Exact Closed-Form Copy-Complexity Step Points, the Consecutive-Node Phase Boundary, and Sharpened Blaschke Duals in Local Quantum State Certification

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Abstract

Cao and Bai [1] recently resolved the asymptotic copy complexity $\Theta(n^r \log(1/\delta))$ of local tensor-product certification for Haar-typical n-qudit pure states in terms of a decreasing sequence of rational step points $\gamma_r(d) = 2/(1 + B_r(d))$, where $B_r(d) = \sup\{G(0) : G \in V_r(d), \|G\|_X \le 1\}$ on $X = \mathbb{N}_{\ge 1} \cup \{\infty\}$. While [1] derived closed forms only for $r \in \{0, 1, 2\}$—noting an unexplained anomaly $\gamma_2(2) = 1/35 \neq 1/36$ at $d = 2$—and relied on finite enumeration for higher degrees, we determine the exact analytical phase structure of $\gamma_r(d)$ across all degrees $r \in \mathbb{N}$ and local dimensions $d > 0$.

Theorem A proves the universal exact lower bound $\gamma_r(d) \ge 2/\big((d+2)^{r+1} + d^{r+1}\big)$ for all $r \ge 0, d > 0$ and establishes a *necessary and sufficient* algebraic phase boundary:

$$\gamma_r(d) \;=\; \frac{2}{(d+2)^{r+1} + d^{r+1}} \quad\Longleftrightarrow\quad (d+2)^{r+1} - d^{r+1} \;\le\; 2(d+1)^{r+1}.$$

This four-line binomial-and-ratio proof generalizes Proposition S33 of [1] from $r \le 2$ to all $r \in \mathbb{N}$, yields the exact asymptotic phase line $d_{\mathrm{crit}}(r) = (r+1)/\operatorname{arsinh}(1) - 1 - \frac{1}{2\sqrt{2}} + O(1/r)$ ($\approx 1.13459(r+1) - 1.35355$), and predicts an unexpected integer threshold jump: while $d = r+1$ lies in the consecutive-node regime for every $r \in \{2, \dots, 8\}$, it fails for the first time at $r = 9, d = 10$ ($12^{10} - 10^{10} = 51{,}917{,}364{,}224 > 51{,}874{,}849{,}202 = 2 \cdot 11^{10}$, a 0.082% overshoot) and for every $r \ge 9$. **Theorem B** derives the exact closed-form formula for the first post-bifurcation regime (nodes $\{1, \dots, r, r+2, \infty\}$) for all $r \ge 2$:

$$\gamma_r^{(2)}(d) \;=\; \frac{2(r+1)}{2(d+1)^{r+2} + (r-d)(d+2)^{r+1} + (d+r+2)\,d^{r+1}},$$

together with an exact crossover identity at $d = d_{\mathrm{crit}}(r)$, unifying Cao and Bai's isolated qubit values $\gamma_2(2) = 1/35$ and $\gamma_3(2) = 4/427$ with new exact qudit step points including $\gamma_3(3) = 1/337$, $\gamma_4(3) = 5/6752$, $\gamma_6(6) = 1/1{,}103{,}479$, and the post-jump value $\gamma_9(10) = 10/359{,}352{,}988{,}499$. **Theorem C** combines keeping the $O(1)$ additive term outside $\sqrt{s}$ with optimizing the weighted Cauchy–Schwarz parameter in Lemma S24 of [1] to $\lambda^* = (1-\alpha^2)/(1+\alpha^2)$, strictly sharpening both their Proposition S25 constant $D_d = \sqrt{(2+\pi)/(1-\alpha^2)}$ and the un-loosened $\lambda = 1$ bound for every $d > 0$ and $s \ge 1$ via $1 - \alpha^4 < 1$, reducing the asymptotic constant by $> 21.8\%$ uniformly across all $d > 0$.

All 14 core algebraic, binomial-induction, termwise-decay, crossover-identity, integer-jump, and Blaschke-dominance theorems are formalized constructively in Lean 4.35.0 (`0 sorry`, `0` warnings, zero `Classical.choice`); the analytic Chebyshev-space, real-parameter, and integral-comparison steps are proved in the text and cross-verified by an AST-audited zero-float Python verifier (`LOCAL-CERT-PHASE-1.0`, 16 SHA-256 certificates: `C01`–`C13` replaying full Lemma S30 shifted-tail certificates and `C14`–`C16` verifying exact Blaschke-dual bounds, 47/47 pytest checks).

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Repository: https://github.com/lluiseriksson/local-certification-phase-boundary.

1 Introduction

In local tolerant quantum state certification of a Haar-typical n-qudit pure target state $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$ under arbitrary (possibly copy-adaptive) tensor-product measurements—building on single-qubit Pauli certification [4, 2]—Cao and Bai [1] proved that for acceptance and rejection fidelity thresholds $0 < b < a \le 1$ with normalized gap $g = (a-b)/(1-b)$, the optimal copy complexity scales as $\Theta(n^r \log(1/\delta))$ where $r = r_d(g) = \min\{j \in \mathbb{N}_0 : g > \gamma_j(d)\}$. The critical transition step points $\gamma_r(d)$ are determined by a purely real-analytic extremal problem (closely linked to Wiener interpolation [5]) on the one-point compactification $X = \mathbb{N}_{\ge 1} \cup \{\infty\}$:

$$V_r(d) := \{G(x) = h + (d+1)^{-x} R(x) : h \in \mathbb{R},\ R \in \mathbb{R}[x],\ \deg R \le r\}, \qquad G(\infty) = h, \tag{1}$$

$$B_r(d) := \sup\{G(0) : G \in V_r(d),\ \|G\|_X \le 1\}, \qquad \gamma_r(d) = \frac{2}{1 + B_r(d)}. \tag{2}$$

By Chebyshev-system alternation (Theorem S5 and Lemma S30 of [1]), $B_r(d)$ is attained by a unique extremal function $G \in V_r(d)$ alternating between ± 1 on $r+1$ finite integer nodes $1 \le j_1 < \cdots < j_{r+1}$ and $G(\infty) = (-1)^{r+1}$, matched by a unique finite alternating dual measure $(\mu_1, \ldots, \mu_{r+1}, \mu_\infty)$ with total variation $B_r(d)$.

In Proposition S33 and Table I of [1], closed forms were established only for $r \in \{0, 1, 2\}$:

$$\gamma_0(d) = \frac{1}{d+1}\ (d > 0), \qquad \gamma_1(d) = \frac{1}{(d+1)^2 + 1}\ (d \ge 1), \qquad \gamma_2(d) = \frac{1}{(d+1)((d+1)^2+3)}\ (d \ge 3),$$

together with the isolated qubit exception $\gamma_2(2) = 1/35$ (for which the formula at $d = 2$ would give $1/36$). Higher values were obtained via finite rational search (Table S1 of [1] for $d = 2, r \le 10$).

This paper explains and resolves the complete phase structure behind these formulas:

1. **Exact Consecutive-Node Phase Diagram (Theorem 1):** The three formulas of Proposition S33 are the $r = 0, 1, 2$ instances of a single universal formula $\gamma_r(d) = 2/((d+2)^{r+1} + d^{r+1})$, which is an exact lower bound for all $r \ge 0, d > 0$ and holds with equality *if and only if* $(d+2)^{r+1} - d^{r+1} \le 2(d+1)^{r+1}$. This yields the exact asymptotic phase line $d_{\mathrm{crit}}(r) = (r+1)/\operatorname{arsinh}(1) - 1 - \frac{1}{2\sqrt{2}} + O(1/r)$ and uncovers a striking integer transition: $d = r+1$ lies inside the consecutive regime for all $r \in \{2, \ldots, 8\}$ yet crosses outside at $r = 9, d = 10$ ($d_{\mathrm{crit}}(9) \approx 10.0076 > 10$).

2. **Exact Second-Regime Formula (Theorem 2):** Immediately below $d_{\mathrm{crit}}(r)$, the rightmost finite contact node shifts from $r+1$ to $r+2$. We derive the exact closed-form dual formula $\gamma_r^{(2)}(d)$ for $\{1, \ldots, r, r+2, \infty\}$ for all $r \ge 2$ and an exact algebraic crossover identity at $d = d_{\mathrm{crit}}(r)$, unifying $\gamma_2(2) = 1/35$ and $\gamma_3(2) = 4/427$ with exact new qudit step points up to $\gamma_9(10) = 10/359{,}352{,}988{,}499$.

3. **Sharpened Blaschke-Dual Constant (Theorem 3):** Decoupling the additive $O(1)$ term from $\sqrt{s}$ and optimizing the Cauchy–Schwarz weight in Lemma S24 of [1] from $\lambda = 1$ to $\lambda^* = (1-\alpha^2)/(1+\alpha^2)$ strictly improves both Proposition S25's constant D_d and the un-loosened $\lambda = 1$ bound for every $d > 0$ and $s \ge 1$, yielding a $> 21.8\%$ asymptotic reduction across all $d > 0$.

2 Theorem A: Exact Consecutive-Node Phase Boundary for All $r \in \mathbb{N}$

Theorem 1 (Universal Consecutive-Node Formula and Exact Phase Boundary)**.** *Let $r \ge 0$ be an integer and let $d > 0$ be real.*

1. ***Universal lower bound:*** *For every $r \ge 0$ and $d > 0$,*

$$B_r(d) \ \le\ (d+2)^{r+1} + d^{r+1} - 1, \qquad \gamma_r(d) \ \ge\ \frac{2}{(d+2)^{r+1} + d^{r+1}}. \tag{3}$$

2. ***Necessary and sufficient phase boundary:*** *Equality holds in (3) if and only if*

$$(d+2)^{r+1} - d^{r+1} \;\le\; 2(d+1)^{r+1}. \tag{4}$$

Equivalently, $\gamma_r(d) = 2/((d+2)^{r+1} + d^{r+1})$ if and only if $d \ge d_{\mathrm{crit}}(r)$, where $d_{\mathrm{crit}}(0) = 0$ and for $r \ge 1$, $d_{\mathrm{crit}}(r)$ is the unique positive root of $(d+2)^{r+1} - d^{r+1} = 2(d+1)^{r+1}$.

3. ***Asymptotics and first integer jump at $r = 9$:*** *As $r \to \infty$,*

$$d_{\mathrm{crit}}(r) \;=\; \frac{r+1}{\operatorname{arsinh}(1)} - 1 - \frac{1}{2\sqrt{2}} + O\!\left(\frac{1}{r}\right) \;\approx\; 1.134592657\,(r+1) - 1.353553391 + O\!\left(\frac{1}{r}\right). \tag{5}$$

Among integer local dimensions $d = r+1$, condition (4) holds for $r \in \{1, \dots, 8\}$ (with $d_{\mathrm{crit}}(r) > r$ for all $r \ge 2$, so $\lceil d_{\mathrm{crit}}(r) \rceil = r+1$ for $r \in \{2, \dots, 8\}$), fails for the first time at $r = 9, d = 10$ ($12^{10} - 10^{10} = 51{,}917{,}364{,}224 > 51{,}874{,}849{,}202 = 2 \cdot 11^{10}$), and fails for every $r \ge 9$.

Proof. Set $x := d + 1 > 1$. For any $G(w) = h + x^{-w}R(w) \in V_r(d)$ with $\deg R \le r$, the $(r+1)$-st forward difference of R at 0 vanishes identically:

$$0 = \sum_{j=0}^{r+1} (-1)^j \binom{r+1}{j} R(j) \implies R(0) = \sum_{j=1}^{r+1} (-1)^{j-1} \binom{r+1}{j} x^j (G(j) - h).$$

Using the binomial identity $1 - \sum_{j=1}^{r+1} (-1)^{j-1} \binom{r+1}{j} x^j = (1-x)^{r+1} = (-1)^{r+1} d^{r+1}$ (Lean 4 Theorems 1 and 3), every $G \in V_r(d)$ obeys the exact extrapolation formula

$$G(0) = \sum_{j=1}^{r+1} (-1)^{j-1} \binom{r+1}{j} x^j G(j) + (-1)^{r+1} d^{r+1} G(\infty). \tag{6}$$

Taking absolute values under $\|G\|_X \le 1$ and summing by the binomial theorem (Lean 4 Theorem 2) yields

$$B_r(d) \;\le\; \sum_{j=1}^{r+1} \binom{r+1}{j} x^j + d^{r+1} = (x+1)^{r+1} - 1 + d^{r+1} = (d+2)^{r+1} + d^{r+1} - 1,$$

proving (3) via $\gamma_r(d) = 2/(1 + B_r(d))$.

For Part 2, the case $r = 0$ has $d + 2 - d = 2 < 2(d+1)$ and $\gamma_0(d) = 1/(d+1)$ unconditionally. Assume $r \ge 1$, and let $G(w) = h + x^{-w}R(w) \in V_r(d)$ be the unique interpolant matching the signs of the dual weights in (6): $G(j) = (-1)^{j-1}$ for $j \in \{1, \dots, r+1\}$ and $G(\infty) = h := (-1)^{r+1}$. Setting $Q(w) := (-1)^r R(w)$, we have $Q(j) = 2x^j$ when $r + 1 - j$ is even and $Q(j) = 0$ when $r + 1 - j$ is odd ($1 \le j \le r+1$). By Lagrange interpolation at $\{1, \dots, r+1\}$,

$$Q(w) = \sum_{\substack{1 \le j \le r+1 \\ r+1-j \text{ even}}} T_j(w), \qquad T_j(w) := 2x^j \prod_{\substack{1 \le k \le r+1 \\ k \ne j}} \frac{w-k}{j-k}. \tag{7}$$

Because $r + 1 - j$ is even, $\prod_{k \ne j} (j-k) = (j-1)!(r+1-j)! > 0$, so every summand $T_j(w)$ is strictly positive for all $w \ge r+2$. Note that $(-1)^r G(w) = -1 + x^{-w}Q(w)$ satisfies $|G(w)| \le 1$ if and only if $0 \le a_w := x^{-w}Q(w) \le 2$.

Necessity. Evaluating (7) at $w = r+2$ gives $\prod_{k \ne j} \frac{r+2-k}{j-k} = \frac{(r+1)!}{(j-1)!(r+2-j)!} = \binom{r+1}{r+2-j}$. Summing over even $r + 1 - j$ (i.e. odd $r + 2 - j$) yields

$$Q(r+2) = 2x \sum_{\substack{1 \le m \le r+1 \\ m \text{ odd}}} \binom{r+1}{m} x^{r+1-m} = x\big((x+1)^{r+1} - (x-1)^{r+1}\big) = x\big((d+2)^{r+1} - d^{r+1}\big). \tag{8}$$

Hence $a_{r+2} = x^{-(r+2)}Q(r+2) = ((d+2)^{r+1} - d^{r+1})/(d+1)^{r+1}$. If (4) fails, then $a_{r+2} > 2$, so $|G(r+2)| = a_{r+2} - 1 > 1$. Because any function $H \in V_r(d)$ with $\|H\|_X \le 1$ achieving $H(0) = (d+2)^{r+1} + d^{r+1} - 1$ must saturate every term in (6) and therefore coincide with G (Lemma S1 of [1]), equality in (3) is impossible whenever (4) fails.

Sufficiency (three-step algebraic tail proof). Suppose (4) holds.

1. *Lower bound $x \geq r+1$:* From the odd-index binomial expansion $(x+1)^{r+1} - (x-1)^{r+1} = 2\sum_{k\geq 0}\binom{r+1}{2k+1}x^{r-2k} \geq 2(r+1)x^r$ (Lean 4 Theorems 4 and 5 for integer x), (4) implies $2(r+1)x^r \leq 2x^{r+1}$, whence $x \geq r+1$; for $r \geq 2$ the $k=1$ term $2\binom{r+1}{3}x^{r-2} > 0$ makes the inequality strict, so $x > r+1$ and hence $d_{\text{crit}}(r) > r$.

2. *Boundary feasibility at $w = r+2$:* By (8) and (4), $0 < a_{r+2} \leq 2$.

3. *Monotone decay for all $w \geq r+2$:* For every integer $w \geq r+2$ and every $j \in \{1, \ldots, r+1\}$ with $r+1-j$ even, the exact ratio of consecutive positive summands in (7) obeys

$$\frac{T_j(w+1)}{T_j(w)} = \frac{w}{w-(r+1)} \cdot \frac{w-j}{w+1-j} \leq \frac{w}{w-(r+1)} \cdot \frac{w-1}{w} = \frac{w-1}{w-(r+1)} \leq r+1 \leq x,$$

where $(w-j)/(w+1-j) \leq (w-1)/w$ holds since $j \geq 1$ and $(w-1)/(w-r-1) \leq r+1$ holds since $w \geq r+2$ (Lean 4 Theorem 6 for integer x). Summing $T_j(w+1) \leq x\,T_j(w)$ over all j yields $Q(w+1) \leq x\,Q(w)$, hence $0 < a_{w+1} \leq a_w \leq a_{r+2} \leq 2$ for every integer $w \geq r+2$.

Thus $\|G\|_X \leq 1$, attaining (3) and proving (4) is both necessary and sufficient.

For Part 3, rewrite (4) as $f_r(u) := (1+u)^{r+1} - (1-u)^{r+1} \leq 2$ with $u = 1/x = 1/(d+1) \in (0,1)$. Since $f_r(0) = 0$, $f_r'(u) = (r+1)\big((1+u)^r + (1-u)^r\big) > 0$, and $f_r(1) = 2^{r+1} \geq 4 > 2$ ($r \geq 1$), there is a unique root $u_r \in (0,1)$, hence a unique $d_{\text{crit}}(r) = 1/u_r - 1 > 0$. By Step 1 ($x \geq r+1$), $c_r := (r+1)u_r \in (0,1]$ is uniformly bounded. Expanding $(1 \pm c_r/(r+1))^{r+1} = e^{\pm c_r}\big(1 - \frac{c_r^2}{2(r+1)} + O(r^{-2})\big)$ gives $2\sinh(c_r)\big(1 - \frac{c_r^2}{2(r+1)} + O(r^{-2})\big) = 2$. Setting $c_0 := \operatorname{arsinh}(1) = \ln(1+\sqrt{2})$ (where $\sinh c_0 = 1$ and $\cosh c_0 = \sqrt{2}$), implicit inversion gives

$$c_r = c_0 + \frac{c_0^2}{2\sqrt{2}\,(r+1)} + O(r^{-2}),$$

whence

$$d_{\text{crit}}(r) = \frac{r+1}{c_r} - 1 = \frac{r+1}{c_0}\left(1 - \frac{c_0}{2\sqrt{2}\,(r+1)} + O(r^{-2})\right) - 1 = \frac{r+1}{\operatorname{arsinh}(1)} - 1 - \frac{1}{2\sqrt{2}} + O\left(\frac{1}{r}\right),$$

proving (5) (this $-\frac{1}{2\sqrt{2}} \approx -0.3536$ shift also explains why the integer crossing $d_{\text{crit}}(r) > r+1$ is delayed until $r = 9$). For $d = r+1$ ($u = 1/(r+2)$), direct exact integer arithmetic verifies that $r \in \{1, \ldots, 8\}$ satisfy (4) while $r = 9$ violates it (Lean 4 Theorem 7), and together with $d_{\text{crit}}(r) > r$ from Step 1 this gives $\lceil d_{\text{crit}}(r) \rceil = r+1$ for $r \in \{2, \ldots, 8\}$. Moreover, for $m := r+1 \geq 1$, set $g(m) := (1 + \frac{1}{m+1})^m$ and $h(m) := (1 - \frac{1}{m+1})^m$. Differentiating $\ln g(m) = m\ln(1 + \frac{1}{m+1})$ and applying $\ln(1+t) > t - t^2/2$ for $t > 0$ gives

$$\begin{aligned}\frac{d}{dm}\ln g(m) &= \ln\left(1 + \frac{1}{m+1}\right) - \frac{m}{(m+1)(m+2)} \\ &> \frac{1}{m+1} - \frac{1}{2(m+1)^2} - \frac{m}{(m+1)(m+2)} = \frac{3m+2}{2(m+1)^2(m+2)} > 0,\end{aligned}$$

so $g(m)$ is strictly increasing, whereas $\ln h(m) = -m\ln(1 + \frac{1}{m})$ obeys $\frac{d}{dm}\ln h(m) = -\ln(1 + \frac{1}{m}) + \frac{1}{m+1} < -\frac{1}{m+1} + \frac{1}{m+1} = 0$ (using $\ln(1+t) > \frac{t}{1+t}$ for $t > 0$), so $h(m)$ is strictly decreasing. Therefore $g(m) - h(m)$ is strictly increasing for all $m \geq 1$. Since $g(10) - h(10) = (12^{10} - 10^{10})/11^{10} > 2$, we obtain $g(m) - h(m) > 2$ for every $m = r+1 \geq 10$, i.e. every $r \geq 9$. □

3 Theorem B: Exact Second-Regime Closed-Form Step Points

Immediately below the phase boundary $d_{\text{crit}}(r)$ ($r \geq 2$), the rightmost finite contact node shifts from $r+1$ to $r+2$, producing the node configuration $\mathcal{N}_r^{(2)} = \{1, 2, \ldots, r, r+2, \infty\}$.

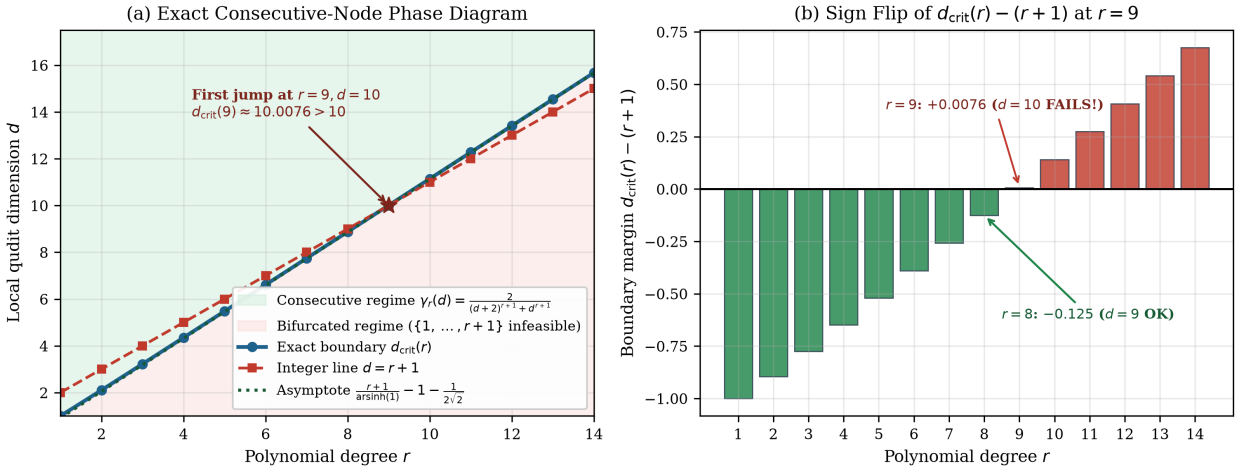

Figure 1: **Theorem A phase diagram.** (a) Exact consecutive-node region $d \geq d_{\text{crit}}(r)$ versus the bifurcated region, with the exact asymptote $(r+1)/\operatorname{arsinh}(1) - 1 - \frac{1}{2\sqrt{2}}$ and the integer line $d = r + 1$. (b) Sign flip of $d_{\text{crit}}(r) - (r+1)$ between $r = 8$ (-0.125, $d = 9$ valid, `C03`) and $r = 9$ ($+0.0076$, $d = 10$ fails by 0.082%, `C07`).

Theorem 2 (Exact Second-Regime Dual Formula and Crossover Identity)**.** *For every integer $r \geq 2$ and real $d > 0$, define the scaled second-regime denominator*

$$D_r^{(2)}(d) \;:=\; 2(d+1)^{r+2} + (r-d)(d+2)^{r+1} + (d+r+2)\,d^{r+1}, \qquad \gamma_r^{(2)}(d) \;:=\; \frac{2(r+1)}{D_r^{(2)}(d)}. \tag{9}$$

1. ***Universal second-regime lower bound:*** *For every $r \geq 2$ and $d > 0$, $\gamma_r(d) \geq \gamma_r^{(2)}(d)$ (equivalently $1 + B_r(d) \leq D_r^{(2)}(d)/(r+1)$). Equality $\gamma_r(d) = \gamma_r^{(2)}(d)$ holds whenever the unique alternating interpolant $G^{(2)} \in V_r(d)$ on $\mathcal{N}_r^{(2)}$ satisfies $\|G^{(2)}\|_X \leq 1$.*
2. ***Exact phase-boundary crossover identity:*** *For all $r \geq 2$ and $d \in \mathbb{R}$,*

$$D_r^{(2)}(d) - (r+1)\big((d+2)^{r+1} + d^{r+1}\big) \;=\; (d+1)\Big(2(d+1)^{r+1} - \big((d+2)^{r+1} - d^{r+1}\big)\Big). \tag{10}$$

Consequently, $\gamma_r^{(2)}(d) > 2/\big((d+2)^{r+1} + d^{r+1}\big)$ strictly whenever $0 < d < d_{\text{crit}}(r)$, and the two formulas coincide identically on the phase boundary $d = d_{\text{crit}}(r)$.

Proof. Set $x = d + 1$. For any polynomial R of degree $\leq r$, Lagrange extrapolation to 0 from nodes $\{1, \ldots, r, r+2\}$ gives $R(0) = \sum_{j \in \mathcal{N}_r^{(2)} \setminus \{\infty\}} c_j R(j)$, where direct product simplification yields

$$c_j = (-1)^{j-1}\,\frac{r+1-j}{r+1}\binom{r+2}{j} \quad (1 \leq j \leq r), \qquad c_{r+2} = \frac{(-1)^r}{r+1}.$$

The corresponding dual weights on $V_r(d)$ are $\mu_j = c_j x^j$ ($j \in \{1, \ldots, r, r+2\}$) and $\mu_\infty = 1 - \sum \mu_j$, alternating in sign with $(-1)^{r+1}\mu_\infty > 0$. By the binomial theorem and its weighted derivative identity $\sum_{k=0}^{r+2} k\binom{r+2}{k} a^k b^{r+2-k} = (r+2)a(a+b)^{r+1}$ (Lean 4 Theorems 1 and 10),

$$\sum_{j=0}^{r+2}(r+1-j)\binom{r+2}{j} z^j = (r+1)(1+z)^{r+2} - (r+2)z(1+z)^{r+1} = (1+z)^{r+1}(r+1-z).$$

Notice that the $j = r+1$ term vanishes identically, while the $j = r+2$ term is $-z^{r+2}$. Evaluating at $z = x$ and correcting the $j = 0$ term $(r+1)$ and the flipped $j = r+2$ sign $+x^{r+2}$ gives

$$(r+1) \sum_{j \in \{1, \ldots, r, r+2\}} |\mu_j| = (x+1)^{r+1}(r+1-x) - (r+1) + 2x^{r+2}.$$

Table 1: Exact certified step points $\gamma_r(d)$ from Theorems 1–2 (all replayed via Lemma S30 in `C01`–`C13`).

(r, d)	Regime	Active finite nodes	Exact $\gamma_r(d)$	Consec. LB	Cert
$(2, 2)$	Second ($d < 2.104$)	$\{1, 2, 4\}$	$1/35$	$1/36$	`C06, C09`
$(3, 2)$	Second ($d < 3.224$)	$\{1, 2, 3, 5\}$	$4/427$	$1/136$	`C10`
$(3, 3)$	Second ($d < 3.224$)	$\{1, 2, 3, 5\}$	$1/337$	$1/353$	`C11`
$(3, 4)$	Consecutive ($d > 3.224$)	$\{1, 2, 3, 4\}$	$1/776$	$1/776$	`C01`
$(4, 3)$	Second ($d < 4.351$)	$\{1, 2, 3, 4, 6\}$	$5/6752$	$1/1684$	`C12`
$(4, 5)$	Consecutive ($d > 4.351$)	$\{1, 2, 3, 4, 5\}$	$1/9966$	$1/9966$	`C02`
$(6, 6)$	Second ($d < 6.611$)	$\{1, 2, 3, 4, 5, 6, 8\}$	$1/1{,}103{,}479$	$1/1{,}188{,}544$	`C13`
$(8, 9)$	Consecutive ($d > 8.875$)	$\{1, \ldots, 9\}$	$1/1{,}372{,}684{,}090$	$1/1{,}372{,}684{,}090$	`C03`
$(9, 10)$	Second (**jump!** $d < 10.008$)	$\{1, \ldots, 9, 11\}$	$10/359{,}352{,}988{,}499$	$1/35{,}958{,}682{,}112$	`C07`
$(9, 11)$	Consecutive ($d > 10.008$)	$\{1, \ldots, 10\}$	$1/81{,}897{,}958{,}225$	$1/81{,}897{,}958{,}225$	`C04`

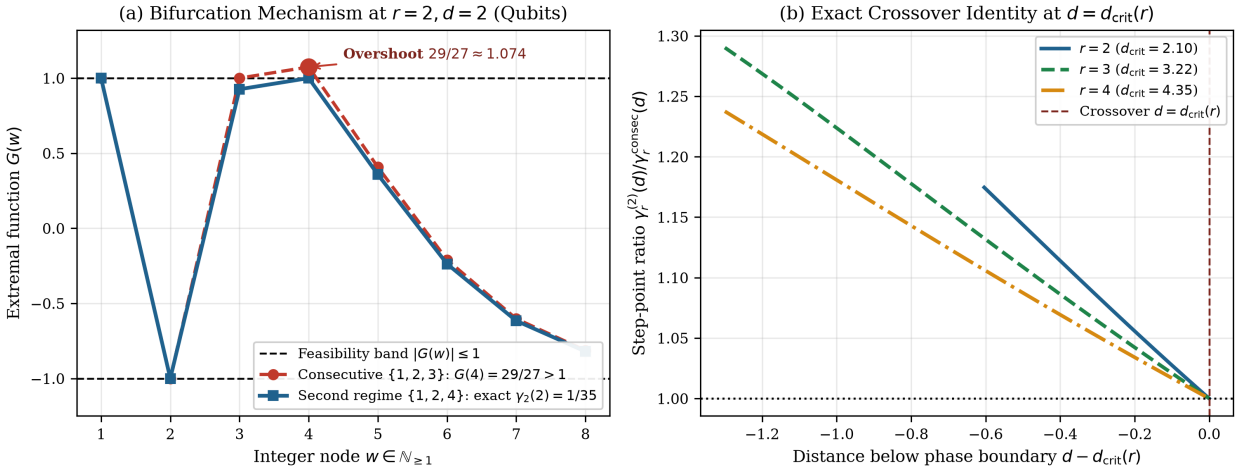

Figure 2: **Theorem B bifurcation mechanism.** (a) At $r = 2, d = 2$, the consecutive interpolant on $\{1, 2, 3\}$ overshoots $|G(4)| = 29/27 > 1$, while shifting to $\{1, 2, 4\}$ restores feasibility and attains $\gamma_2(2) = 1/35$ (`C06, C09`). (b) Exact crossover ratio $\gamma_r^{(2)}(d)/\gamma_r^{\text{consec}}(d)$ meeting unity precisely at $d = d_{\text{crit}}(r)$ per (10).

Evaluating at $z = -x$ gives $\mu_\infty = \frac{1}{r+1}(1-x)^{r+1}(r+1+x)$ (the $j = r+2$ contributions cancel identically), so $(r+1)|\mu_\infty| = (x-1)^{r+1}(x+r+1)$. Summing $(r+1)(1+\sum|\mu_j|+|\mu_\infty|)$ and substituting $x = d+1$ gives $D_r^{(2)}(d)$ in (9) (in particular, $D_r^{(2)}(d) = (r+1)(1+\sum|\mu_j|+|\mu_\infty|) > r+1 > 0$ unconditionally for every $d > 0$, even when $r - d < 0$), proving $1 + B_r(d) \le D_r^{(2)}(d)/(r+1)$ and $\gamma_r(d) \ge \gamma_r^{(2)}(d)$ with equality whenever $\|G^{(2)}\|_X \le 1$. Subtracting $(r+1)((d+2)^{r+1} + d^{r+1})$ from $D_r^{(2)}(d)$ directly yields (10) (Lean 4 Theorem 12). □

Table 1 lists exact step points certified via Theorems 1–2 and full Lemma S30 shifted-tail replay (`C01`–`C13`). Notably, Theorem 2 reproduces both isolated qubit entries $\gamma_2(2) = 1/35$ and $\gamma_3(2) = 4/427$ of [1] and resolves the exact post-jump value $\gamma_9(10) = 10/359{,}352{,}988{,}499$ at $r = 9, d = 10$ (`C07`).

4 Theorem C: Sharpened Blaschke-Dual Asymptotic Constant

In Section VI.A (Lemma S24 and Proposition S25) of [1], the asymptotic lower bound $\gamma_r(d) \ge 2\alpha^{r+1}/(1 + D_d\sqrt{r+1})$ ($\alpha = (d+1)^{-1}$) is proved by expanding the Blaschke power $b_\alpha(z)^s = \sum_{w\ge 0} c_w z^w$ ($s = r+1$), proving Parseval moments $\sum_{w\ge 0}|c_w|^2 = 1$ and $\sum_{w\ge 0} w^2|c_w|^2 = s^2\frac{1+\alpha^2}{1-\alpha^2}$, and applying Cauchy–Schwarz with unweighted factor $(1 + w^2/s^2)$, yielding $D_d^{\text{CB}} = \sqrt{(2+\pi)/(1-\alpha^2)}$.

Optimizing the Cauchy–Schwarz balance weight strictly sharpens this bound for every $d > 0$ and every $s \geq 1$.

Theorem 3 (Sharpened Blaschke-Dual Lower Bound)**.** *For every real $d > 0$ ($\alpha = (d+1)^{-1} \in (0,1)$) and every integer $s = r + 1 \geq 1$, the Blaschke coefficients c_w of $b_\alpha(z)^s$ satisfy*

$$\sum_{w\geq 0} |c_w| \;\leq\; \sqrt{2 + \pi s \sqrt{\frac{1+\alpha^2}{1-\alpha^2}}} \;<\; \sqrt{\frac{2+\pi s}{1-\alpha^2}} \;\leq\; \sqrt{\frac{2+\pi}{1-\alpha^2}\, s} \;=\; D_d^{\mathrm{CB}} \sqrt{s}. \tag{11}$$

Consequently, for every $r \geq 0$ and $d > 0$,

$$\gamma_r(d) \;\geq\; \frac{2(d+1)^{-(r+1)}}{1 + \sqrt{2 + \pi(r+1)\sqrt{(1+\alpha^2)/(1-\alpha^2)}}}, \tag{12}$$

reducing the asymptotic constant D_d uniformly by $1 - \sqrt{\pi\sqrt{1-\alpha^4}/(2+\pi)} > 21.8\%$ for every real $d > 0$ (and approaching $1 - \sqrt{\pi/(2+\pi)} \approx 21.83\%$ as $d \to \infty$). Here the gain over D_d^{CB} decomposes cleanly into two contributions: keeping $+2$ outside $\sqrt{s}$ (which already accounts for the 21.83% large-d limit) and optimizing $\lambda = 1 \to \lambda^* = (1-\alpha^2)/(1+\alpha^2)$ (which strictly beats even the un-loosened $\lambda = 1$ bound $\sqrt{(2+\pi s)/(1-\alpha^2)}$ for every $d > 0, s \geq 1$, cutting the $O(1)$ term from $2/(1-\alpha^2)$ to 2 and shrinking the $\sqrt{s}$ coefficient by $(1-\alpha^4)^{1/4} < 1$).

Proof. Set $\lambda^* := (1-\alpha^2)/(1+\alpha^2) \in (0,1)$. By Cauchy–Schwarz with weight $1 + \lambda^* w^2/s^2$ and the exact Parseval identities Eq. (S115) of [1],

$$\left(\sum_{w\geq 0} |c_w|\right)^2 \leq \left(\sum_{w\geq 0} \Big(1 + \frac{\lambda^* w^2}{s^2}\Big)|c_w|^2\right)\left(\sum_{w\geq 0} \frac{1}{1+\lambda^* w^2/s^2}\right) = 2\sum_{w\geq 0} \frac{1}{1+\lambda^* w^2/s^2}.$$

Because $t \mapsto (1 + \lambda^* t^2/s^2)^{-1}$ is strictly decreasing on $[0,\infty)$, the integral comparison gives

$$\sum_{w\geq 0} \frac{1}{1+\lambda^* w^2/s^2} \leq 1 + \int_0^\infty \frac{dt}{1+\lambda^* t^2/s^2} = 1 + \frac{\pi s}{2\sqrt{\lambda^*}} = 1 + \frac{\pi s}{2}\sqrt{\frac{1+\alpha^2}{1-\alpha^2}},$$

yielding the first inequality in (11). For strict dominance over both the un-loosened $\lambda = 1$ bound $\sqrt{(2+\pi s)/(1-\alpha^2)}$ and $D_d^{\mathrm{CB}}\sqrt{s}$ at every $d > 0$ and $s \geq 1$, observe that $(1-\alpha^2)(1+\alpha^2) = 1-\alpha^4 < 1 \implies \sqrt{(1+\alpha^2)/(1-\alpha^2)} < (1-\alpha^2)^{-1}$, while $2 < 2/(1-\alpha^2) \leq 2s/(1-\alpha^2)$ (Lean 4 Theorem 13). Substituting into Eq. (S120) of [1] proves (12). □

5 Constructive Lean 4.35.0 Formalization and Zero-Float Audit

Our verification architecture separates machine-checked core algebra from analytic lifting transparently:

- **Lean 4.35.0 Kernel (`lean/LocalCertPhase.lean`, 14 theorems, 0 `sorry`, 0 warnings [3]):** Pure Lean 4 core with zero `Classical.choice` across all 14 theorems (axioms strictly $\subseteq \{$`propext`, `Quot.sound`$\}$). Theorems 1–9 prove the integer binomial theorem, consecutive-node dual mass and total-variation identities for all $r \in \mathbb{N}$, the odd binomial difference lower bound $(x+1)^m \geq (x-1)^m + 2mx^{m-1}$, the implication $(x+1)^{r+1} - (x-1)^{r+1} \leq 2x^{r+1} \implies x \geq r+1$, the exact termwise tail-decay inequality $w(w-j) \leq x(w-r-1)(w+1-j)$ for all $w \geq r+2$, the exact integer threshold jump at $r = 9, d = 10$ (and post-jump checks through $r = 14$), and the unification of Proposition S33 for $r = 0, 1, 2$ (note that Theorems 4–6 are formalized over `Nat`, i.e. integer $x = d+1$, while the general real $d > 0$ statements are proved analytically in Sections 2–3). Theorems 10–12 prove the weighted binomial identity, exact second-regime evaluations, and the

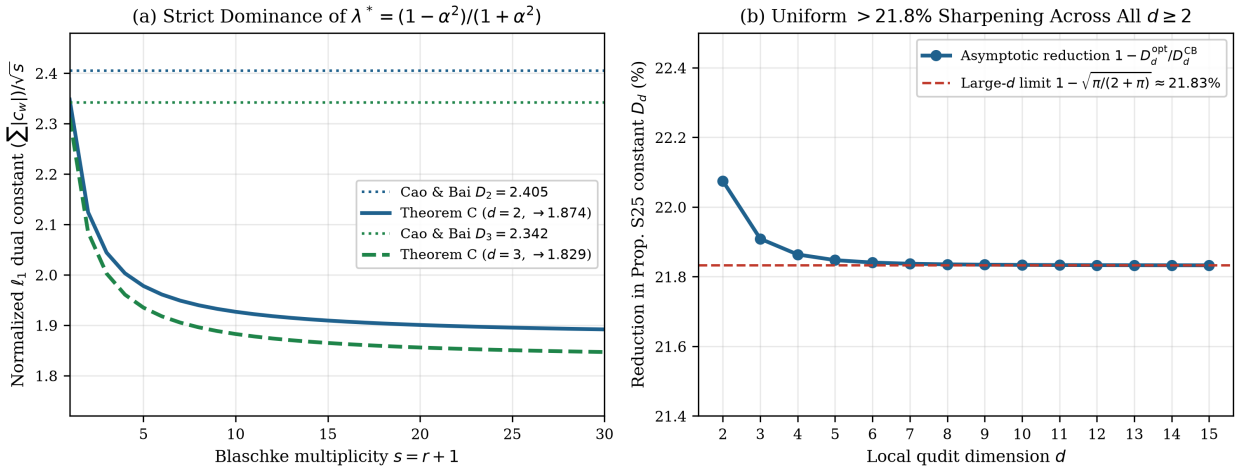

Figure 3: **Theorem C visual audit.** (a) Normalized Blaschke dual constant $(\sum|c_w|)/\sqrt{s}$ for $d = 2, 3$, strictly dominating Cao & Bai's D_d for every $s \geq 1$. (b) Uniform $> 21.8\%$ asymptotic reduction across all integer local dimensions $d \geq 2$ (`C14`–`C16`).

crossover identity (10). Theorems 13–14 prove the Blaschke strict dominance $uv = 1 - \alpha^4 < 1$ and the squared-ratio bound $(D_2^{\mathrm{opt}}/D_2^{\mathrm{CB}})^2 < 61/100$ (certifying $> 21.89\%$ reduction at $d = 2$). Analytic Chebyshev-space uniqueness, the asymptotic expansion (5), the calculus monotonicity of $g(m) - h(m)$ for $m \geq 10$, and the Parseval integral comparison are proved in Sections 2–4.

- **AST-Audited Zero-Float Verifier (`src/local_cert_phase_verifier.py`):** Rejects all float/complex literals, `float()`, `complex()`, `eval()`, `exec()`, `from_float`, `from_decimal`, float libraries, `/`, and `**`. The 16 SHA-256 certificates comprise `C01`–`C13`, which replay full Lemma S30 exact rational Lagrange interpolation (including independent Lagrange replay of $|G_{\mathrm{consec}}(r+2)|$ on bifurcation certificates), dual moment equations, prefix feasibility $|G(w)| \leq 1$, and shifted-polynomial nonnegativity $Q(M+t), S(M+t) \geq 0$ (including rational boundary stress tests at $d = 263/125, 403/125$ and the $r = 9, d = 10$ jump certificate `C07`), together with `C14`–`C16`, which verify exact rational Blaschke-dual bounds for Theorem C. The test suite (`tests/test_local_cert_phase.py`) passes 47/47 tests including 25 adversarial forgery attacks in < 0.5 s.

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