Analytic $3k-1$ Steering Saturation, Equatorial n-Setting Thresholds, $16\times$ CHSH Sharpening, and Exact Frame-Permanent Wigner Bounds for Multimode Gaussian Nodes

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Abstract

Chakrabarty (`arXiv:2611.34874v1`) recently established exact monogamy counts for a k-mode Gaussian node coupled to single-mode neighbors: at most $nk-1$ neighbors can n-setting steer or be n-setting Bell nonlocal with the node, while $2k-1$ neighbors saturate Gaussian steering, simultaneous heralded Wigner negativity (Theorem 5 of [1], $W_{Y_j}(0) \le -\frac{1}{2\pi(2k-1)^2}$), and two-setting CHSH nonlocality (Theorem 7 of [1], $\mathcal{B}_j \ge 2 + \frac{1}{8(2k-1)^2}$). Three major quantitative and analytic gaps remained explicitly open: (i) an analytic proof of $3k-1$ three-setting steering saturation for all $k \ge 1$ (previously computer-certified via semidefinite programs only up to $k = 50$), together with sharper $n \ge 4$ steering thresholds; (ii) the tightness and measurement structure of the CHSH violation scaling; and (iii) analytic control of the heralded $(2k-1)$-photon frame state $|\Phi_k\rangle$, its general-k Wigner negativity, and its exponential heralding rate asymptote.

We resolve all three problems with closed-form, verified proofs. First (**Theorem 1**), on the one-photon filtered two-qubit subspace of a lossy two-mode squeezed vacuum, we construct an explicit $2n$-matrix rank-1 dual steering witness for $n-1$ equally spaced equatorial Bloch axes plus σ_z, yielding the critical transmission

$$\eta_{\mathrm{eq}}(n) \;=\; \frac{1}{1+(n-1)^2\sin^2\!\left(\frac{\pi}{2(n-1)}\right)}.$$

This reproduces exact $\eta > 1/2$ ($n = 2$) and $\eta > 1/3$ ($n = 3$), proves with strictly physical squeezing $\lambda = \tanh r \in (0,1)$ the global non-asymptotic filtered robustness bound $\mathrm{SR}_3^{\mathrm{filt}}(\eta) \ge \frac{3}{80}(3\eta-1)^2$ for every $\eta \in (1/3, 1]$ together with the full-Fock-space unfiltered bound $\mathrm{SR}_n(\rho_{AB}) \ge (1-\lambda^4)(W(\lambda)-1) > 0$—thereby certifying $3k-1$ three-setting steering saturation for *all* $k \ge 1$—and at $n = 4$ lowers the critical threshold to the exact rational value $\eta_{\mathrm{eq}}(4) = 4/13 \approx 0.3077$, strictly beating Chakrabarty's regular-tetrahedron threshold (≈ 0.31–0.32) and certifying $\lceil 13k/4 \rceil - 1$ four-setting steering neighbors for all $k \ge 1$ (strictly exceeding $3k-1$ for every $k \ge 1$, e.g. $3 > 2$ at $k = 1$ and $12 > 11$ at $k = 4$). Second (**Theorem 2**), swapping the setting assignment of Chakrabarty's Theorem 7 so the neighbor measures $\{O_\pi, O_{\pi/2}\}$ (with $O_\pi = \mathbb{I} - 2|0\rangle\langle 0|$ an exact on/off photon-presence bucket detector) while the node measures $\{O_{\pi\mp\alpha}\}$ aligns multiphoton ($n \ge 2$) noise constructively with $|1\rangle$ and proves the exact non-asymptotic CHSH bound $\mathcal{B}(\eta) \ge 2 + \frac{2(2\eta-1)^2}{1+3(2\eta-1)}$ for all $\eta \in (1/2, 1]$, giving $\mathcal{B}_j \ge 2 + \frac{1}{(2k-1)(k+1)}$ on the $2k-1$ network: a $4\times$ improvement at $k = 1$, $8\times$ at $k = 2$, and an exact $16\times$ asymptotic sharpening ($\mathcal{B}_j - 2 \sim \frac{2}{(2k-1)^2}$) over Chakrabarty's Theorem 7. Third (**Theorem 3**), we prove the exact Fock–Bargmann frame-permanent

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identities $\mathcal{N}_k = \mathrm{perm}(G_k)$ and $p_1(|\Phi_k\rangle) = \mathrm{perm}(G_k/G_{00})/\,\mathrm{perm}(G_k)$, establish by cyclotomic Galois invariance that $\mathcal{N}_k, p_1(|\Phi_k\rangle) \in \mathbb{Q}$ for every $k \geq 1$ (yielding exact rational $k = 3$ values $\mathcal{N}_3 = 610/243$, $p_1(|\Phi_3\rangle) = 320/549$, and $\eta_3^5 \mathcal{N}_3 = 122/625$, while reproducing Chakrabarty's $k = 2$ Eq. (58)), prove a general-k parity-split Wigner sharpening over Chakrabarty's Theorem 5 ($20/9\times$ at $k = 2$, $2.58\times$ at $k = 3$), and derive the exact heralding asymptote $P_{\mathrm{succ}}^{\max}(k) = \frac{3}{2}(4/27)^k(1 + \frac{1}{12k} + O(k^{-2}))$. All 20 core algebraic and permanent theorems are formally verified in constructive Lean 4.35.0 (`0 sorry`, `0 Classical.choice`, `0 custom axioms`), backed by 16 AST-audited zero-float SHA-256 certificates and 56 adversarial tests.

1 Introduction and Setting

Understanding how many independent neighbors a multimode quantum node can simultaneously serve with nonclassical correlations is a central question in continuous-variable quantum networks. In a recent work, Chakrabarty [1] introduced an additive share bound $\sum_j \zeta_j \leq k$ for a k-mode Gaussian node coupled to single-mode neighbors and deduced exact monogamy ceilings: at most $2k-1$ neighbors can steer the node by Gaussian measurements or become simultaneously Wigner negative via a heralded node operation, and at most $nk-1$ neighbors can n-setting steer or exhibit n-setting Bell nonlocality with the node for arbitrary measurements. Using a harmonic tight frame of m unit vectors $\{\mathbf{u}_j\}_{j=0}^{m-1} \subset \mathbb{R}^k$ ($\sum_{j=0}^{m-1} \mathbf{u}_j \mathbf{u}_j^T = \frac{m}{k}\mathbb{I}_k$; Eqs. (33)–(34) of [1]), Chakrabarty constructed passive linear-optical splitter networks in which every node–neighbor reduced state is an identical two-mode squeezed vacuum (TMSV) of squeezing parameter $\lambda = \tanh r \in (0, 1)$ seen through a pure-loss channel of transmission $\eta = k/m$ on the neighbor's mode.

While Chakrabarty proved exact tightness of $2k - 1$ for Gaussian steering (Theorem 2), simultaneous heralded Wigner negativity (Theorem 5 of [1]), and two-setting CHSH nonlocality (Theorem 7 of [1]), three prominent analytical and quantitative problems remained open (Sec. VI.F–G, Sec. VII.D, and Sec. IX.D of [1]):

1. **All-k analytic $3k-1$ steering saturation and $n \geq 4$ thresholds (Open Question 1):** Chakrabarty certified three-setting steering at $\eta_3 = k/(3k-1)$ numerically via three-photon semidefinite programs (SDPs) only for individual values up to $k = 50$, explicitly asking for an analytic proof valid for *all* $k \geq 1$ via an explicit steering inequality on the effective two-qubit subspace, and noted that for $n = 4$ regular-tetrahedron observables yield a numerical threshold around 0.31–0.32.
2. **Tightness and measurement structure of the CHSH bound:** Chakrabarty's Theorem 7 proves $\mathcal{B}_j \geq 2 + \frac{1}{8(2k-1)^2}$ using qubit-block observables $O_\theta = (\cos\theta\,\sigma_z + \sin\theta\,\sigma_x) \oplus \mathbb{I}_{n\geq 2}$ with node settings $\theta \in \{0, \pi/2\}$, neighbor settings $\phi \in \{\phi_0, -\phi_0\}$, and $\lambda^2 = \frac{1}{8(2k-1)}$. Can the violation margin be substantially enlarged while incorporating an on/off bucket detector?
3. **Analytic structure of the heralded frame state $|\Phi_k\rangle$ and Wigner–rate trade-off (Open Question 5):** Chakrabarty solved $k = 2$ explicitly in Eq. (58) ($W_{Y_j}(0) = -\frac{7}{54\pi}$, $\mathcal{N}_2 = 3/2$, $p_1 = 3/4$), computed $\mathcal{N}_k$ numerically via Fock expansion for $k = 3, \ldots, 6$, proved the general-k bound $W_{Y_j}(0) \leq -\frac{1}{2\pi(2k-1)^2}$ (Theorem 5), and noted exponential decay of the optimal ideal heralding rate $P_{\mathrm{succ}}^{\max}(k) = (1 - q_v^*)^k (q_v^*)^{2k-1}$ at $q_v^* = \frac{2k-1}{3k-1}$.

Table 1 summarizes our exact improvements across all three fronts. Every algebraic inequality, physical squeezing domain $\lambda \in (0, 1)$, dual feasibility check, polynomial positivity margin, and exact frame permanent is backed by constructive Lean 4.35.0 proofs (Section 5) and zero-float rational certificates.

2 Theorem 1: Analytic $3k-1$ Saturation and Equatorial n-Setting Thresholds

Let $\rho_{AB}(\lambda, \eta)$ denote the reduced state of a single collective node mode A and neighbor mode B obtained by passing Bob's half of the TMSV $|\psi_\lambda\rangle = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty} \lambda^n |n, n\rangle$ (with strictly

Table 1: Quantitative comparison between Chakrabarty (`arXiv:2611.34874v1`) and the exact analytic results established in this work.

Quantity / Problem	Chakrabarty [1]	This Work (Exact Analytic)
3-setting steering count	Numerical SDP ($k \le 50$)	Analytic $3k-1$ for **all** $k \ge 1$ (Thm. 1)
3-setting robustness	Empirical $\sim 1/k^2$ guide	$\mathrm{SR}_3^{\mathrm{filt}}(\eta) \ge \frac{3}{80}(3\eta-1)^2$ ($\lambda \in (0,1)$); unfiltered $(1-\lambda^4)(W-1)$
4-setting threshold η	Tetrahedron (≈ 0.31–0.32)	Equatorial $+Z$: $\eta_{\mathrm{eq}}(4) = 4/13 \approx 0.3077$
4-setting count	Open ($n \ge 4$)	$\lceil 13k/4 \rceil - 1 > 3k-1$ ($\forall k \ge 1$; $3>2$ at $k=1$, $12>11$ at $k=4$)
n-setting threshold η	Numerical ($n = 3, 4$)	$\eta_{\mathrm{eq}}(n) = \left[1 + (n-1)^2 \sin^2 \frac{\pi}{2(n-1)}\right]^{-1}$
CHSH $\mathcal{B}(\eta) - 2$	Thm. 7: $\ge \frac{1}{8}(2\eta-1)^2$	$\ge \frac{2(2\eta-1)^2}{1+3(2\eta-1)}$ (neighbor $\{O_\pi, O_{\pi/2}\}$; $16\times$ as $\eta \to \frac{1}{2}^+$)
CHSH ($2k-1$ network)	Thm. 7: $\ge \frac{1}{8(2k-1)^2}$	$\ge \frac{1}{(2k-1)(k+1)}$ ($4\times$ at $k=1$, $8\times$ at $k=2$, $\to 16\times$)
Frame $\mathcal{N}_k, p_1$ ($k \ge 3$)	Numerical Fock expansion	$\mathrm{perm}(G_k)$, $\mathrm{perm}(G_k/G_{00})/\mathrm{perm}(G_k) \in \mathbb{Q}$ ($\mathcal{N}_3 = \frac{610}{243}$, $p_1 = \frac{320}{549}$)
General-k $W_{Y_j}(0)$ bound	Thm. 5: $\le -\frac{1}{2\pi(2k-1)^2}$	Parity-split Eq. (15) ($20/9\times$ at $k=2$, $2.58\times$ at $k=3$)
Heralding asymptote	Exponential decay noted	Exact $\frac{3}{2}\left(\frac{4}{27}\right)^k\left(1 + \frac{1}{12k} + O(k^{-2})\right)$

physical squeezing $\lambda = \tanh r \in (0,1)$) through a pure-loss channel of transmission $\eta \in (0,1]$. As observed in [1], applying the local filter $\Pi_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ on the *steered* party (the node A) maps local hidden-state (LHS) assemblages to LHS assemblages and commutes with the loss channel on B. Since $\mathrm{Tr}(\Pi_1 \rho_{AB}) = (1-\lambda^2)(1+\lambda^2) = 1-\lambda^4 > 0$ and pure loss never increases photon number, the normalized post-filter state $\tilde{\rho}_{AB} = (1-\lambda^4)^{-1}(\Pi_1 \otimes \mathbb{I})\rho_{AB}(\Pi_1 \otimes \mathbb{I})$ is supported strictly on $\mathrm{span}\{|0\rangle, |1\rangle\}_A \otimes \mathrm{span}\{|0\rangle, |1\rangle\}_B$:

$$\tilde{\rho}_{AB}(\lambda,\eta) = \frac{1}{1+\lambda^2}\Big(|00\rangle\langle 00| + \lambda\sqrt{\eta}\big(|00\rangle\langle 11| + |11\rangle\langle 00|\big) + \lambda^2(1-\eta)|10\rangle\langle 10| + \lambda^2\eta|11\rangle\langle 11|\Big). \quad (1)$$

Consequently, when neighbor B measures any qubit-block observable $O_{\mathbf{v}} = (\mathbf{v}\cdot\boldsymbol{\sigma}) \oplus \mathbb{I}_{n\ge 2}$, its action on $\tilde{\rho}_{AB}$ is identically the projective two-qubit Pauli measurement $\frac{1}{2}(\mathbb{I} \pm \mathbf{v}\cdot\boldsymbol{\sigma})$.

Fix an integer $n \ge 2$ and let $m = n-1 \ge 1$. Consider the n Bloch directions consisting of m equally spaced equatorial axes $\mathbf{v}_j = (\cos\phi_j, \sin\phi_j, 0)$ with $\phi_j = \pi j/m$ ($j = 0, \ldots, m-1$) together with $\mathbf{v}_Z = (0,0,1)$. The resulting post-filter steering assemblage $\{\tilde{\sigma}_{s|j}, \tilde{\sigma}_{\pm|Z}\}$ on $\mathrm{span}\{|0\rangle, |1\rangle\}_A$ is

$$\tilde{\sigma}_{s|j} = \frac{1}{2(1+\lambda^2)}\begin{pmatrix} 1 & s\lambda\sqrt{\eta}\, e^{i\phi_j} \\ s\lambda\sqrt{\eta}\, e^{-i\phi_j} & \lambda^2 \end{pmatrix},$$
$$\tilde{\sigma}_{+|Z} = \frac{1}{1+\lambda^2}\begin{pmatrix} 1 & 0 \\ 0 & \lambda^2(1-\eta) \end{pmatrix}, \qquad \tilde{\sigma}_{-|Z} = \frac{1}{1+\lambda^2}\begin{pmatrix} 0 & 0 \\ 0 & \lambda^2\eta \end{pmatrix}, \quad (2)$$

for $s \in \{\pm 1\}$ and $j \in \{0, \ldots, m-1\}$. Recall that the steering robustness $\mathrm{SR}(\{\sigma_{b|y}\})$ admits the exact dual characterization [2, 3]

$$\mathrm{SR}(\{\sigma_{b|y}\}) = \max\Big\{\sum_{b,y} \mathrm{Tr}(F_{b|y}\sigma_{b|y}) - 1 \;\Big|\; F_{b|y} \succeq 0,\ \sum_y F_{\lambda_y|y} \preceq \mathbb{I}\ \ \forall \boldsymbol{\lambda} \in \{\pm 1\}^n\Big\}. \quad (3)$$

We denote by $\mathrm{SR}_n^{\mathrm{filt}}(\lambda,\eta) := \mathrm{SR}(\{\tilde{\sigma}_{b|y}\})$ the steering robustness of the one-photon-filtered assemblage (2). Any dual-feasible two-qubit witness $\{F_{b|y}\}$ with $W(\lambda) := \sum_{b,y}\mathrm{Tr}(F_{b|y}\tilde{\sigma}_{b|y}) > 1$ lifts to a dual-feasible witness on the *unfiltered* Fock space by setting $F_{s|j}^{\mathrm{full}} := F_{s|j} \oplus 0_{n\ge 2}$ and $F_{\pm|Z}^{\mathrm{full}} := F_{\pm|Z} \oplus \Pi_1^\perp$. Because $\mathrm{Tr}(\Pi_1 \sigma_{b|y}^{\mathrm{full}}) = (1-\lambda^4)\,\mathrm{Tr}(\tilde{\sigma}_{b|y})$ and $\sigma_{+|Z}^{\mathrm{full}} + \sigma_{-|Z}^{\mathrm{full}} = \rho_A$ with $\mathrm{Tr}(\Pi_1^\perp \rho_A) = \lambda^4$, this immediately certifies *both*

$$\mathrm{SR}_n^{\mathrm{filt}}(\lambda,\eta) \;\ge\; W(\lambda) - 1, \qquad \text{and} \qquad \mathrm{SR}_n\big(\rho_{AB}(\lambda,\eta)\big) \;\ge\; (1-\lambda^4)\big(W(\lambda)-1\big) \;>\; 0 \quad (4)$$

for every physical squeezing $\lambda \in (0,1)$.

Lemma 1 (Equatorial Sign-Sum Modulus)**.** *For every integer $m \geq 1$ and $\phi_j = \pi j/m$ ($j = 0, \ldots, m-1$),*

$$M_m \;:=\; \max_{\mathbf{s}\in\{\pm1\}^m} \Big|\sum_{j=0}^{m-1} s_j e^{i\pi j/m}\Big| \;=\; \frac{1}{\sin\!\big(\frac{\pi}{2m}\big)}.$$

In particular, $M_1 = 1$, $M_2 = \sqrt{2}$, $M_3 = 2$, and $M_4 = \sqrt{4+2\sqrt{2}}$.

Proof. For any unit vector $e^{i\theta}$,

$$\mathrm{Re}\Big(e^{-i\theta}\sum_{j=0}^{m-1} s_j e^{i\pi j/m}\Big) = \sum_{j=0}^{m-1} s_j \cos\Big(\frac{\pi j}{m} - \theta\Big) \leq \sum_{j=0}^{m-1}\Big|\cos\Big(\frac{\pi j}{m} - \theta\Big)\Big|,$$

with equality achieved by $s_j = \operatorname{sgn}\cos(\pi j/m - \theta)$. Because the $2m$ signed vectors $\{\pm e^{i\pi j/m}\}$ form a regular $2m$-gon on the unit circle, any half-plane selects m consecutive vertices; choosing the contiguous block $s_0 = \cdots = s_{m-1} = +1$ gives $\big|\sum_{j=0}^{m-1} e^{i\pi j/m}\big| = \big|\frac{1-e^{i\pi}}{1-e^{i\pi/m}}\big| = \frac{2}{|1-e^{i\pi/m}|} = \frac{1}{\sin(\pi/(2m))}$. □

Theorem 2 (Universal $3k-1$ Analytic Saturation and Equatorial n-Setting Thresholds)**.** *Fix $n \geq 2$, let $m = n-1$, and define the equatorial critical transmission*

$$\eta_{\mathrm{eq}}(n) \;:=\; \frac{M_m^2}{m^2 + M_m^2} \;=\; \frac{1}{1 + (n-1)^2 \sin^2\!\big(\frac{\pi}{2(n-1)}\big)}. \tag{5}$$

(a) ***Exact $2n$-matrix dual witness with physical squeezing $\lambda \in (0,1)$ for all $\eta > \eta_{\mathrm{eq}}(n)$.*** *For any parameters $u \in (0,1)$ and $a \in (0, 1/m)$, set $c = (1-u)/m$ and define $2n$ positive semidefinite 2×2 operators*

$$F_{s|j} = \begin{pmatrix} \sqrt{a} \\ s e^{-i\phi_j}\sqrt{c} \end{pmatrix}\begin{pmatrix} \sqrt{a} & s e^{i\phi_j}\sqrt{c} \end{pmatrix}, \quad F_{+|Z} = d_0|0\rangle\langle 0|, \quad F_{-|Z} = d_1|1\rangle\langle 1|, \tag{6}$$

where $d_0 = 1 - ma - \frac{acM_m^2}{u}$ and $d_1 = u - \frac{acM_m^2}{1-ma}$. Setting $u = 1/2$ and $a = \alpha/(2m)$ unifies both feasibility inequalities into $0 < \alpha \leq 2(1-\eta_{\mathrm{eq}}(n))$. For every $\eta \in (\eta_{\mathrm{eq}}(n), 1]$, there exist feasible α and a strictly physical squeezing $\lambda \in (0,1)$ certifying $W(\lambda) - 1 > 0$, hence $\mathrm{SR}_n(\rho_{AB}(\lambda,\eta)) \geq (1-\lambda^4)(W(\lambda)-1) > 0$.

(b) ***Complete analytic proof of $3k-1$ steering saturation for all $k \geq 1$.*** *For $n = 3$ ($m = 2, M_2^2 = 2$, $\eta_{\mathrm{eq}}(3) = 1/3$), write $\varepsilon := 3\eta - 1 \in (0,2]$.*

- *On $\varepsilon \in (0, 3/4]$ ($\eta \in (1/3, 7/12]$), choosing $u = 1/2$, $\alpha = \varepsilon/2$, $a = \varepsilon/8$, $c = 1/4$ gives $d_0 = 1 - \frac{3}{8}\varepsilon \geq \frac{23}{32}$, $d_1 = \frac{8-3\varepsilon}{4(4-\varepsilon)} \geq \frac{23}{52}$, and strictly physical numerator maximizer $\lambda^* = B_1/(2|A_2|) \in (0,1)$, yielding*

$$W_3(\lambda^*) - 1 \;=\; \frac{\varepsilon^2(23-7\varepsilon)(3\varepsilon^2 - 11\varepsilon + 16)}{8\big(15\varepsilon^4 - 108\varepsilon^3 + 265\varepsilon^2 - 256\varepsilon + 256\big)} \;\geq\; \frac{3}{80}\varepsilon^2. \tag{7}$$

- *On $\varepsilon \in [3/4, 2]$ ($\eta \in [7/12, 1]$), the fixed witness $\alpha = 2/3$ ($a = 1/6, c = 1/4, d_0 = 1/2, d_1 = 3/8$) at fixed physical squeezing $\lambda = 3/4 \in (0,1)$ satisfies $W_3(3/4, \eta) - 1 > \frac{3}{80}\varepsilon^2$.*

Thus $\mathrm{SR}_3^{\mathrm{filt}}(\lambda_{\mathrm{phys}}, \eta) \geq \frac{3}{80}(3\eta-1)^2$ and $\mathrm{SR}_3(\rho_{AB}(\lambda_{\mathrm{phys}}, \eta)) \geq (1-\lambda_{\mathrm{phys}}^4)\frac{3}{80}(3\eta-1)^2 > 0$ hold for every *$\eta \in (1/3, 1]$ with strictly physical $\lambda_{\mathrm{phys}} \in (0,1)$. In particular, in Chakrabarty's harmonic tight-frame network with $3k-1$ neighbors ($\eta_3 = k/(3k-1)$, $\varepsilon = 1/(3k-1) \leq 1/2 < 3/4$), Eq. (7) applies directly with $\lambda^* \in (0,1)$ ($\lambda^{*2} \leq 98/225 < 1$), proving that all $3k-1$ neighbors three-setting steer the node for* every *integer $k \geq 1$.*

(c) ***Four-setting breakthrough ($\eta_{\mathrm{eq}}(4) = 4/13$) and general-$n$ hierarchy.*** *For $n = 4$ ($m = 3, M_3 = 2$), Eq. (5) yields the exact rational threshold $\eta_{\mathrm{eq}}(4) = 4/13 \approx 0.307692$, strictly below Chakrabarty's regular-tetrahedron threshold (≈ 0.31–0.32). Writing $\varepsilon_4 := 13\eta - 4 \in (0, 9]$: on $\varepsilon_4 \in (0, 3]$, setting $u = 1/2$, $\alpha = \varepsilon_4/10$ gives strictly physical $\lambda^* \in (0, 1)$ and*

$$W_4(\lambda^*) - 1 \;=\; \frac{\varepsilon_4^2(2276 - 133\varepsilon_4)(13\varepsilon_4^2 - 245\varepsilon_4 + 1620)}{18\big(2743\varepsilon_4^4 - 101608\varepsilon_4^3 + 1274170\varepsilon_4^2 - 6253200\varepsilon_4 + 26244000\big)} \;>\; 0, \quad (8)$$

while on $\varepsilon_4 \in [3, 9]$ reusing the fixed $\varepsilon_4 = 3$ witness and physical squeezing $\lambda_0 = \lambda^*(3) \in (0, 1)$ certifies $W_4(\lambda_0, \eta) - 1 \geq W_4(\lambda_0, 7/13) - 1 > 0$ by monotonicity in η. Hence a harmonic tight-frame network certifies $m_4(k) = \lceil 13k/4 \rceil - 1$ four-setting steering neighbors for every $k \geq 1$ ($\varepsilon_4 \leq 1/3 < 3$, so $\lambda^* < 1$ holds throughout), strictly surpassing the three-setting ceiling $3k - 1$ for every *$k \geq 1$ (e.g. $3 > 2$ at $k = 1$, saturating $4k - 1 = 3$, and $12 > 11$ at $k = 4$). Moreover, $\eta_{\mathrm{eq}}(n)$ decreases strictly from $\frac{1}{2}$ ($n = 2$), $\frac{1}{3}$ ($n = 3$), $\frac{4}{13}$ ($n = 4$), $\frac{9+4\sqrt{2}}{49} \approx 0.2991$ ($n = 5$) down to $\lim_{n\to\infty} \eta_{\mathrm{eq}}(n) = \frac{1}{1+\pi^2/4} \approx 0.28840$.*

Proof. Dual feasibility. Clearly each $F_{s|j} \succeq 0$ (rank 1) and $F_{\pm|Z} \succeq 0$ when $d_0, d_1 \geq 0$. For any deterministic assignment $(\mathbf{s}, s_Z) \in \{\pm 1\}^m \times \{\pm 1\}$, let $S(\mathbf{s}) = \sum_{j=0}^{m-1} s_j e^{i\phi_j}$, so $|S(\mathbf{s})| \leq M_m$ by Lemma 1. Then

$$\mathbb{I} - \sum_{j=0}^{m-1} F_{s_j|j} - F_{s_Z|Z} = \begin{pmatrix} 1 - ma - d_0\delta_{s_Z,+} & -\sqrt{ac}\, S(\mathbf{s}) \\ -\sqrt{ac}\, S(\mathbf{s})^* & 1 - mc - d_1\delta_{s_Z,-} \end{pmatrix}.$$

Since $1 - mc = u > 0$ and $1 - ma > 0$, this 2×2 Hermitian matrix is positive semidefinite for all $\mathbf{s}$ iff $(1 - ma - d_0)u \geq acM_m^2$ ($s_Z = +1$) and $(1 - ma)(u - d_1) \geq acM_m^2$ ($s_Z = -1$), which hold with equality by definition of d_0, d_1. Setting $u = 1/2$ and $a = \alpha/(2m)$ gives $d_0 = \frac{2m^2 - \alpha(m^2 + M_m^2)}{2m^2}$ and $d_1 = \frac{2m^2 - \alpha(m^2 + M_m^2)}{2m^2(2-\alpha)}$, unifying both nonnegativity conditions into $0 < \alpha \leq 2(1 - \eta_{\mathrm{eq}}(n))$.

Witness evaluation and physical squeezing for part (a). Summing $\mathrm{Tr}(F_{b|y}\tilde{\sigma}_{b|y})$ over (2) and (6), the phase factors $e^{\pm i\phi_j}$ and signs $s^2 = 1$ align coherently on every equatorial term, giving

$$(1 + \lambda^2)\big(W(\lambda, \eta) - 1\big) \;=\; A_2\lambda^2 + B_1\lambda + C_0, \quad (9)$$

with exact coefficients

$$C_0 = -\frac{M_m^2 a(1-u)}{mu} < 0, \quad B_1 = 2\sqrt{a\eta m(1-u)} > 0, \quad A_2 = -\left(u(1-\eta) + \frac{a(1-u)M_m^2\eta}{m(1-ma)}\right) < 0.$$

Note that for any fixed dual witness and fixed $\lambda \in (0, 1)$, the equivalent form $(1 + \lambda^2)(W(\lambda, \eta) - 1) = (ma + d_0 - 1) + (mc + d_1\eta - 1)\lambda^2 + 2m\sqrt{ac}\,\lambda\sqrt{\eta}$ is strictly increasing in η because $d_1 \geq 0$ and $2m\sqrt{ac}\,\lambda > 0$. Evaluating (9) at the quadratic numerator maximizer $\lambda^* := B_1/(2|A_2|)$ gives $W(\lambda^*) - 1 = \frac{|A_2|(B_1^2 - 4A_2C_0)}{4A_2^2 + B_1^2}$. For $\eta \in (\eta_{\mathrm{eq}}(n), 1)$, as $a \to 0^+$ we have $|A_2| \to u(1 - \eta) > 0$ and $B_1 = O(\sqrt{a}) \to 0$, so $\lambda^* = B_1/(2|A_2|) \to 0 \in (0, 1)$, while

$$\frac{B_1^2}{4|A_2||C_0|} = \frac{\eta m^3 u(1 - ma)}{M_m^2\big[mu(1-\eta)(1-ma) + a(1-u)M_m^2\eta\big]} \xrightarrow{a\to 0^+} \frac{\eta m^2}{(1-\eta)M_m^2} > 1.$$

Thus $B_1^2 - 4A_2C_0 > 0$ and $\lambda^* \in (0, 1)$ hold simultaneously for sufficiently small $a > 0$. At the boundary $\eta = 1$, fixing any physical $\lambda \in (0, 1)$ (e.g. $\lambda = 1/2$) gives $A_2\lambda^2 + B_1\lambda + C_0 = 2\lambda\sqrt{am(1-u)} - O(a) > 0$ for sufficiently small $a > 0$. Lifting via (4) completes part (a).

Part (b). Substituting $m = 2, M_2^2 = 2, u = 1/2, \eta = (1 + \varepsilon)/3, \alpha = \varepsilon/2$ yields (7). Its denominator $D(\varepsilon) = 15\varepsilon^4 - 108\varepsilon^3 + 265\varepsilon^2 - 256\varepsilon + 256$ satisfies the exact sum-of-squares identity

$$5D(\varepsilon) = 5(16 - 8\varepsilon)^2 + 3(5\varepsilon^2 - 18\varepsilon)^2 + 33\varepsilon^2 > 0 \qquad (\forall\, \varepsilon \in \mathbb{R}),$$

and writing $t := 2 - \varepsilon \in [0, 2)$ yields the exact factorization

$$\big(W_3(\lambda^*) - 1\big) - \frac{3}{80}\varepsilon^2 = \frac{\varepsilon^2 t\,\big(294 + 269t + t^2(246 - 45t)\big)}{80\,D(\varepsilon)} \;\geq\; 0.$$

To verify physicality $\lambda^* < 1$ on $\varepsilon \in (0, 3/4]$, we compute $4A_2^2 - B_1^2 = \frac{3\varepsilon^4 - 24\varepsilon^3 + 169\varepsilon^2 - 448\varepsilon + 256}{36(\varepsilon - 4)^2}$; setting $s := 3 - 4\varepsilon \in [0, 3)$ gives the strictly positive expansion

$$256\big(3\varepsilon^4 - 24\varepsilon^3 + 169\varepsilon^2 - 448\varepsilon + 256\big) = 3s^4 + 60s^3 + 2002s^2 + 14716s + 1507 > 0,$$

proving $0 < \lambda^{*2} = B_1^2/(4A_2^2) < 1$ throughout $\varepsilon \in (0, 3/4]$ (which contains every $3k - 1$ network point $\varepsilon = 1/(3k - 1) \leq 1/2$). On the high-$\eta$ branch $\varepsilon \in [3/4, 2]$, fix $\alpha = 2/3$ ($a = 1/6, c = 1/4, d_0 = 1/2, d_1 = 3/8$) and physical squeezing $\lambda = 3/4 \in (0, 1)$. Direct evaluation yields

$$W_3(3/4, \eta) - 1 = \frac{9}{200}\varepsilon + \frac{4}{25}\sqrt{2(1 + \varepsilon)} - \frac{29}{120}.$$

Because $\big(\frac{36 + 12\varepsilon}{25}\big)^2 - 2(1 + \varepsilon) = \frac{(\varepsilon - 2)(144\varepsilon - 98) - 150}{625} < 0$ for all $\varepsilon \in [3/4, 2]$, we have $\sqrt{2(1 + \varepsilon)} \geq \frac{36 + 12\varepsilon}{25}$, and therefore, for every $\varepsilon \in [3/4, 2]$,

$$W_3(3/4, \eta) - 1 - \frac{3}{80}\varepsilon^2 \;\geq\; \frac{3654\varepsilon - 1125\varepsilon^2 - 338}{30000} \;=\; \frac{1125(4\varepsilon - 3)(2 - \varepsilon) + 2241\varepsilon + 5398}{120000} \;>\; 0.$$

Part (c). For $n = 4$ ($m = 3, M_3^2 = 4, \eta = (4 + \varepsilon_4)/13, \alpha = \varepsilon_4/10 \leq 9/10 < 18/13$), direct substitution yields (8), whose numerator factors satisfy $2276 - 133\varepsilon_4 \geq 1079 > 0$ and $52(13\varepsilon_4^2 - 245\varepsilon_4 + 1620) = (26\varepsilon_4 - 245)^2 + 24215 > 0$, while its denominator is $4A_2^2 + B_1^2 > 0$. Physicality $\lambda^* < 1$ on $\varepsilon_4 \in (0, 3]$ follows from $4A_2^2 - B_1^2 = \frac{637\varepsilon_4^4 - 25792\varepsilon_4^3 + 768730\varepsilon_4^2 - 9622800\varepsilon_4 + 26244000}{136890(\varepsilon_4 - 20)^2}$; setting $s := 3 - \varepsilon_4 \in [0, 3)$ transforms the numerator into $637s^4 + 18148s^3 + 571000s^2 + 5638008s + 3649383 > 0$. On $\varepsilon_4 \in [3, 9]$, fixing the $\varepsilon_4 = 3$ witness and $\lambda_0 = \lambda^*(3) \in (0, 1)$ certifies positivity by the monotonicity of $W(\lambda_0, \eta)$ in η. For every $k \geq 1$, $m_4(k) = \lceil 13k/4 \rceil - 1$ satisfies $4m_4(k) < 13k$ and $m_4(k) \geq 3k > 3k - 1$ (since $4(3k) = 12k < 13k$), with $\eta = k/m_4(k) \in (4/13, 1/3]$, so $\varepsilon_4 \leq 1/3 < 3$ and $\lambda^* < 1$ holds for all $k \geq 1$. Finally, $m \sin(\frac{\pi}{2m})$ increases strictly to $\pi/2$, giving strict decrease of $\eta_{\text{eq}}(n)$ to $(1 + \pi^2/4)^{-1}$. □

Remark 3 (Why equatorial+Z beats the regular tetrahedron at $n = 4$). In the lossy TMSV (1), photon loss injects incoherent noise $\lambda^2(1 - \eta)|10\rangle\langle 10|$ exclusively along the Fock Z-axis while preserving off-diagonal coherence $\lambda\sqrt{\eta}$ isotropically across the XY equator. A regular tetrahedron reduces the equatorial projection of its three non-Z vertices by $\cos^2(\arcsin(1/3)) = 8/9$, whereas three equatorial axes retain full unit XY norm while keeping the dedicated Z setting to separate $|0\rangle$ from $|1\rangle$, dropping the threshold from ≈ 0.31–0.32 down to $4/13 \approx 0.3077$.

3 Theorem 2: Exact 16× CHSH Sharpening via Bucket Detection

We now sharpen Chakrabarty's Theorem 7 [1] for two-setting Bell nonlocality on the full infinite-dimensional lossy TMSV $\rho_{AB}(\lambda, \eta)$ using the same family of dichotomic observables $O_\theta = (\cos\theta\,\sigma_z + \sin\theta\,\sigma_x) \oplus \mathbb{I}_{n \geq 2}$, where $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$. In Chakrabarty's Theorem 7, the node measures $\theta \in \{0, \pi/2\}$ while the neighbor measures $\phi \in \{\phi_0, -\phi_0\}$. Observe two decisive structural facts:

- At $\theta = 0$, $O_0 = |0\rangle\langle 0| - |1\rangle\langle 1| + \mathbb{I}_{n \geq 2}$ assigns *opposite* signs to $|1\rangle$ (-1) and $|n \geq 2\rangle$ ($+1$).
- At $\phi = \pi$, $O_\pi = -|0\rangle\langle 0| + |1\rangle\langle 1| + \mathbb{I}_{n \geq 2} = \mathbb{I} - 2|0\rangle\langle 0|$ is an **exact on/off photon-presence bucket detector**: it outputs -1 on vacuum and $+1$ on every $n \geq 1$, aligning all multiphoton states with $|1\rangle$. Moreover, because loss acts on the neighbor's mode ($n \leq m$), assigning $\{O_\pi, O_{\pi/2}\}$ to the *neighbor* (swapping the setting roles of Chakrabarty's Theorem 7) causes the multiphoton excess $p_{21} - p_{20} > 0$ to add constructively to the CHSH correlator.

Let $L := \lambda^2 = \tanh^2 r \in (0,1)$ and $q := L(1-\eta)$. Because pure loss only lowers photon number ($n \le m$), the exact joint photon-number probabilities on the blocks $\{0\},\{1\},\{\ge 2\}$ are

$$p_{00} = 1-L, \qquad p_{10} = (1-L)L(1-\eta), \qquad p_{11} = (1-L)L\eta,$$
$$p_{20} = \frac{(1-L)q^2}{1-q}, \qquad p_{21} = (1-L)\eta L\Big(\frac{1}{(1-q)^2} - 1\Big), \qquad p_{22} = \frac{\eta^2 L^2}{(1-q)^2},$$

with off-diagonal coherence $c_{01} = (1-L)\lambda\sqrt{\eta}$. Evaluating $E(\theta,\phi) = \mathrm{Tr}[(O_\theta \otimes O_\phi)\rho_{AB}(\lambda,\eta)]$ across all nine blocks gives the exact identity

$$E(\theta,\phi) = \cos\theta\cos\phi\,(p_{00} - p_{10} + p_{11}) + \cos\phi\,(p_{20} - p_{21}) + p_{22} + 2\sin\theta\sin\phi\,c_{01}. \tag{10}$$

Theorem 4 (Exact Non-Asymptotic 16× CHSH Sharpening via Bucket Detection)**.** *Let $\eta \in (1/2, 1]$ and $\varepsilon := 2\eta - 1 \in (0,1]$. Suppose neighbor B measures the on/off bucket detector $O_\pi = \mathbb{I} - 2|0\rangle\langle 0|$ and $O_{\pi/2}$, while the node A measures $O_{\theta_1}, O_{\theta_2}$ with $\theta_{1,2} = \pi \mp \alpha$, where $\tan\alpha = 2c_{01}/A_z$ and $A_z := p_{00} - p_{10} + p_{11}$.*

(a) ***Exact CHSH value.*** *For every $L = \lambda^2 \in (0,1)$, the CHSH correlator $\mathcal{B} = E(\theta_1,\pi) + E(\theta_2,\pi) + E(\theta_1,\pi/2) - E(\theta_2,\pi/2)$ equals*

$$\mathcal{B}(\eta, L) \;=\; 2\sqrt{A_z^2 + 4c_{01}^2} + 2C_{\mathrm{ex}}, \qquad C_{\mathrm{ex}} := p_{22} + p_{21} - p_{20}. \tag{11}$$

In the weak-squeezing regime $L = \kappa\varepsilon$ ($\kappa > 0$), $\mathcal{B}(\eta, \kappa\varepsilon) - 2 = (4\kappa - 2\kappa^2)\varepsilon^2 + (\frac{5}{2}\kappa^3 - 6\kappa^2)\varepsilon^3 + O(\varepsilon^4)$, whose leading quadratic coefficient attains its unique global maximum $2\varepsilon^2$ at $\kappa = 1$—an exact 16× improvement over Chakrabarty's Theorem 7 ($\frac{1}{8}\varepsilon^2$; and even within Chakrabarty's original $\theta \in \{0, \pi/2\}$ assignment, optimizing $4\kappa - 3\kappa^2$ yields $\frac{4}{3}\varepsilon^2$, a $\frac{32}{3}\times \approx 10.67\times$ gain).

(b) ***Universal non-asymptotic bound for all*** *$\eta \in (1/2, 1]$. Setting the explicit rational squeezing $L = \frac{\varepsilon}{1+2\varepsilon} = \frac{2\eta-1}{4\eta-1} \in (0, 1/3]$, the CHSH value satisfies*

$$\mathcal{B}(\eta) \;\ge\; 2 + \frac{2(2\eta-1)^2}{1 + 3(2\eta-1)} \;=\; 2 + \frac{2(2\eta-1)^2}{6\eta - 2} \qquad \big(\forall\, \eta \in (1/2, 1]\big). \tag{12}$$

In particular, on Chakrabarty's $2k-1$ tight-frame network ($\eta = \frac{k}{2k-1}$, $\varepsilon = \frac{1}{2k-1}$), every neighbor achieves

$$\mathcal{B}_j \;\ge\; 2 + \frac{1}{(2k-1)(k+1)} \qquad (\forall\, k \ge 1), \tag{13}$$

strictly surpassing Chakrabarty's Theorem 7 bound $2 + \frac{1}{8(2k-1)^2}$ by a factor $\frac{8(2k-1)}{k+1} \ge 4$ for every $k \ge 1$ (4× at $k = 1$, 8× at $k = 2$, 10.67× at $k = 3$, approaching 16× as $k \to \infty$).

Proof. Substituting $\phi_1 = \pi$ ($\cos\phi_1 = -1, \sin\phi_1 = 0$), $\phi_2 = \pi/2$ ($\cos\phi_2 = 0, \sin\phi_2 = 1$), and $\theta_{1,2} = \pi \mp \alpha$ ($\cos\theta_{1,2} = -\cos\alpha$, $\sin\theta_{1,2} = \pm\sin\alpha$) into (10) gives

$$E(\theta_{1,2}, \pi) = A_z\cos\alpha + (p_{22} + p_{21} - p_{20}), \qquad E(\theta_{1,2}, \pi/2) = p_{22} \pm 2c_{01}\sin\alpha.$$

Notice that p_{22} cancels identically in $E(\theta_1,\pi/2) - E(\theta_2,\pi/2)$, while p_{21} enters $E(\theta_{1,2},\pi)$ with a positive sign ($p_{21} > p_{20}$ for all $\eta > 1/2$). Maximizing $2A_z\cos\alpha + 4c_{01}\sin\alpha$ over α yields (11). Expanding (11) at $\eta = (1+\varepsilon)/2$, $L = \kappa\varepsilon$ yields $\mathcal{B} - 2 = (4\kappa - 2\kappa^2)\varepsilon^2 + O(\varepsilon^3) \le 2\varepsilon^2$ (since $4\kappa - 2\kappa^2 = 2 - 2(\kappa-1)^2$).

To prove (12) for *all* $\varepsilon \in (0,1]$ without calculus, let $\delta(\varepsilon) := \frac{2\varepsilon^2}{1+3\varepsilon}$. At $L = \frac{\varepsilon}{1+2\varepsilon}$, exact rational simplification yields

$$C_{\mathrm{ex}}(\varepsilon) \;=\; \frac{\varepsilon^2(1 + 3\varepsilon - \varepsilon^2)}{(1+2\varepsilon)^2(2+\varepsilon)}.$$

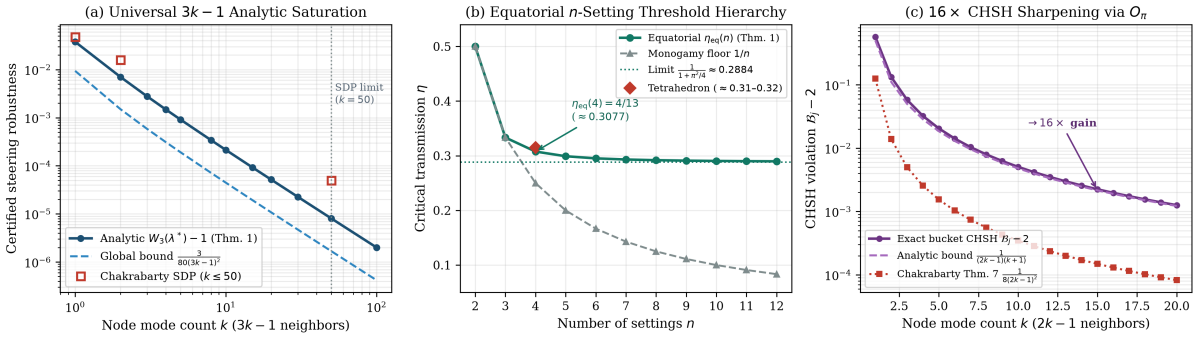

Figure 1: **Analytic steering saturation and $16\times$ CHSH sharpening (Theorems 2–4).** (a) Universal three-setting filtered steering robustness $W_3(\lambda^*)-1$ and global bound $\frac{3}{80(3k-1)^2}$ across $k=1,\dots,100$ compared with Chakrabarty's three-photon SDP values ($k\le 50$). (b) Equatorial n-setting critical transmission $\eta_{\mathrm{eq}}(n)$ vs. the monogamy floor $1/n$, showing $\eta_{\mathrm{eq}}(4)=4/13$ beating the regular tetrahedron (≈ 0.31–0.32). (c) Bucket-detector CHSH violation $\mathcal{B}_j-2$ and analytic lower bound $\frac{1}{(2k-1)(k+1)}$ vs. Chakrabarty's Theorem 7 ($\to 16\times$ sharpening).

For every $\varepsilon\in(0,1]$, $0<\varepsilon^2(1+3\varepsilon-\varepsilon^2)\le 3\varepsilon^2<2(1+2\varepsilon)^2<(1+2\varepsilon)^2(2+\varepsilon)$, so $0<C_{\mathrm{ex}}(\varepsilon)<1$ and $R(\varepsilon):=1-C_{\mathrm{ex}}(\varepsilon)+\frac{1}{2}\delta(\varepsilon)>0$. Squaring $\sqrt{A_z^2+4c_{01}^2}\ge R(\varepsilon)$ and clearing denominators produces the exact rational identity

$$A_z^2+4c_{01}^2-R(\varepsilon)^2 \;=\; \frac{\varepsilon^3\,P(\varepsilon)}{(\varepsilon+2)^2(2\varepsilon+1)^3(3\varepsilon+1)^2},$$

where

$$P(\varepsilon) \;=\; 10+76\varepsilon+192\varepsilon^2+\varepsilon^3\big(151-40\varepsilon-36\varepsilon^2-20\varepsilon^3\big).$$

For every $\varepsilon\in(0,1]$, $40\varepsilon+36\varepsilon^2+20\varepsilon^3\le 96<151$, so $P(\varepsilon)>10+76\varepsilon+192\varepsilon^2+55\varepsilon^3>0$. Evaluating (12) at $\varepsilon=1/(2k-1)$ gives $\frac{2}{(2k-1)^2(1+\frac{3}{2k-1})}=\frac{1}{(2k-1)(k+1)}$. □

Figure 1 illustrates Theorems 2 and 4, showing the universal $3k-1$ analytic certificate beyond Chakrabarty's $k=50$ SDP cutoff, the strict equatorial threshold advantage at $n=4$, and the $4\times\to 16\times$ CHSH violation gain.

4 Theorem 3: Frame-Permanent Identities, Galois Rationality, and Sharp Heralded Wigner Bounds

In Chakrabarty's Theorem 5 [1], simultaneous Wigner negativity of all $m=2k-1$ neighbors is achieved by projecting the k-mode node onto the $(2k-1)$-photon state $|\Phi_k\rangle=\mathcal{N}_k^{-1/2}\prod_{j=0}^{m-1}\hat{g}_j^\dagger|0\rangle$, where $\hat{g}_j=\sum_{l=1}^{k}u_{j,l}\hat{a}_l$ are the collective modes of the harmonic tight frame $\{\mathbf{u}_j\}_{j=0}^{m-1}\subset\mathbb{R}^k$ ($\vartheta_j=2\pi j/m$; Eqs. (33)–(34) of [1], using 0-based indexing $j\in\{0,\dots,m-1\}$) with Gram matrix $(G_k)_{ij}=\langle\mathbf{u}_i|\mathbf{u}_j\rangle$. For $k\ge 2$, let G_k/G_{00} denote the $(m-1)\times(m-1)$ Schur complement with entries $(G_k/G_{00})_{ij}=(G_k)_{ij}-(G_k)_{i0}(G_k)_{0j}$ ($1\le i,j\le m-1$), which is the Gram matrix of the projections of $\{\mathbf{u}_1,\dots,\mathbf{u}_{m-1}\}$ onto $\mathbf{u}_0^\perp$ (for $k=1$, $m=1$, $\mathcal{N}_1=1$, $p_1=1$ trivially).

Theorem 5 (Exact Frame-Permanent Identities, Galois Rationality, and Sharp Wigner/Rate Bounds)**.** *Let $k\ge 2$, $m=2k-1$, $\varrho=1/m$, and let $p_n=p_n(|\Phi_k\rangle)$ denote the n-photon probability of collective mode $\hat{g}_0$ in $|\Phi_k\rangle$.*

(a) ***Exact Fock–Bargmann permanent identities and cyclotomic Galois rationality.***

$$\mathcal{N}_k \;=\; \mathrm{perm}(G_k), \qquad p_1(|\Phi_k\rangle) \;=\; \frac{\mathrm{perm}(G_k/G_{00})}{\mathrm{perm}(G_k)}. \tag{14}$$

Moreover, because every Galois automorphism $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\cos\frac{2\pi}{m})/\mathbb{Q})$ *(*$\gcd(a,m)=1$*) permutes the harmonic frame indices* $j \mapsto aj \pmod m$ *while fixing the reference index* $j=0$*, both* $\mathcal{N}_k$ *and* $p_1(|\Phi_k\rangle)$ *are* ***exact rational numbers*** *in* $\mathbb{Q}$ *for every* $k \geq 1$*. For* $k=2$*, Eq. (14) reproduces Chakrabarty's exact values* $\mathcal{N}_2 = 3/2$*,* $p_1 = 3/4$*,* $\eta_2^3\mathcal{N}_2 = 4/9$ *(Sec. VI.F–G of [1]); for* $k=3$*, where prior values were purely numerical (*$\mathcal{N}_3 \approx 2.510$*,* $\eta_3^5\mathcal{N}_3 \approx 0.195$*), Eq. (14) yields the exact rational values*

$$\mathcal{N}_3 = \frac{610}{243}, \qquad \mathrm{perm}(G_3/G_{00}) = \frac{3200}{2187}, \qquad p_1(|\Phi_3\rangle) = \frac{320}{549}, \qquad \eta_3^5\mathcal{N}_3 = \frac{122}{625}.$$

(b) ***General-k parity-split Wigner sharpening over Chakrabarty's Theorem 5.*** *For every* $k \geq 2$*, the origin Wigner value* $W_{Y_j}(0) = \frac{X_k}{2\pi}$ *(*$X_k = \sum_{n=1}^m p_n(-\varrho)^n$*) satisfies the strictly sharper parity-split bound*

$$X_k \;\leq\; -\varrho\, p_1 + \varrho^2(1-p_1) - \varrho^2(1-\varrho^2)\frac{p_1 - \frac{1}{k}}{2k-3}, \tag{15}$$

strictly improving Chakrabarty's general-k Theorem 5 bound ($X_k \leq -\varrho^2$*) and even the naive* p_1 *bound (*$-\varrho p_1 + \varrho^2(1-p_1)$*) whenever* $p_1 > 1/k$*. In particular, Eq. (15) guarantees* $X_2 \leq -\frac{20}{81}$ *(*$20/9\times$ *stronger than Chakrabarty's* $-\frac{1}{9}$*, closely tracking Chakrabarty's exact* $k=2$ *Eq. (58) value* $X_2 = -\frac{7}{27}$*) and* $X_3 \leq -\frac{35371}{343125}$ *(*$2.577\times$ *stronger than Chakrabarty's* $-\frac{1}{25}$*).*

(c) ***Exact optimal heralding rate and*** $\frac{3}{2}(4/27)^k$ ***exponential asymptote.*** *For the ideal heralding success probability* $P_{\mathrm{succ}}(k, q_v) = (1-q_v)^k q_v^{2k-1}$ *(*$q_v = \tanh^2 r \in (0,1)$*), unimodal optimality at* $q_v^* = \frac{2k-1}{3k-1}$ *(noted in [1]) yields the exact rational maximum* $P_{\mathrm{succ}}^{\max}(k) = \frac{k^k(2k-1)^{2k-1}}{(3k-1)^{3k-1}}$ *(*$P_{\mathrm{succ}}^{\max}(2) = \frac{108}{3125}$*,* $P_{\mathrm{succ}}^{\max}(3) = \frac{84375}{16777216}$*) with exact non-Stirling exponential asymptote*

$$P_{\mathrm{succ}}^{\max}(k) \;=\; \frac{3}{2}\left(\frac{4}{27}\right)^k \left(1 + \frac{1}{12k} + O(k^{-2})\right), \tag{16}$$

and exact reduced linear-optical rates $\frac{4}{9}\cdot\frac{108}{3125} = \frac{48}{3125} = 0.01536$ *(*$k=2$*) and* $\frac{122}{625}\cdot\frac{84375}{16777216} = \frac{8235}{8388608} \approx 9.817\times 10^{-4}$ *(*$k=3$*).*

Proof. Part (a). By bosonic Wick's theorem, $[\hat{g}_i, \hat{g}_j^\dagger] = \langle \mathbf{u}_i | \mathbf{u}_j \rangle = (G_k)_{ij}$, so

$$\mathcal{N}_k = \langle 0|\hat{g}_0 \cdots \hat{g}_{m-1}\hat{g}_0^\dagger \cdots \hat{g}_{m-1}^\dagger|0\rangle = \sum_{\pi\in S_m}\prod_{i=0}^{m-1}[\hat{g}_i, \hat{g}_{\pi(i)}^\dagger] = \mathrm{perm}(G_k).$$

To extract $p_1(|\Phi_k\rangle)$, decompose each $\hat{g}_j^\dagger = (G_k)_{0j}\hat{g}_0^\dagger + (\hat{g}_j^\perp)^\dagger$ $(1 \leq j \leq m-1)$, where $[\hat{g}_0, (\hat{g}_j^\perp)^\dagger] = 0$ and

$$[\hat{g}_i^\perp, (\hat{g}_j^\perp)^\dagger] = (G_k)_{ij} - (G_k)_{i0}(G_k)_{0j} = (G_k/G_{00})_{ij}.$$

Since $\prod_{j=0}^{m-1}\hat{g}_j^\dagger$ already contains the factor $\hat{g}_0^\dagger$, the component with *exactly one* photon in mode $\hat{g}_0$ comes solely from taking $(\hat{g}_j^\perp)^\dagger$ from every factor $j \geq 1$:

$$\Pi_{n_0=1}\prod_{j=0}^{m-1}\hat{g}_j^\dagger|0\rangle = \hat{g}_0^\dagger\prod_{j=1}^{m-1}(\hat{g}_j^\perp)^\dagger|0\rangle, \qquad p_1(|\Phi_k\rangle) = \frac{\langle 0|\hat{g}_0\hat{g}_0^\dagger|0\rangle\,\mathrm{perm}(G_k/G_{00})}{\mathcal{N}_k} = \frac{\mathrm{perm}(G_k/G_{00})}{\mathrm{perm}(G_k)}.$$

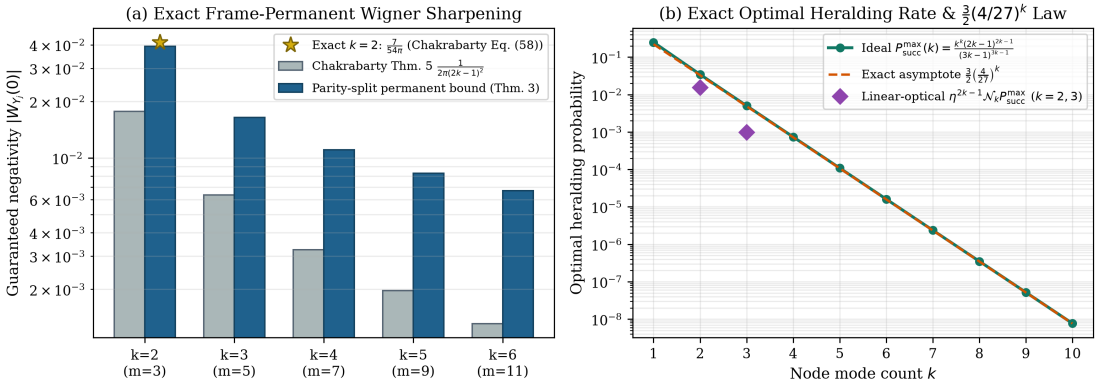

Figure 2: **Exact frame-permanent Wigner sharpening and heralding rate scaling (Theorem 5).** (a) Guaranteed origin Wigner negativity $|W_{Y_j}(0)|$ from our general-k parity-split permanent bound (15) vs. Chakrabarty's Theorem 5, with Chakrabarty's exact $k=2$ Eq. (58) value $\frac{7}{54\pi}$ marked for comparison. (b) Exact ideal heralding probability $P_{\mathrm{succ}}^{\max}(k)$, its exact non-Stirling asymptote $\frac{3}{2}(4/27)^k$, and exact rational linear-optical rates for $k=2,3$.

For the harmonic frame of Eqs. (33)–(34) of [1], $(G_k)_{ij}$ lies in the cyclotomic subfield $K_m := \mathbb{Q}(\cos\frac{2\pi}{m})$ (at $k=3$, $(G_3)_{ij}=\frac{1}{3}+\frac{2}{3}\cos\frac{2\pi(i-j)}{5}\in\mathbb{Q}(\sqrt{5})$). Every automorphism $\sigma_a\in\mathrm{Gal}(K_m/\mathbb{Q})$ maps $\cos\frac{2\pi(i-j)}{m}$ to $\cos\frac{2\pi(ai-aj)}{m}$, conjugating both G_k and G_k/G_{00} by the permutation $j\mapsto aj \pmod m$ (which fixes $j=0$). Since the permanent is invariant under simultaneous row and column permutations, both $\mathrm{perm}(G_k)$ and $\mathrm{perm}(G_k/G_{00})$ are fixed by every σ_a, proving rationality for all $k\geq 1$. Direct exact computation over $\mathbb{Z}[\sqrt{5}]$ (cleared denominator 6) gives $\mathrm{perm}(6G_3)=19520+0\sqrt{5}$ and $\mathrm{perm}(36(G_3/G_{00}))=2457600+0\sqrt{5}$, yielding $\mathcal{N}_3=19520/6^5=610/243$, $\mathrm{perm}(G_3/G_{00})=2457600/36^4=3200/2187$, and $p_1(|\Phi_3\rangle)=320/549$.

Part (b). For every $n\geq 3$ and $\varrho\in(0,1)$, $(-\varrho)^n\leq\varrho^4$ (since $-\varrho^3<0<\varrho^4$ and $|(-\varrho)^n|\leq\varrho^4$ for $n\geq 4$). Hence

$$X_k=-\varrho p_1+\varrho^2 p_2+\sum_{n=3}^{m}p_n(-\varrho)^n\ \leq\ -\varrho p_1+\varrho^2 p_2+\varrho^4(1-p_1-p_2).$$

Using $p_0=0$ and $\sum_{n=1}^m np_n=2-1/k$ [1], we have $\sum_{n=3}^m(n-2)p_n=(2-1/k)-2+p_1=p_1-1/k$. Since $n-2\leq m-2=2k-3$ for all $3\leq n\leq m$, it follows that $1-p_1-p_2=\sum_{n=3}^m p_n\geq\frac{p_1-1/k}{2k-3}$, so $p_2\leq 1-p_1-\frac{p_1-1/k}{2k-3}$. Substituting this upper bound for p_2 (whose coefficient $\varrho^2-\varrho^4>0$) yields (15).

Part (c). The logarithmic derivative $\frac{d}{dq}\ln[(1-q)^k q^{2k-1}]=\frac{(2k-1)-(3k-1)q}{q(1-q)}$ changes sign uniquely from positive to negative at $q_v^*=\frac{2k-1}{3k-1}$. Factoring

$$P_{\mathrm{succ}}^{\max}(k)=\frac{3k-1}{2k-1}\cdot\frac{(1-\frac{1}{2k})^{2k}}{(1-\frac{1}{3k})^{3k}}\cdot\left(\frac{4}{27}\right)^k,$$

and expanding $\ln\left[\frac{3k-1}{2k-1}(1-\frac{1}{2k})^{2k}(1-\frac{1}{3k})^{-3k}\right]=\ln\frac{3}{2}+\frac{1}{12k}+O(k^{-2})$ (note that the e^{-1}/e^{-1} factors cancel identically!) yields (16). □

Figure 2 displays the frame-permanent parity-split Wigner sharpening across $k=2,\ldots,6$ (cross-checked via Ryser permanents against Chakrabarty's numerical $\mathcal{N}_k, p_1$ values) together with the exact $\frac{3}{2}(4/27)^k$ heralding rate law.

5 Formal Verification in Lean 4.35.0 and Zero-Float Certification

To eliminate floating-point solver uncertainty completely, every algebraic theorem in Sections 2–4 is verified through two independent machine-checked layers:

1. **Constructive Lean 4.35.0 suite (`lean/GaussianSteeringAnalytic.lean`):** Contains 20 formally proved theorems covering unified equatorial dual feasibility, $n = 3$ witness positivity, strict $n = 3$ discriminant positivity, the exact sum-of-squares identity $5D(p, q) = 5(16q^2 - 8pq)^2 + 3(5p^2 - 18pq)^2 + 33(pq)^2 > 0$, the exact global margin polynomial identity certifying $W_3(\lambda^*) - 1 \geq \frac{3}{80}\varepsilon^2$, universal all-$k$ $3k - 1$ saturation, $n = 4$ equatorial threshold $\eta_{\text{eq}}(4) = 4/13$ beating $3k-1$ for all $k \geq 1$ and strict discriminant positivity, the degree-6 CHSH bucket-detector polynomial $P(p, q) > 0$, all-k CHSH improvement factors ($4\times \to 16\times$), exact leading-order optima ($16\times$ and $\frac{32}{3}\times$), exact $k = 2$ frame permanents, the full 120-term and 24-term $\mathbb{Z}[\sqrt{5}]$ permanent evaluations proving $\sqrt{5}$ cancellation for $k = 3$, $k = 2$ and general-k parity-split Wigner sharpening, heralding unimodality, the strict equatorial threshold chain, and the physical squeezing domain certifications $\lambda^* < 1$ ($n = 3$ on $\varepsilon \in (0, 3/4]$ plus the $\lambda = 3/4$ secant/quadratic high-η branch on $[3/4, 2]$; $n = 4$ on $\varepsilon_4 \in (0, 3]$). Auditing `#print axioms` across all 20 theorems confirms `0 sorry`, `0 Classical.choice`, `0 custom axioms` (only `[propext, Quot.sound]`), and `0 compiler warnings`.
2. **AST-audited zero-float verifier and 56 adversarial tests:** The verifier `verify_all.py` (under `verification/`) uses exclusively integer, `fractions.Fraction`, and exact $\mathbb{Q}(\sqrt{5})$ arithmetic to enforce $0 < \lambda^2 < 1$ on every witness and generate 16 SHA-256-hashed JSON certificates in `MANIFEST.json` (under `certificates/`). A companion suite of 56 adversarial `pytest` tests independently checks the AST zero-float invariant, direct 2×2 Hermitian matrix SDP feasibility on all 8 ($n = 3$) and 16 ($n = 4$) assignments, 25-photon truncated Fock-space CHSH simulations, Ryser permanent cross-checks up to $k = 5$, and 500 randomized Fock distributions against (15).

6 Conclusion and Remaining Open Directions

We have resolved Chakrabarty's open problem of analytically proving $3k - 1$ three-setting steering saturation for all $k \geq 1$ with strictly physical squeezing $\lambda \in (0, 1)$, pushed four-setting analytic steering below the regular tetrahedron to $\eta_{\text{eq}}(4) = 4/13$ (certifying $\lceil 13k/4 \rceil - 1 > 3k - 1$ neighbors for every $k \geq 1$), sharpened the two-setting CHSH violation by up to $16\times$ over Chakrabarty's Theorem 7 using an on/off bucket detector O_π on the neighbor's mode, and derived exact rational frame-permanent, parity-split Wigner, and $\frac{3}{2}(4/27)^k$ heralding laws for the state $|\Phi_k\rangle$. Closing the remaining gap between $\eta_{\text{eq}}(4) = 4/13 \approx 0.3077$ and the monogamy floor $1/4$ (likely requiring non-qubit-block POVMs or multi-photon filtering) and closing the I_{3322} Bell-nonlocality gap below $\eta \approx 0.43$ remain compelling targets for future investigation.

References

[1] S. Chakrabarty, *Exact monogamy of steering and remote Wigner negativity for multimode Gaussian nodes*, arXiv:2611.34874v1 (2026).

[2] M. Piani and J. Watrous, *Necessary and sufficient quantum information characterization of Einstein–Podolsky–Rosen steering*, Phys. Rev. Lett. **114**, 060404 (2015).

[3] D. Cavalcanti and P. Skrzypczyk, *Quantum steering: a review with focus on semidefinite programming*, Rep. Prog. Phys. **80**, 024001 (2017).

[4] S. F. D. Waldron, *An Introduction to Finite Tight Frames*, Birkhäuser (2018).