.

Proposition 6 (Strict decrease allows isolated zeros of W)**.** *For $F_b(t) = \sin t$ (positive on $(0, \pi)$) and $F_a(t) = \frac{1}{4}\sin(2t) + \frac{1}{8}\sin(4t)$, we have*

$$\frac{F_a(t)}{F_b(t)} = \cos^3 t,$$

which is strictly decreasing on $(0, \pi)$, yet $W(t) = 2\sin^2 t \cdot (-3\cos^2 t \sin t) = -6\sin^3 t \cos^2 t$ vanishes at $t = \pi/2$. In general, for real-analytic F_A, F_B with $F_B > 0$ on $(0, \pi)$, strict decrease of F_A/F_B on $(0, \pi)$ is equivalent to $W(t) \le 0$ on $(0, \pi)$ with $W \not\equiv 0$, whereas $W(t) < 0$ on all of $(0, \pi)$ is the strictly stronger assertion that $(F_A/F_B)'(t) < 0$ everywhere on $(0, \pi)$.

Proof. Using $\sin(2t) = 2\sin t \cos t$ and $\sin(4t) = 4\sin t \cos t(2\cos^2 t - 1)$,

$$\frac{F_a(t)}{\sin t} = \tfrac{1}{2}\cos t + \tfrac{1}{2}\cos t(2\cos^2 t - 1) = \cos^3 t.$$

Hence $(F_a/F_b)'(t) = -3\cos^2 t \sin t \le 0$ with unique zero at $t = \pi/2$ in $(0, \pi)$, so F_a/F_b is strictly decreasing while $W(\pi/2) = 0$. For real-analytic F_A/F_B on the connected interval $(0, \pi)$, if $W \le 0$ and $W \not\equiv 0$ then the zeros of $(F_A/F_B)'$ are isolated, so F_A/F_B is strictly decreasing; conversely, strict decrease forces $(F_A/F_B)' \le 0$ and $(F_A/F_B)' \not\equiv 0$. □

Second, global negativity on $(0, \pi)$ strictly implies near-π negativity, and no reverse local-to-global implication follows from the generic side conditions alone (see Proposition 9 below). We therefore state the global conjecture as a strengthening of the original near-π problem, not as an equivalence:

Conjecture 7 (Global Wronskian negativity for the Bessel arrays). For every $\beta > 0$:

(a) (*Weak/strict-decrease form*) $W(t) \le 0$ for all $t \in (0, \pi)$ and $W \not\equiv 0$ (equivalently, by Theorem 5 and real-analyticity, F_A/F_B is strictly decreasing on $(0, \pi)$).

(b) (*Strong/pointwise form*) $W(t) < 0$ for all $t \in (0, \pi)$ (equivalently, $(F_A/F_B)'(t) < 0$ everywhere on $(0, \pi)$).

4 Counterexamples to naive sine-series monotonicity and local-to-global lifting

For power series with positive coefficients converging on a common interval $(0, R)$ with nonvanishing denominator, the Biernacki–Krzyż principle [1] asserts that monotonicity of a_m/b_m transfers to monotonicity of $\sum a_m x^m / \sum b_m x^m$. No unrestricted analogue holds for sine series, even on finite supports where $b_m > 0$ decreases, a_m/b_m strictly increases on the support $\{1, \dots, K\}$, and $F_b > 0$ on $(0, \pi)$.

Proposition 8 (Counterexample to unrestricted sine-series Biernacki–Krzyż)**.** *Let $a = (3, 4, 3)$ and $b = (3, 2, 1)$ on indices $m \in \{1, 2, 3\}$ (with $a_m = b_m = 0$ for $m \geq 4$). On the support $\{1, 2, 3\}$, b_m is strictly decreasing and the coefficient ratio $a_m/b_m = (1, 2, 3)$ is strictly increasing; moreover*

$$F_b(t) = 3\sin t + 2\sin 2t + \sin 3t = 2\sin t\,(2\cos^2 t + 2\cos t + 1) > 0 \qquad \forall t \in (0, \pi),$$

since $2y^2 + 2y + 1 = 2(y + \frac{1}{2})^2 + \frac{1}{2} > 0$. Yet at $t = 2\pi/3$,

$$F_a = -\tfrac{\sqrt{3}}{2}, \quad F_b = \tfrac{\sqrt{3}}{2}, \quad F_a' = \tfrac{7}{2}, \quad F_b' = -\tfrac{1}{2}, \qquad F_a' F_b - F_a F_b' = \tfrac{3\sqrt{3}}{2} > 0,$$

so $W(2\pi/3) = 3\sqrt{3} > 0$ and F_a/F_b is strictly increasing *at $t = 2\pi/3$.*

Proof. At $t = 2\pi/3$: $\sin t = \frac{\sqrt{3}}{2}$, $\sin 2t = -\frac{\sqrt{3}}{2}$, $\sin 3t = 0$; $\cos t = -\frac{1}{2}$, $\cos 2t = -\frac{1}{2}$, $\cos 3t = 1$. Hence $F_a = (3-4)\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$, $F_b = (3-2)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$, $F_a' = -\frac{3}{2} - 4 + 9 = \frac{7}{2}$, and $F_b' = -\frac{3}{2} - 2 + 3 = -\frac{1}{2}$. (Mere coefficient decrease does not in general force sine positivity, e.g. $\sin t + \sin 2t < 0$ near π; see [2] for Vietoris-type criteria, which is why we verify $F_b > 0$ directly by factorization.) □

Proposition 9 (Near-π negativity does not force global negativity)**.** *Let $a = (6, 10, 24)$ and $b = (6, 5, 2)$ on indices $m \in \{1, 2, 3\}$ (zero for $m \geq 4$). Then on $\{1, 2, 3\}$, b_m is strictly decreasing, $a_m/b_m = (1, 2, 12)$ is strictly increasing, and*

$$F_b(t) = 6\sin t + 5\sin 2t + 2\sin 3t = \sin t\,(8\cos^2 t + 10\cos t + 4) > 0 \qquad \forall t \in (0, \pi),$$

since $10^2 - 4 \cdot 8 \cdot 4 = -28 < 0$. Near $t = \pi$, setting $t = \pi - \varepsilon$ yields $W(\pi - \varepsilon) = -8\varepsilon^3 + O(\varepsilon^5) < 0$ for all sufficiently small $\varepsilon > 0$. Nevertheless, at $t = 2\pi/3$, $W(2\pi/3) = 51\sqrt{3} > 0$. Thus any proof of Conjecture 7 from near-π negativity must invoke Bessel-specific structure beyond these generic side conditions.

Proof. Using $\sin 2t = 2\sin t \cos t$ and $\sin 3t = \sin t(4\cos^2 t - 1)$ gives $F_b(t)/\sin t = 6 + 10\cos t + 2(4\cos^2 t - 1) = 8\cos^2 t + 10\cos t + 4 > 0$. Similarly $F_a(t)/\sin t = 96\cos^2 t + 20\cos t - 18$. Writing $y = \cos t \in (-1, 1)$, the quotient $q(y) = F_a/F_b = (96y^2 + 20y - 18)/(8y^2 + 10y + 4)$ has derivative

$$q'(y) = \frac{2(400y^2 + 528y + 130)}{(8y^2 + 10y + 4)^2} = \frac{4(200y^2 + 264y + 65)}{(8y^2 + 10y + 4)^2}.$$

Since $dy/dt = -\sin t < 0$, the sign of $W(t) = 2F_b(t)^2 q'(\cos t)(-\sin t)$ is opposite to the sign of $p(y) := 200y^2 + 264y + 65$. At $y = -1$ ($t = \pi$), $p(-1) = 1 > 0$, confirming $W(\pi - \varepsilon) = -8\varepsilon^3 + O(\varepsilon^5) < 0$ for small $\varepsilon > 0$. At $t = 2\pi/3$ ($y = -1/2$), $p(-1/2) = 50 - 132 + 65 = -17 < 0$, and $F_b(2\pi/3) = (\sqrt{3}/2)(2 - 5 + 4) = \sqrt{3}/2$, giving $W(2\pi/3) = 2 \cdot \frac{3}{4} \cdot \frac{4(-17)}{1^2} \cdot (-\frac{\sqrt{3}}{2}) = 51\sqrt{3} > 0$. □

Remark 10 (Structural moral and numerical precision). Propositions 8 and 9 show that coefficient-ratio monotonicity and denominator positivity alone cannot certify the sign of W, nor lift local near-π negativity to $(0, \pi)$. Quantitatively, high-precision scans of the Bessel arrays (archived at `docs/BF2-ATTACK-NOTES.md`) display parity-mirror cancellation between even- and odd-index sums of relative magnitude $\approx e^{-c\beta}$ ($c \approx 2.1$, corresponding to roughly $c\beta/\ln 10 \approx 0.91\beta$ lost decimal digits; e.g. $W \approx -2.9 \cdot 10^{-8}$ at $\beta = 8$ and $-5.0 \cdot 10^{-29}$ at $\beta = 32$ on sampled grids). Maintaining $\geq \beta + 20$ (conservatively $2.2\beta + 20$) working digits prevents catastrophic cancellation on tested grids, though grid samples never constitute a continuum proof. Theorem 5 already realizes F_B as an exact circle-convolution derivative; a corresponding positive kernel representation of $-W$ (or total positivity of $(A; B)$ against the sine kernel) remains the natural analytic route to Conjecture 7.

5 Verification

Theorem 1 (finite form, arbitrary K) is machine-checked in Lean 4 / Mathlib (`WronskianIdentity.lean`, accompanying `papers/wronskian-reduction` in https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME): declarations `bracket_eq_two_G`, `double_sum_eq_wronskian`, `pair_sum_eq_double_sum`, and `wronskian_identity`, each with zero `sorry` and axiom readout `[propext, Classical.choice, Quot.sound]`. Symbolically, Theorem 1 was checked in SymPy for generic arrays at $K \leq 5$ (exact algebraic zero after trigonometric expansion), numerically evaluated at $K = 7$ at rational inputs with 40-digit precision (residual $< 10^{-35}$, a high-precision numerical check rather than exact rational arithmetic), and compared against the direct Bessel double sum at $K = 90$, $\beta = 8$, $t \in \{0.5, 1.5, 2.6, \pi - 0.1\}$. Propositions 6, 8, and 9 are exact algebraic evaluations (also cross-checked in SymPy), and the convolution identity in Theorem 5 was independently cross-checked by 30-digit numerical quadrature at $\beta \in \{0.5, 2.0\}$.

Scope and Disclosure

This v2 revision addresses all material objections raised by v1's independent assessments (`ARR-2026-5FKM2GGK348CPTE9` v1). Conjecture 7 remains open, and no claim is made regarding four-dimensional gauge theory or a Yang–Mills mass gap. Large language models assisted manuscript preparation under author supervision; author Lluis Eriksson administers AIRR (https://airr.science/), so no independent editorial admission is claimed.

References

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