A Wronskian Identity for Antisymmetrized Sine Sums:
unconditional denominator positivity via circular convolution,
sharp monotonicity distinctions, and counterexamples to naive sine-series principles

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Abstract

For real arrays $(a_m)_{m\ge1}$ and $(b_m)_{m\ge1}$ that are finitely supported, or summable with $\sum_{m\ge1} m(|a_m|+|b_m|)<\infty$, we prove the exact Wronskian identity

$$\sum_{1\le m<n} (a_m b_n - a_n b_m)\big[(m-n)\sin((m+n)t)+(m+n)\sin((n-m)t)\big] \;=\; 2\,\big[F_a'(t)\,F_b(t) - F_a(t)\,F_b'(t)\big],$$

where $F_x(t)=\sum_{m\ge1} x_m \sin(mt)$, with explicit Fubini justification of the rearrangement in the infinite-series case. Applying this identity to the modified-Bessel coefficient arrays (A_m, B_m) from a companion two-dimensional lattice gauge theory surface-positivity problem, we prove *unconditionally* for every $\beta>0$ that $F_B(t) = -(f^{*4})'(t)/(2I_1(\beta)^4) > 0$ on $(0,\pi)$, where $f(t)=e^{\beta\cos t}$ and f^{*4} is its fourfold normalized circular convolution (Theorem 5). We separate precisely between: (a) strict decrease of the real-analytic ratio F_A/F_B on $(0,\pi)$ (equivalent to $W\le 0$ and $W\not\equiv 0$), (b) strict pointwise negativity $W(t)<0$ on $(0,\pi)$ (strictly stronger, as shown by $F_b=\sin t$, $F_a=\frac14\sin 2t+\frac18\sin 4t$ with $F_a/F_b=\cos^3 t$ and $W(\pi/2)=0$), and (c) local negativity near $t=\pi$ (implied by (a) in the real-analytic Bessel family, but strictly weaker under generic side conditions, as shown by $a=(6,10,24)$, $b=(6,5,2)$). We also prove nonmonotonicity of F_a/F_b for $a=(3,4,3)$, $b=(3,2,1)$ via two explicit opposite signs ($W(\pi/2)=-32<0$ and $W(2\pi/3)=3\sqrt3>0$), refuting any unrestricted sine-series analogue of the Biernacki–Krzyż principle even when $b_m>0$ decreases, a_m/b_m strictly increases on the support, and $F_b>0$ on $(0,\pi)$. The finite Wronskian identity (arbitrary K) is machine-checked in Lean 4/Mathlib (`WronskianIdentity.lean`, release `yang-mills-archive-v1.0`, commit `6b54eca695e96b3038ceda18276c4e28fb45983f`), and every counterexample is an exact algebraic evaluation. The global Bessel sign conjecture remains open.

1 The Wronskian identity

Theorem 1 (Wronskian identity)**.** *Let $(a_m)_{m\ge1}$ and $(b_m)_{m\ge1}$ be real sequences satisfying $\sum_{m\ge1} m(|a_m|+|b_m|)<\infty$ (in particular, any finitely supported sequences). Set $F_a(t)=\sum_{m\ge1} a_m\sin(mt)$ and $F_b(t)=\sum_{m\ge1} b_m\sin(mt)$. Then $F_a, F_b\in C^1(\mathbb{R})$ with termwise derivatives $F_a'(t)=\sum_{m\ge1} m a_m\cos(mt)$ and $F_b'(t)=\sum_{m\ge1} m b_m\cos(mt)$, the double series below converges absolutely and uniformly on $\mathbb{R}$, and for all $t\in\mathbb{R}$,*

$$\sum_{1\le m<n} (a_m b_n - a_n b_m)\big[(m-n)\sin((m+n)t)+(m+n)\sin((n-m)t)\big] = 2\,\big[F_a'(t)F_b(t) - F_a(t)F_b'(t)\big]. \tag{1}$$

Proof. Since $|a_m \sin(mt)| \le |a_m| \le m|a_m|$ and $|m a_m \cos(mt)| \le m|a_m|$, the Weierstrass M-test yields uniform convergence of F_a, F_b, F_a', F_b' on $\mathbb{R}$ and validates termwise differentiation. For any $m, n \ge 1$, the product-to-sum identities give

$$\begin{aligned}(m-n)\sin((m+n)t) + (m+n)\sin((n-m)t) \\ = 2\left[m\cos(mt)\sin(nt) - n\sin(mt)\cos(nt)\right] =: 2\,G(m,n).\end{aligned}$$

Note that $|G(m,n)| \le m+n$, $G(n,m) = -G(m,n)$, and $G(n,n) = 0$. Moreover,

$$\sum_{m,n\ge1} |a_m b_n|\,(m+n) = \Big(\sum_{m\ge1} m|a_m|\Big)\Big(\sum_{n\ge1}|b_n|\Big) + \Big(\sum_{m\ge1}|a_m|\Big)\Big(\sum_{n\ge1} n|b_n|\Big) < \infty,$$

so every double sum below is absolutely convergent and may be rearranged freely by Fubini's theorem. Using the joint antisymmetry of $(a_m b_n - a_n b_m)$ and $G(m,n)$ together with $G(n,n) = 0$,

$$\begin{aligned}\sum_{1\le m<n} (a_m b_n - a_n b_m)\,2G(m,n) &= \sum_{m,n\ge1} (a_m b_n - a_n b_m)\,G(m,n) = 2\sum_{m,n\ge1} a_m b_n\,G(m,n) \\ &= 2\Big[\Big(\sum_{m\ge1} m a_m\cos(mt)\Big)\Big(\sum_{n\ge1} b_n\sin(nt)\Big) \\ &\qquad - \Big(\sum_{m\ge1} a_m\sin(mt)\Big)\Big(\sum_{n\ge1} n b_n\cos(nt)\Big)\Big] \\ &= 2\left[F_a'(t)F_b(t) - F_a(t)F_b'(t)\right].\end{aligned}$$

□

Remark 2. Throughout this paper we define the scaled Wronskian with the factor of 2 matching the right-hand side of (1):

$$W(t) := 2\big[F_a'(t)F_b(t) - F_a(t)F_b'(t)\big].$$

Wherever $F_b(t) \ne 0$, the quotient rule gives $W(t) = 2\,F_b(t)^2\,\frac{d}{dt}\big(F_a(t)/F_b(t)\big)$. Thus on any open interval I where $F_b \ne 0$, pointwise strict negativity $W(t) < 0$ on I is equivalent to $(F_a/F_b)'(t) < 0$ on I, which is *strictly stronger* than strict decrease of F_a/F_b on I (see Proposition 6).

2 The Bessel arrays and unconditional denominator positivity

We recall the modified-Bessel coefficient arrays from the companion note [4] (AIRR record `ARR-2026-72G1SSCNNG8YDVC2`; programme context in [5]). For $\beta > 0$, let $I_\mu(\beta) = \sum_{j=0}^{\infty} \frac{(\beta/2)^{\mu+2j}}{j!\,\Gamma(\mu+j+1)}$ be the modified Bessel function of the first kind, set $r_k = I_{k+1}(\beta)/I_1(\beta)$ for $k \ge 0$, and define for $k \ge 1$,

$$A_k = r_{k-1}^2 \sum_{j\in\{k-2,\,k\},\,j\ge0} (j+1) r_j^2, \qquad B_k = k\,r_{k-1}^4 = k\,\frac{I_k(\beta)^4}{I_1(\beta)^4}, \qquad c_{mn} = A_m B_n - A_n B_m.$$

Lemma 3 (Non-asymptotic super-exponential decay)**.** *For every $\beta > 0$ and every integer $k \ge 0$,*

$$0 < I_k(\beta) \le \frac{(\beta/2)^k}{k!}\,I_0(\beta).$$

Consequently, for every integer $k \ge 1$,

$$0 < r_{k-1} = \frac{I_k(\beta)}{I_1(\beta)} \le \frac{I_0(\beta)}{I_1(\beta)}\,\frac{(\beta/2)^k}{k!}.$$

In particular, for each fixed $\beta > 0$,

$$A_m = O_\beta\left(m^3\left[\frac{(\beta/2)^m}{m!}\right]^4\right), \qquad B_m = O_\beta\left(m\left[\frac{(\beta/2)^m}{m!}\right]^4\right) \qquad (m \to \infty),$$

so $\sum_{m\geq 1} m^r(A_m + B_m) < \infty$ for every $r \geq 0$ (in particular $r = 1$). Moreover, F_A and F_B extend to entire (hence real-analytic on $\mathbb{R}$) 2π-periodic odd functions.

Proof. Since $\binom{k+j}{j} = \frac{(k+j)!}{k!\,j!} \geq 1$ for all $k, j \geq 0$, we have $(k+j)! \geq k!\,j!$, whence

$$I_k(\beta) = \sum_{j=0}^{\infty} \frac{(\beta/2)^{k+2j}}{j!\,(k+j)!} \leq \frac{(\beta/2)^k}{k!} \sum_{j=0}^{\infty} \frac{(\beta/2)^{2j}}{(j!)^2} = \frac{(\beta/2)^k}{k!}\, I_0(\beta).$$

Positivity follows termwise for $\beta > 0$. For $m \geq 1$, $B_m = m r_{m-1}^4$ immediately gives $B_m = O_\beta(m[(\beta/2)^m/m!]^4)$. For A_m with $m \geq 2$, using $(\beta/2)^{m-1}/(m-1)! = (2m/\beta)\,(\beta/2)^m/m!$ gives $r_{m-2}^2 = O_\beta(m^2[(\beta/2)^m/m!]^2)$ while $r_m^2 = O_\beta(m^{-2}[(\beta/2)^m/m!]^2)$, so $(m-1)r_{m-2}^2 + (m+1)r_m^2 = O_\beta(m^3[(\beta/2)^m/m!]^2)$ and multiplying by r_{m-1}^2 yields $A_m = O_\beta(m^3[(\beta/2)^m/m!]^4)$. Since $|\sin(mz)| \leq e^{m|\operatorname{Im} z|}$ on $\mathbb{C}$ and $m^3[(\beta/2)^m/m!]^4$ decays faster than e^{-Rm} for every $R > 0$, both series converge locally uniformly on $\mathbb{C}$ to entire 2π-periodic odd functions. □

We now prove unconditionally that $F_B(t) > 0$ on $(0, \pi)$ for every $\beta > 0$.

Lemma 4 (Circular convolution preserves even strict decrease)**.** *Let $g, h \in C^1(\mathbb{R})$ be 2π-periodic and even, and suppose $g'(s) < 0$ and $h'(s) < 0$ for all $s \in (0, \pi)$. Define their normalized circular convolution*

$$(g * h)(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\, h(t-s)\, ds.$$

Then $g * h \in C^1(\mathbb{R})$ is 2π-periodic, even, and satisfies $(g * h)'(t) < 0$ for all $t \in (0, \pi)$.

Proof. Periodicity and evenness follow by the substitution $s \mapsto -s$. By periodicity we may write $(g*h)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t-u)\, h(u)\, du$, differentiate under the integral sign, and change variables back to $s = t - u$:

$$(g * h)'(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g'(s)\, h(t-s)\, ds = \frac{1}{2\pi}\int_{0}^{\pi} g'(s)\,\big[h(t-s) - h(t+s)\big]\, ds,$$

using $g'(-s) = -g'(s)$ because g is even. Because h is 2π-periodic, even, and strictly decreasing on $[0, \pi]$, there is a strictly increasing function $\tilde{h} : [-1, 1] \to \mathbb{R}$ such that $h(\theta) = \tilde{h}(\cos\theta)$ for all $\theta \in \mathbb{R}$. For any $t, s \in (0, \pi)$,

$$\cos(t-s) - \cos(t+s) = 2\sin t\, \sin s > 0,$$

so $h(t-s) - h(t+s) = \tilde{h}(\cos(t-s)) - \tilde{h}(\cos(t+s)) > 0$. Together with $g'(s) < 0$ on $(0, \pi)$, the integrand is strictly negative on all of $(0, \pi)$, giving $(g * h)'(t) < 0$. □

Theorem 5 (Unconditional denominator positivity)**.** *For every $\beta > 0$ and every $t \in (0, \pi)$,*

$$F_B(t) = \sum_{k=1}^{\infty} k\,\frac{I_k(\beta)^4}{I_1(\beta)^4}\,\sin(kt) = -\frac{(f^{*4})'(t)}{2\,I_1(\beta)^4} \;>\; 0,$$

where $f(t) := e^{\beta\cos t}$ and $f^{*4} := (f * f) * (f * f)$.

Proof. By the Jacobi–Anger expansion [3, 10.35.2], $f(t) = e^{\beta\cos t} = \sum_{k\in\mathbb{Z}} I_k(\beta)e^{ikt}$ with $I_{-k}(\beta) = I_k(\beta)$, converging uniformly on $\mathbb{R}$. Under normalized circular convolution, Fourier coefficients multiply, so the k-th Fourier coefficient of f^{*4} is $I_k(\beta)^4$:

$$f^{*4}(t) = I_0(\beta)^4 + 2\sum_{k=1}^{\infty} I_k(\beta)^4\cos(kt).$$

By Lemma 3, termwise differentiation is justified uniformly on $\mathbb{R}$ and yields

$$-(f^{*4})'(t) = 2\sum_{k=1}^{\infty} k\, I_k(\beta)^4\sin(kt) = 2\, I_1(\beta)^4\, F_B(t).$$

For $\beta > 0$, $f \in C^\infty(\mathbb{R})$ is 2π-periodic, even, and satisfies $f'(t) = -\beta\sin t\, e^{\beta\cos t} < 0$ on $(0,\pi)$. Applying Lemma 4 to (f, f) yields f^{*2} even, 2π-periodic, C^∞, and strictly decreasing on $(0,\pi)$; applying Lemma 4 once more to (f^{*2}, f^{*2}) gives $(f^{*4})'(t) < 0$ on $(0,\pi)$. Dividing by $2I_1(\beta)^4 > 0$ gives $F_B(t) > 0$ on $(0,\pi)$. □

3 Sharp monotonicity distinctions and the global conjecture

Let $W(t) := 2[F_A'(t)F_B(t) - F_A(t)F_B'(t)]$ be the Wronskian of Theorem 1 at $(a,b) = (A,B)$. In the companion note [4], the surface-positivity problem asks for $W(t) < 0$ in a punctured left neighborhood of $t = \pi$, and proves that the coefficient ratio $\varphi_m = A_m/B_m$ is strictly increasing ($c_{mn} < 0$ for all $m < n$). By Theorem 5, F_A/F_B is real-analytic and well-defined on all of $(0,\pi)$, with $(F_A/F_B)' = W/(2F_B^2)$.

We clarify the exact implication structure among three candidate sign statements:

(b) *Pointwise global negativity:* $W(t) < 0$ for all $t \in (0,\pi)$.

(a) *Weak global negativity / strict decrease:* $W(t) \le 0$ for all $t \in (0,\pi)$ and $W \not\equiv 0$ (equivalently, for real-analytic F_A, F_B with $F_B > 0$, F_A/F_B is strictly decreasing on $(0,\pi)$).

(c) *Near-π local negativity:* there exists $\delta > 0$ such that $W(t) < 0$ for all $t \in (\pi-\delta,\pi)$.

Clearly (b) $\implies$ (a). Moreover, in the Bessel setting (where W is entire by Lemma 3, hence analytic across $t = \pi$), if (a) holds then W cannot vanish identically and its real zeros near π are isolated, so (a) $\implies$ (c) as well. Thus within the Bessel family we have the forward chain (b) $\implies$ (a) $\implies$ (c). Under generic finite-support hypotheses ($b_m > 0$ decreasing, a_m/b_m strictly increasing, $F_b > 0$ on $(0,\pi)$), neither reverse implication holds: Proposition 6 shows (a) $\not\Rightarrow$ (b), and Proposition 9 shows (c) $\not\Rightarrow$ (a). (These finite examples rule out generic reverse implications, not Bessel-specific ones.)

Proposition 6 (Strict decrease allows isolated zeros of W)**.** *For $F_b(t) = \sin t$ (positive on $(0,\pi)$) and $F_a(t) = \frac{1}{4}\sin(2t) + \frac{1}{8}\sin(4t)$, we have*

$$\frac{F_a(t)}{F_b(t)} = \cos^3 t,$$

which is strictly decreasing on $(0,\pi)$, yet $W(t) = 2\sin^2 t\cdot(-3\cos^2 t\sin t) = -6\sin^3 t\cos^2 t$ vanishes at $t = \pi/2$. In general, for real-analytic F_A, F_B with $F_B > 0$ on $(0,\pi)$, strict decrease of F_A/F_B on $(0,\pi)$ is equivalent to $W(t) \le 0$ on $(0,\pi)$ with $W \not\equiv 0$, whereas $W(t) < 0$ on all of $(0,\pi)$ is the strictly stronger assertion that $(F_A/F_B)'(t) < 0$ everywhere on $(0,\pi)$.

Proof. Using $\sin(2t) = 2\sin t\cos t$ and $\sin(4t) = 4\sin t\cos t(2\cos^2 t - 1)$,

$$\frac{F_a(t)}{\sin t} = \tfrac{1}{2}\cos t + \tfrac{1}{2}\cos t(2\cos^2 t - 1) = \cos^3 t.$$

Hence $(F_a/F_b)'(t) = -3\cos^2 t\sin t \le 0$ with unique zero at $t = \pi/2$ in $(0,\pi)$, so F_a/F_b is strictly decreasing while $W(\pi/2) = 0$. For real-analytic F_A/F_B on the connected interval $(0,\pi)$, if $W \le 0$ and $W \not\equiv 0$ then the zeros of $(F_A/F_B)'$ are isolated, so F_A/F_B is strictly decreasing; conversely, strict decrease forces $(F_A/F_B)' \le 0$ and $(F_A/F_B)' \not\equiv 0$. □

Conjecture 7 (Global Wronskian negativity for the Bessel arrays). For every $\beta > 0$:

(a) (*Weak/strict-decrease form*) $W(t) \le 0$ for all $t \in (0,\pi)$ and $W \not\equiv 0$ (equivalently, by Theorem 5 and real-analyticity, F_A/F_B is strictly decreasing on $(0,\pi)$).

(b) (*Strong/pointwise form*) $W(t) < 0$ for all $t \in (0,\pi)$ (equivalently, $(F_A/F_B)'(t) < 0$ everywhere on $(0,\pi)$).

4 Counterexamples to naive sine-series monotonicity and local-to-global lifting

For power series with positive coefficients converging on a common interval $(0,R)$ with nonvanishing denominator, the Biernacki–Krzyż principle [1] asserts that monotonicity of a_m/b_m transfers to monotonicity of $\sum a_m x^m / \sum b_m x^m$. No unrestricted analogue holds for sine series in either direction of monotonicity, even on finite supports where $b_m > 0$ decreases, a_m/b_m strictly increases on $\{1,\dots,K\}$, and $F_b > 0$ on $(0,\pi)$.

Proposition 8 (Counterexample to unrestricted sine-series Biernacki–Krzyż)**.** *Let $a = (3,4,3)$ and $b = (3,2,1)$ on indices $m \in \{1,2,3\}$ (with $a_m = b_m = 0$ for $m \ge 4$). On the support $\{1,2,3\}$, b_m is strictly decreasing and the coefficient ratio $a_m/b_m = (1,2,3)$ is strictly increasing; moreover*

$$F_b(t) = 3\sin t + 2\sin 2t + \sin 3t = 2\sin t\,(2\cos^2 t + 2\cos t + 1) > 0 \qquad \forall t \in (0,\pi),$$

since $2y^2 + 2y + 1 = 2(y + \frac{1}{2})^2 + \frac{1}{2} > 0$. Nevertheless,

$$W(\pi/2) = -32 < 0 \qquad \text{and} \qquad W(2\pi/3) = 3\sqrt{3} > 0,$$

so F_a/F_b is strictly decreasing at $t = \pi/2$ and strictly increasing at $t = 2\pi/3$, hence non-monotone on $(0,\pi)$.

Proof. At $t = \pi/2$: $\sin t = 1$, $\sin 2t = 0$, $\sin 3t = -1$, $\cos t = 0$, $\cos 2t = -1$, $\cos 3t = 0$, so $F_a(\pi/2) = 3 - 3 = 0$, $F_b(\pi/2) = 3 - 1 = 2$, $F_a'(\pi/2) = -8$, and $F_b'(\pi/2) = -4$, giving $W(\pi/2) = 2[(-8)\cdot 2 - 0\cdot(-4)] = -32 < 0$. At $t = 2\pi/3$: $\sin t = \frac{\sqrt{3}}{2}$, $\sin 2t = -\frac{\sqrt{3}}{2}$, $\sin 3t = 0$, $\cos t = -\frac{1}{2}$, $\cos 2t = -\frac{1}{2}$, $\cos 3t = 1$, so $F_a = (3-4)\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$, $F_b = (3-2)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$, $F_a' = -\frac{3}{2} - 4 + 9 = \frac{7}{2}$, and $F_b' = -\frac{3}{2} - 2 + 3 = -\frac{1}{2}$, giving $F_a'F_b - F_aF_b' = \frac{3\sqrt{3}}{2}$ and $W(2\pi/3) = 3\sqrt{3} > 0$. (Mere coefficient decrease does not in general force sine positivity, e.g. $\sin t + \sin 2t < 0$ near π; see [2] for Vietoris-type criteria, which is why we verify $F_b > 0$ directly by factorization.) □

Proposition 9 (Near-π negativity does not force global negativity)**.** *Let* $a = (6, 10, 24)$ *and* $b = (6, 5, 2)$ *on indices* $m \in \{1, 2, 3\}$ *(zero for* $m \geq 4$*). Then on* $\{1, 2, 3\}$*,* b_m *is strictly decreasing,* $a_m/b_m = (1, 2, 12)$ *is strictly increasing, and*

$$F_b(t) = 6\sin t + 5\sin 2t + 2\sin 3t = \sin t\,(8\cos^2 t + 10\cos t + 4) > 0 \qquad \forall t \in (0, \pi),$$

since $10^2 - 4 \cdot 8 \cdot 4 = -28 < 0$*. Near* $t = \pi$*, setting* $t = \pi - \varepsilon$ *yields* $W(\pi - \varepsilon) = -8\varepsilon^3 + O(\varepsilon^5) < 0$ *for all sufficiently small* $\varepsilon > 0$*. Nevertheless, at* $t = 2\pi/3$*,* $W(2\pi/3) = 51\sqrt{3} > 0$*. Thus any proof of Conjecture 7 from near-*π *negativity must invoke Bessel-specific structure beyond these generic side conditions.*

Proof. Using $\sin 2t = 2\sin t\cos t$ and $\sin 3t = \sin t(4\cos^2 t - 1)$ gives $F_b(t)/\sin t = 6 + 10\cos t + 2(4\cos^2 t - 1) = 8\cos^2 t + 10\cos t + 4 > 0$. Similarly $F_a(t)/\sin t = 96\cos^2 t + 20\cos t - 18$. Writing $y = \cos t \in (-1, 1)$, the quotient $q(y) = F_a/F_b = (96y^2 + 20y - 18)/(8y^2 + 10y + 4)$ has derivative

$$q'(y) = \frac{2(400y^2 + 528y + 130)}{(8y^2 + 10y + 4)^2} = \frac{4(200y^2 + 264y + 65)}{(8y^2 + 10y + 4)^2}.$$

Since $dy/dt = -\sin t < 0$, the sign of $W(t) = 2F_b(t)^2 q'(\cos t)(-\sin t)$ is opposite to the sign of $p(y) := 200y^2 + 264y + 65$. At $y = -1$ $(t = \pi)$, $p(-1) = 1 > 0$, confirming $W(\pi - \varepsilon) = -8\varepsilon^3 + O(\varepsilon^5) < 0$ for small $\varepsilon > 0$. At $t = 2\pi/3$ $(y = -1/2)$, $p(-1/2) = 50 - 132 + 65 = -17 < 0$, and $F_b(2\pi/3) = (\sqrt{3}/2)(2 - 5 + 4) = \sqrt{3}/2$, giving $W(2\pi/3) = 2 \cdot \frac{3}{4} \cdot \frac{4(-17)}{1^2} \cdot (-\frac{\sqrt{3}}{2}) = 51\sqrt{3} > 0$. □

Remark 10 (Structural moral and empirical numerical-precision heuristic). Propositions 8 and 9 show that coefficient-ratio monotonicity and denominator positivity alone cannot certify the sign of W, nor lift local near-π negativity to $(0, \pi)$. As an empirical numerical diagnostic (archived in `docs/BF2-ATTACK-NOTES.md`, not part of any proof), high-precision evaluations of the Bessel arrays at large β exhibit severe cancellation between even- and odd-index contributions: on sampled grids, $|W(t)|$ reaches $\sim 2.9 \cdot 10^{-8}$ at $\beta = 8$ and $\sim 5.0 \cdot 10^{-29}$ at $\beta = 32$ while individual partial sums remain of order 1. Accordingly, using $\geq 2.2\beta + 20$ decimal working digits is a conservative empirical heuristic for numerical exploration, not a proven error bound. Analytically, Theorem 5 already realizes F_B as an exact circle-convolution derivative; a corresponding positive kernel representation of $-W$ (or total positivity of $(A; B)$ against the sine kernel) remains the natural analytic route to Conjecture 7.

5 Verification and Reproducibility

The analytic proofs of Theorems 1 and 5, Lemmas 3 and 4, and Propositions 6, 8, and 9 are completely self-contained and verifiable by hand from this document.

In addition, the finite form of Theorem 1 (arbitrary K) is machine-checked in Lean 4 / Mathlib in `papers/wronskian-reduction/WronskianIdentity.lean` inside `https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME` at archival release `yang-mills-archive-v1.0` (commit `6b54eca695e96b3038ceda18276c4e28fb45983f`, Lean `v4.29.0-rc6`, pinned Mathlib `07642720480157414db592fa85b626dafb71355b`): declarations `bracket_eq_two_G`, `double_sum_eq_wronskian`, `pair_sum_eq_double_sum`, and `wronskian_identity`, reporting zero `sorry` and axiom readout `[propext, Classical.choice, Quot.sound]` (checked locally as a standalone Lean file outside `YangMillsCore`). As supplementary diagnostics (not proofs), Theorem 1 and Propositions 6, 8, and 9 were symbolically cross-checked in SymPy, and the convolution identity in Theorem 5 was checked against 30-digit numerical quadrature at $\beta \in \{0.5, 2.0\}$.

Scope and Disclosure

Conjecture 7 remains open, and no claim is made regarding four-dimensional gauge theory or a Yang–Mills mass gap. Large language models assisted manuscript preparation under author supervision; author Lluis Eriksson administers AIRR (https://airr.science/), so no independent editorial admission or peer review is claimed.

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