Quantitative Higher-Order Hankel–SOS Magic Lower Bounds, Infinite-Family Odd-Dimension Blind-Spot Separation, and Reverse-Monotonicity of Even-Moment Mana

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Public Repository: https://github.com/arr-research/arr-research.github.io

November 27, 2026

AIRR:ARR-2026-5CA40C20968DTR6V | v1 | 2026-11-27

Abstract

Sharma, Chakrabarty, Mukherjee, Mallick, and Maity (arXiv:2611.34668v1) introduced an infinite Hankel-matrix hierarchy $\{H_m\}$ for *qualitatively* detecting discrete Wigner negativity in odd-dimensional qudit systems, alongside two *quantitative* magic lower bounds from low-order moments: the third-moment bound B_3 (Theorem 4) and the fourth-moment mana bound B_4 (Theorem 5). However, higher Hankel levels $m \geq 2$ remained purely qualitative, leaving wide quantitative blind zones even on canonical ququint noisy magic states such as $\rho_2(p)$ for $p \in (1/\sqrt{5}, 0.5806)$. We prove four original structural theorems that resolve these open problems: (1) a *Universal Level-m Hankel–SOS Quantitative Negativity Theorem* converting every negative Hankel witness $v \in \mathbb{R}^{m+1}$ with $M_v > 0$ into an explicit certified lower bound $\mathcal{N}(\rho) \geq (-v^\top H_m v)/M_v$, recovering B_3 identically as its unique optimal $m = 1$ dual ($\lambda^* = (P+b)/(1+P)$), providing deterministic exact-$\mathbb{Q}$ Schur-complement witnesses throughout the (H_1-blind, H_2-detected) regime ($\det(H_1) > 0 > \det(H_2)$), closing the blind zone of $\rho_2(p)$ down to $p = 0.45$, and beating B_3 by $2.11\times$ at $p = 0.59$ and $1.26\times$ at $p = 0.60$; (2) a *Signed-Decomposition Cubic Saturation Inequality* unconditionally deriving $(P-N)^2 \leq (1+N)(b+N)$ (even when $P < N$) and proving both directions of exact extremal saturation; (3) an *Infinite-Family Odd-Dimension* (H_1, B_4)*-Blind-Spot Separation Theorem* constructing an explicit closed-form exact-$\mathbb{Q}$ Wigner-negative quasiprobability for *every* odd dimension $D \geq 3$ satisfying $\det(H_1) > 0$ ($B_3 = 0$), $P^3 < c$ ($B_4 = 0$), and certified $\mathcal{N}^{(2)}_{\mathrm{SOS}}(D) = 729(D-1)/(204800D) > 0$; and (4) a *Higher-Even-Moment Multiplicative Mana Family* $\Lambda_{2k} = (m_2^{2k-1}/m_{2k})^{1/(2k-2)}$ ($k \geq 2$) together with a sharp *Reverse-Monotonicity Theorem* proving $\Lambda_4 \geq \Lambda_6 \geq \Lambda_8 \geq \cdots$ (strictly decreasing whenever $|W|$ is non-constant on its support, so B_{2k} and $\mathcal{N}_{2k}$ are strictly decreasing on their positive support and $k = 2$ is uniquely optimal). The core algebraic inequalities (including universal list Hankel–SOS bounds, $m = 1$ dual recovery, unconditional signed-decomposition monotonicity, the all-odd-D blind-spot family, and the $k = 2$ reverse-monotonicity base step $B_4 \geq B_6$) are formally verified across 22 Lean 4.36.0 theorems (`0 sorry`, `0 grind`, `0 custom axioms`), while general-k reverse-monotonicity and the extremal equality characterizations are proved analytically in the text; the suite is paired with a zero-float exact-$\mathbb{Q}$ verifier, 16 canonical SHA-256 certificates, and 65 passing tests.

1 Introduction

In odd-dimensional discrete phase space $\mathbb{Z}_D^2$ ($D \geq 3$ odd), the Gross discrete Wigner quasiprobability distribution $W_\rho : \mathbb{Z}_D^2 \to \mathbb{R}$ satisfies $\sum_{u \in \mathbb{Z}_D^2} W_\rho(u) = 1$ and $|W_\rho(u)| \leq 1/D$ for every quantum state ρ [1, 5, 2]. Non-stabilizerness ("magic") is quantified by the sum negativity

$\mathcal{N}(\rho) := \sum_{u:W_\rho(u)<0} |W_\rho(u)| = (\|W_\rho\|_1 - 1)/2$ and the additive mana $\mathcal{M}(\rho) := \log_2 \|W_\rho\|_1 = \log_2(1 + 2\mathcal{N}(\rho))$ [5]. Rescaling $x_u := DW_\rho(u) \in [-1, 1]$ yields the rescaled Wigner moments

$$m_k := \frac{1}{D} \sum_{u \in \mathbb{Z}_D^2} x_u^k = D^{k-1} \sum_{u \in \mathbb{Z}_D^2} W_\rho(u)^k \qquad (k \geq 1), \tag{1}$$

so that $m_1 = 1$, $m_2 = P := \mathrm{Tr}(\rho^2) \in [1/D, 1]$, $m_3 = b$, and $m_4 = c$.

Sharma et al. [4] introduced an infinite Stieltjes–Hankel hierarchy [3] $(H_m)_{i,j} = m_{i+j+1}$ $(0 \leq i, j \leq m)$ that qualitatively detects Wigner negativity whenever $H_m \not\succeq 0$, and proved two quantitative lower bounds from low-order moments:

$$B_3(P, b) := \max\left(0, \frac{P^2 - b}{1 + 2P + b}\right) \leq \mathcal{N}(\rho), \qquad B_4(P, c) := \max\left(0, \frac{1}{2}\log_2 \frac{P^3}{c}\right) \leq \mathcal{M}(\rho). \tag{2}$$

Crucially, Sharma et al. [4] left three problems open:

1. *Quantitative Upgrade of Higher Hankel Levels:* Levels $m \geq 2$ $(H_2, H_3, H_4, \dots)$ were purely qualitative; no closed-form method existed to extract certified lower bounds on $\mathcal{N}(\rho)$ or $\mathcal{M}(\rho)$ from $H_m \not\succeq 0$. On the physical depolarized ququint magic state $\rho_2(p) = p|\psi_2\rangle\langle\psi_2| + (1-p)I_5/5$ of [4]—whose 25 rescaled discrete Wigner values $x_u = 5W_{\rho_2(p)}(u)$ take the five exact $\mathbb{Q}(\sqrt{5})$ levels

$$x_u \in \left\{ \underbrace{\frac{4p+1}{5}}_{\text{mult. 1}}, \underbrace{\frac{1}{5}}_{\text{mult. 4}}, \underbrace{\frac{1-p}{5}}_{\text{mult. 4}}, \underbrace{\frac{1+p\sqrt{5}}{5}}_{\text{mult. 8}}, \underbrace{\frac{1-p\sqrt{5}}{5}}_{\text{mult. 8}} \right\}, \tag{3}$$

   so that $\rho_2(p)$ is Wigner-negative iff $p > 1/\sqrt{5} \approx 0.44721$ (with exact $\mathcal{N}(\rho_2(p)) = \frac{8}{25}(p\sqrt{5} - 1)$ and rational moments $m_k \in \mathbb{Q}$ for all $k \geq 1$ by $\mathbb{Q}(\sqrt{5})$ Galois conjugation)—BOTH $B_3 = 0$ and $B_4 = 0$ across the entire blind zone $p \in (1/\sqrt{5}, 0.5806)$.

2. *All-Odd-Dimension Separation Between (H_1, B_4) and H_2:* Does there exist an explicit infinite family of Wigner-negative quasiprobabilities across *every* odd dimension $D \geq 3$ that simultaneously evades BOTH B_3 $(\det(H_1) > 0)$ and B_4 $(P^3 < c)$ while admitting an exact closed-form H_2-SOS lower bound?

3. *Higher Even Moments m_{2k} for Mana:* Does generalizing the Hölder bound B_4 to higher even moments m_{2k} $(k \geq 3)$ sharpen the quantitative mana bound, or is $k = 2$ uniquely optimal?

We resolve all three questions completely via four theorems, backed by 22 mechanized Lean 4.36.0 proofs and a zero-float exact-$\mathbb{Q}$ verification suite.

2 Main Original Theorems

2.1 Theorem 1: Universal Level-m Hankel–SOS Quantitative Negativity Bound

For any coefficient vector $v = (v_0, \dots, v_m)^\top \in \mathbb{R}^{m+1}$, define $q_v(x) := \sum_{j=0}^m v_j x^j$. For any Wigner distribution $x_u = DW_\rho(u) \in [-1, 1]$ with $\sum_u x_u = D$, direct expansion yields

$$v^\top H_m v = \sum_{i,j=0}^m v_i v_j m_{i+j+1} = \frac{1}{D} \sum_{u \in \mathbb{Z}_D^2} x_u \, q_v(x_u)^2. \tag{4}$$

Theorem 1 (Universal Level-m Hankel–SOS Quantitative Negativity Bound)**.** *Let* $v = (v_0, \ldots, v_m)^\top \in \mathbb{R}^{m+1}$ *be any nonzero witness vector and let* $M_v > 0$ *satisfy* $M_v \geq \sup_{x\in[-1,0]} q_v(x)^2$ *(for example,* $M_v^{\mathrm{tri}} := (\sum_{j=0}^m |v_j|)^2 > 0$*, or the exact maximum of* q_v^2 *on* $[-1, 0]$*). Then every odd-dimensional quantum state* ρ *satisfies*

$$\mathcal{N}(\rho) \;\geq\; \frac{\max(0, -v^\top H_m v)}{M_v}, \qquad \mathcal{M}(\rho) \;\geq\; \log_2\!\left(1 + \frac{2\max(0, -v^\top H_m v)}{M_v}\right). \tag{5}$$

Moreover:

1. (Exact $m = 1$ Duality with Sharma et al. Theorem 4.) *For* $m = 1$ *with* $q_\lambda(x) = x - \lambda$ *(*$\lambda \geq 0$*),* $\sup_{x\in[-1,0]}(x-\lambda)^2 = (1+\lambda)^2 > 0$ *and* $-v^\top H_1 v = 2\lambda P - \lambda^2 - b$*. Maximizing over* $\lambda \geq 0$ *when* $P^2 > b$ *yields the unique closed-form maximizer* $\lambda^* = (P + b)/(1 + P)$*, at which* (5) *recovers* $B_3(P, b) = (P^2 - b)/(1 + 2P + b)$ identically.

2. (Deterministic Schur-Complement H_2-SOS Witnesses.) *Whenever* $\det(H_1) = b - P^2 > 0$ *and* $\det(H_2) < 0$*, the closed-form Schur-complement vector*

$$v_{\mathrm{Schur}} := \left(\frac{Pc - b^2}{b - P^2},\ \frac{Pb - c}{b - P^2},\ 1\right)^\top \in \mathbb{R}^3 \tag{6}$$

   (lying in $\mathbb{Q}^3$ *whenever* $P, b, c \in \mathbb{Q}$*, as for* $\rho_2(p)$ *at rational* p*) satisfies* $v_{\mathrm{Schur}}^\top H_2 v_{\mathrm{Schur}} = \det(H_2)/\det(H_1) < 0$ *identically, certifying strictly positive* $\mathcal{N}(\rho) > 0$ *throughout the entire* (H_1*-blind,* H_2*-detected*) *region* $\det(H_1) > 0 > \det(H_2)$*.*

Proof. Splitting (4) into $x_u \geq 0$ and $x_u < 0$: every $x_u \geq 0$ contributes $x_u q_v(x_u)^2 \geq 0$, while every $x_u < 0$ satisfies $x_u \in [-1, 0)$ so $q_v(x_u)^2 \leq M_v$ and hence $x_u q_v(x_u)^2 \geq -M_v|x_u|$. Summing over $\mathbb{Z}_D^2$ gives $v^\top H_m v \geq -M_v \mathcal{N}(\rho)$ (Lean `T03`), yielding (5) since $M_v > 0$. For $m = 1$ and $q_\lambda(x) = x - \lambda$ ($\lambda \geq 0$), $\max_{[-1,0]} q_\lambda^2 = (1+\lambda)^2$ (Lean `T04`–`T05`). Substituting $\lambda^* = (P+b)/(1+P)$ yields

$$2\lambda^* P - (\lambda^*)^2 - b = \frac{(P^2 - b)(1 + 2P + b)}{(1 + P)^2}, \qquad (1 + \lambda^*)^2 = \frac{(1 + 2P + b)^2}{(1 + P)^2},$$

whose quotient simplifies identically to $(P^2 - b)/(1 + 2P + b) = B_3(P, b)$ (Lean `T06`). For (6), block Gaussian elimination on $H_2 = \begin{pmatrix} H_1 & w \\ w^\top & m_5 \end{pmatrix}$ with $w = (b, c)^\top$ gives $v_{\mathrm{Schur}} = (-H_1^{-1}w, 1)^\top$, whence $v_{\mathrm{Schur}}^\top H_2 v_{\mathrm{Schur}} = m_5 - w^\top H_1^{-1} w = \det(H_2)/\det(H_1) < 0$. □

Applied to the physical ququint state $\rho_2(p)$ of (3), Theorem 1 closes the entire (H_1, B_4)-blind zone sequentially: H_2-Schur certifies strictly positive magic across $p \in (0.4868, 0.5806]$ (`C04`–`C06`, Figure 1), explicit rational H_2 witnesses beat B_3 by $2.11\times$ at $p = 0.59$ and $1.26\times$ at $p = 0.60$ (`C07`), H_3-SOS descends to $p = 0.4525$, and at $p = 9/20 = 0.45$ (where $\det(H_1), \det(H_2), \det(H_3) > 0$ are all blind, `C08`) the explicit degree-4 root-interpolating polynomial $q_4(x) = (x - \frac{14}{25})(x - \frac{2}{5})(x - \frac{1}{5})(x - \frac{11}{100})$ certifies $\mathcal{N}_{\mathrm{SOS}}^{(4)} = 12393/2072994560000 > 0$ (Figure 3).

2.2 Theorem 2: Exact Positive-Part Signed-Decomposition Saturation

Decompose $x_u = x_u^+ - x_u^-$ with $x_u^+ = \max(x_u, 0) \in [0, 1]$, $x_u^- = \max(-x_u, 0) \in [0, 1]$, $N = \mathcal{N}(\rho) = \frac{1}{D}\sum x_u^-$, $\frac{1}{D}\sum x_u^+ = 1 + N$, $u := \frac{1}{D}\sum (x_u^-)^2$, and $v := \frac{1}{D}\sum (x_u^-)^3$.

Theorem 2 (Unconditional Positive-Part Signed-Decomposition Saturation)**.** *For every discrete Wigner distribution* $x \in [-1, 1]^{D^2}$ *with* $\sum_u x_u = D$*, one has* $0 \leq v \leq u \leq \min(N, P)$*,*

$$(P - u)^2 \;\leq\; (1 + N)(b + v) \;\leq\; (1 + N)(b + u), \tag{7}$$

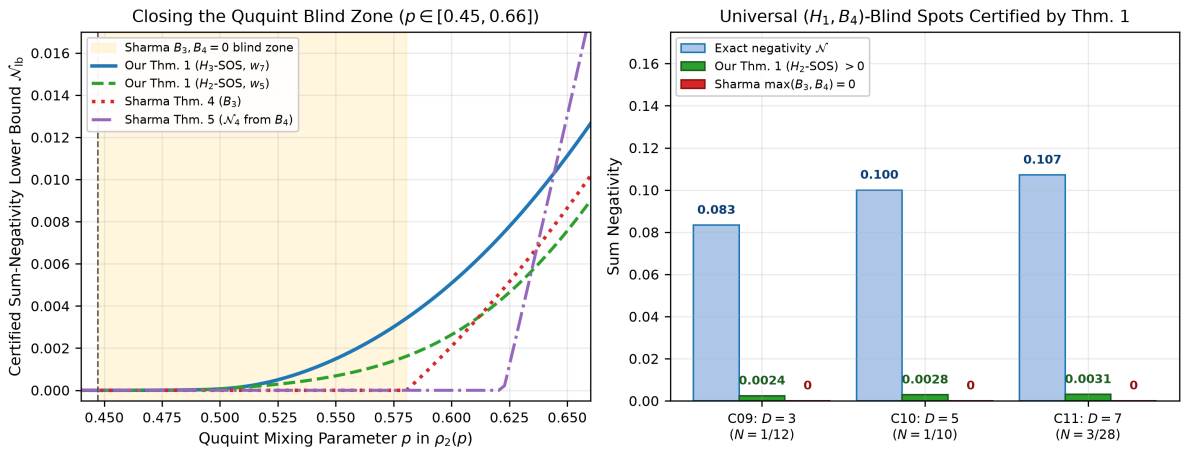

Figure 1: Left: Closing the blind zone on the physical ququint state $\rho_2(p)$ (3) ($p^* = 1/\sqrt{5} \approx 0.4472$): certified sum-negativity lower bounds from Theorem 1 using minimum-eigenvalue witnesses for H_2-SOS (moments up to m_5) and H_3-SOS (up to m_7) vs. Sharma et al.'s B_3 ($p > 0.5806$) and converted $\mathcal{N}_4 = \max(0, \frac{1}{2}(2^{B_4} - 1))$ ($p > 0.6224$) (`C04`–`C07`). Right: Exact rational (H_1, B_4)-blind-spot quasiprobability separation benchmarks $x^{(D)}$ on $D \in \{3, 5, 7\}$ (`C09`–`C11`, Theorem 3), where $\max(B_3, B_4) = 0$ yet Theorem 1 (H_2-SOS) certifies strictly positive negativity.

and UNCONDITIONALLY (even when $P < N$, such as $|S\rangle^{\otimes 3}$ where $P = 1 < N = 49/27$),

$$(P - N)^2 \;\le\; (1 + N)(b + N) \quad\Longleftrightarrow\quad P^2 - b \;\le\; N(1 + 2P + b). \tag{8}$$

Equality $(P - N)^2 = (1 + N)(b + N)$ holds if and only if $x_u^+ \in \{0, \alpha\}$ for a constant $\alpha > 0$ and $x_u^- \in \{0, 1\}$ (achieved by the pure qutrit strange state $|S\rangle$ with $\alpha = 1/2$, $N = B_3 = 1/3$, `C01`).

Proof. Since $\frac{1}{D}\sum (x_u^+)^2 = P - u$ and $\frac{1}{D}\sum (x_u^+)^3 = b + v$, Cauchy–Schwarz on $\sqrt{x_u^+}$ and $(x_u^+)^{3/2}$ yields $(P - u)^2 \le (1 + N)(b + v)$ (Lean `T07`–`T10`). Because $x_u^- \in [0, 1]$, $0 \le (x_u^-)^3 \le (x_u^-)^2 \le x_u^-$, so $0 \le v \le u \le N$; also $P = \frac{1}{D}\sum (x_u^+)^2 + u \ge u$, giving $u \le P$. Direct algebraic expansion gives the exact identity

$$\left[(1 + N)(b + N) - (P - N)^2\right] - \left[(1 + N)(b + u) - (P - u)^2\right] \;=\; (N - u)(1 + 2P - u) \;\ge\; 0,$$

since $N - u \ge 0$ and $1 + 2P - u \ge 1 + P > 0$ (constructively proved in Lean 4.36.0 `T22!`). This yields (8) unconditionally. For the equality characterization: if $x_u^+ \in \{0, \alpha\}$ and $x_u^- \in \{0, 1\}$, then $v = u = N$ and Cauchy–Schwarz holds with equality, giving $(P - N)^2 = (1 + N)(b + N)$. Conversely, equality in (8) forces $(N - u)(1 + 2P - u) = 0$; since $1 + 2P - u \ge 1 > 0$, this forces $u = N$, hence $\sum (x_u^- - (x_u^-)^2) = 0$ so $x_u^- \in \{0, 1\}$ (and thus $v = u = N$), and simultaneously forces equality in Cauchy–Schwarz on $\sqrt{x_u^+}$ and $(x_u^+)^{3/2}$, so x_u^+ is constant on its support ($x_u^+ \in \{0, \alpha\}$). □

2.3 Theorem 3: Infinite-Family Odd-Dimension (H_1, B_4)-Blind-Spot Separation

Theorem 3 (All-Odd-Dimension (H_1, B_4)-Blind-Spot Separation)**.** *For every odd integer $D \ge 3$, define $k := (D+1)/2 \ge 2$ and the explicit exact-$\mathbb{Q}$ quasiprobability vector $x^{(D)} \in ([-1, 1] \cap \mathbb{Q})^{D^2}$ having*

$$k \text{ entries } 1, \qquad 5(k-1) \text{ entries } \tfrac{1}{4}, \qquad 2(k-1) \text{ entries } -\tfrac{1}{8}, \qquad (D-1)(D-3) \text{ entries } 0.$$

Then $x^{(D)}$ is normalized ($\sum_u x_u^{(D)} = D$), strictly Wigner-negative ($\mathcal{N}(x^{(D)}) = \frac{D-1}{8D} > 0$), and satisfies ALL THREE conditions simultaneously for every odd $D \ge 3$:

1. $\det(H_1) = b(D) - P(D)^2 > 0$ *(so Sharma et al. Theorem 2 at $m = 1$ and Theorem 4 yield $B_3 = 0$),*

2. $P(D)^3 < c(D)$ *(so Sharma et al. Theorem 5 yields $B_4 = 0$),*

3. *with witness $q(x) = (x-1)(x-\frac{1}{4})$ ($v = (\frac{1}{4}, -\frac{5}{4}, 1)^\top$, $M_v = \sup_{[-1,0]} q(x)^2 = \frac{25}{4}$), Theorem 1 at $m = 2$ certifies*

$$v^\top H_2(D) v = -\frac{729(D-1)}{32768D} < 0, \qquad \mathcal{N}^{(2)}_{\mathrm{SOS}}(D) = \frac{729(D-1)}{204800D} > 0.$$

Proof. Total entries count $k+5(k-1)+2(k-1)+(D-1)(D-3) = (4D-3)+(D^2-4D+3) = D^2$, with sum $k + \frac{5}{4}(k-1) - \frac{2}{8}(k-1) = 2k-1 = D$ and $\mathcal{N} = \frac{2(k-1)}{8D} = \frac{D-1}{8D} > 0$. Direct evaluation of (1) gives closed-form rational moments for every odd $D \geq 3$:

$$P(D) = \frac{43D+21}{64D}, \qquad b(D) = \frac{275D+237}{512D}, \qquad c(D) = \frac{2089D+2007}{4096D}.$$

Both $\det(H_1) > 0 \iff (43D+21)^2 < (275D+237)(8D)$ and $P(D)^3 < c(D) \iff (43D+21)^3 < (2089D+2007)(64D^2)$ hold for ALL integers $D \geq 3$ (constructively proved for all $D \geq 3$ in Lean 4.36.0 `T21` via $43D + 21 \leq 46D$ for $D \geq 7$ and exact verification for $D \in \{3, 4, 5, 6\}$). Finally, $q(x) = (x-1)(x-\frac{1}{4})$ vanishes on every nonzero positive entry ($x \in \{1, \frac{1}{4}\}$), while $x = 0$ contributes 0 and $x = -\frac{1}{8}$ gives $q(-\frac{1}{8}) = (-\frac{9}{8})(-\frac{3}{8}) = \frac{27}{64}$, so $xq(x)^2 = -\frac{729}{32768}$. Multiplying by $2(k-1)/D = (D-1)/D$ yields $v^\top H_2(D) v = -\frac{729(D-1)}{32768D}$, and dividing by $M_v = q(-1)^2 = \frac{25}{4}$ certifies $\mathcal{N}^{(2)}_{\mathrm{SOS}}(D) = \frac{729(D-1)}{204800D} > 0$ for every odd $D \geq 3$ (evaluating to 243/102400 at $D = 3$, 729/256000 at $D = 5$, and 2187/716800 at $D = 7$, matching `C09`–`C11` identically). □

Remark 1 (Quasiprobability Scope vs. Physical Density Matrices). Theorem 3 (and `C09`–`C11`) establishes strict separation between (H_1, B_4) and H_2-SOS across the abstract simplex of normalized discrete quasiprobabilities $x \in [-1, 1]^{D^2}$ ($\sum_u x_u = D$). For every odd prime $D \geq 3$, $x^{(D)}$ is *provably non-realizable* by any physical density matrix $\rho \succeq 0$: having $k = (D+1)/2 \geq 2$ entries equal to 1 ($W_\rho(u) = 1/D$) forces ρ into the intersection of two phase-point $+1$-eigenspaces (a pure stabilizer state with $P = 1, \mathcal{N} = 0$), contradicting $P(D) < 1$ and $\mathcal{N}(x^{(D)}) > 0$. Genuine *physical-state* (H_1, B_4)-blind-spot separation is instead certified by the depolarized ququint density matrix $\rho_2(p) \succeq 0$ in (3) and `C04`–`C08`.

2.4 Theorem 4: Higher-Even-Moment Multiplicative Mana Family and Reverse-Monotonicity

Generalizing Sharma et al.'s fourth-moment bound B_4 to arbitrary even moments $m_{2k} = \frac{1}{D}\sum_u |x_u|^{2k}$ ($k \geq 2$), define the multiplicative L_1-norm lower bound Λ_{2k}, mana bound B_{2k}, and sum-negativity bound $\mathcal{N}_{2k}$:

$$\Lambda_{2k}(\rho) := \left(\frac{m_2^{2k-1}}{m_{2k}}\right)^{\frac{1}{2k-2}}, \quad B_{2k}(\rho) := \max\big(0, \log_2 \Lambda_{2k}(\rho)\big), \quad \mathcal{N}_{2k}(\rho) := \max\Big(0, \frac{\Lambda_{2k}(\rho) - 1}{2}\Big). \tag{9}$$

Theorem 4 (Higher-Even-Moment Mana Bounds and Reverse-Monotonicity Uniqueness of $k = 2$)**.** *For every odd-dimensional quantum state ρ and every integer $k \geq 2$:*

1. (Universal Multiplicative Lower Bound.) *$\|W_\rho\|_1 \geq \Lambda_{2k}(\rho)$, hence $\mathcal{M}(\rho) \geq B_{2k}(\rho)$ and $\mathcal{N}(\rho) \geq \mathcal{N}_{2k}(\rho)$, with exact multiplicativity $\Lambda_{2k}(\rho^{\otimes n}) = \Lambda_{2k}(\rho)^n$ (overcoming the global ceiling $B_3(\rho^{\otimes n}) \leq 1$, Lean `T15`, `C16`).*

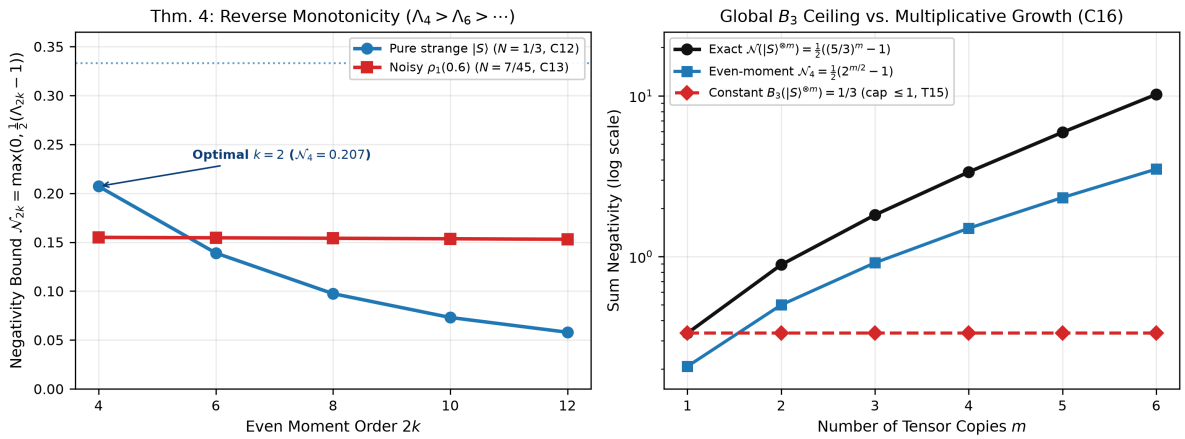

Figure 2: Left: Theorem 4 reverse-monotonicity of the even-moment negativity bound $\mathcal{N}_{2k} = \max(0, \frac{1}{2}(\Lambda_{2k} - 1))$ vs. moment order $2k$ on the pure strange state $|S\rangle$ (`C12`) and noisy $\rho_1(0.6)$ (`C13`), confirming $k = 2$ is uniquely optimal. Right: Constant $B_3(|S\rangle^{\otimes m}) \equiv 1/3$ (bounded by the global $B_3 \leq 1$ ceiling, `T15`) vs. exponential growth of $\mathcal{N}_4(|S\rangle^{\otimes m}) = \frac{1}{2}(2^{m/2} - 1)$ and exact $\mathcal{N}(|S\rangle^{\otimes m})$ across $m = 1..6$ tensor copies (`C16`).

2. (Reverse-Monotonicity — Uniqueness of $k = 2$.) *The sequence* $\{\Lambda_{2k}(\rho)\}_{k\geq 2}$ *is monotonically* non-increasing*:*

$$\Lambda_4(\rho) \;\geq\; \Lambda_6(\rho) \;\geq\; \Lambda_8(\rho) \;\geq\; \cdots \;\geq\; \Lambda_{2k}(\rho) \;\geq\; \Lambda_{2k+2}(\rho) \;\geq\; \cdots \tag{10}$$

Equivalently, in exact integer-exponent form without roots, $m_{2k}^k \leq m_2\, m_{2k+2}^{k-1}$. *Equality* $\Lambda_{2k} = \Lambda_{2k+2}$ *holds if and only if* $|x_u|$ *is constant on its support (e.g., pure stabilizer states,* `C15`*). Whenever* $|x_u|$ *takes at least two distinct nonzero magnitudes,* $\Lambda_4(\rho) > \Lambda_6(\rho) > \Lambda_8(\rho) > \cdots$ strictly*, and consequently* $B_{2k}(\rho)$ *and* $\mathcal{N}_{2k}(\rho)$ *are non-increasing for all* $k \geq 2$ *and* strictly *decreasing on their positive support* $\{k \geq 2 : \Lambda_{2k}(\rho) > 1\}$*!*

Proof. Set $a_u := |x_u| \geq 0$ so $\frac{1}{D}\sum a_u = \|W_\rho\|_1$ and $\frac{1}{D}\sum a_u^{2r} = m_{2r}$. Applying Hölder's inequality with conjugate exponents $p = 2k - 1$ and $q = (2k-1)/(2k-2)$ to $a_u^2 = a_u^{\frac{2k-2}{2k-1}} \cdot a_u^{\frac{2k}{2k-1}}$ gives $m_2 \leq \|W_\rho\|_1^{\frac{2k-2}{2k-1}} m_{2k}^{\frac{1}{2k-1}}$, i.e., $\|W_\rho\|_1 \geq \Lambda_{2k}(\rho)$ (proved for $k = 2$ over arbitrary finite lists in Lean `T12`). For (10), writing $2k = \frac{1}{k}\cdot 2 + \frac{k-1}{k}\cdot(2k+2)$ and applying Hölder's inequality with exponents k and $k/(k-1)$ gives $m_{2k} \leq m_2^{1/k} m_{2k+2}^{(k-1)/k} \iff m_{2k}^k \leq m_2 m_{2k+2}^{k-1}$, which is equivalent to $\Lambda_{2k}(\rho) \geq \Lambda_{2k+2}(\rho)$ (proved for $k = 2$, i.e., $m_4^2 \leq m_2 m_6$, in Lean `T13`–`T14`). Equality holds iff a_u^2 and a_u^{2k+2} are proportional on $\mathrm{supp}(a)$, i.e., a_u is constant on its support; strict inequality $\Lambda_{2k} > \Lambda_{2k+2}$ transfers directly to B_{2k} and $\mathcal{N}_{2k}$ wherever $\Lambda_{2k} > 1$. □

3 Exact-$\mathbb{Q}$ Verification, Lean 4.36.0 Formalization, and Verification Boundary

Verification Boundary.

1. *Formally Proved in Lean 4.36.0* (`lean/WignerMomentMagicCert.lean`, 22 theorems `T01`–`T22`, compiled with `-DwarningAsError=true` and `lake build`, `0 sorry`, `0 grind`, `0 custom axioms`; `T01`–`T06`, `T11`, `T15`–`T22` depend on zero `Classical.choice`, while `T07`–`T10` and `T12`–`T14` use only standard Lean core axioms `[propext, Classical.choice, Quot.sound]`): Universal list Hankel–SOS negativity bound (`T01`–`T05`), exact $m = 1$

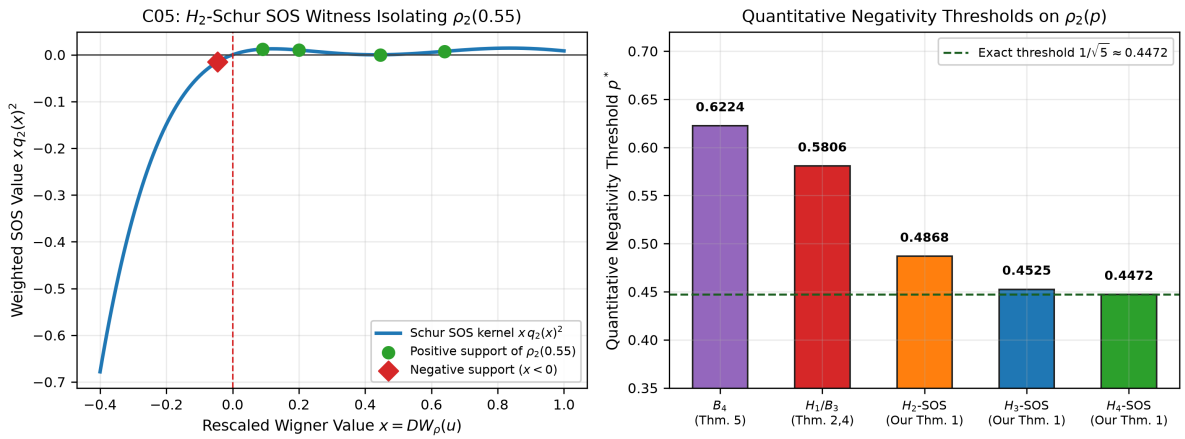

Figure 3: Left: Exact H_2-Schur SOS kernel $x\,q_2(x)^2$ on $\rho_2(0.55)$ (`C05`), staying nonnegative on all four positive Wigner levels of (3) while isolating the negative level $x_- = (1 - 0.55\sqrt{5})/5 < 0$. Right: Quantitative negativity thresholds p^* on $\rho_2(p)$ descending monotonically from Sharma's B_4 (0.6224) and B_3/H_1 (0.5806) through our H_2-SOS (0.4868), H_3-SOS (0.4525), and H_4-SOS (0.4472), converging to the exact physical threshold $1/\sqrt{5} \approx 0.4472$.

Cert ID	Mode / Target	Sharma B_3, B_4	Our Theorem Certified Result
`C01`	Pure strange $\|S\rangle$ $(D = 3)$	$B_3 = 1/3$	$m = 1$ dual exact saturation $\mathcal{N}^{(1)}_{\mathrm{SOS}} = 1/3$
`C02`–`C03`	Noisy $\rho_1(1/2), \rho_1(4/5)$	$B_3 > 0$	Exact $\lambda^* = (P + b)/(1 + P)$: $\mathcal{N}^{(1)}_{\mathrm{SOS}} = B_3$
`C04`–`C06`	Ququint $\rho_2(p)$, $p \in \{0.50, 0.55, 0.58\}$	$B_3 = B_4 = 0$	H_2-Schur certifies $\mathcal{N}^{(2)}_{\mathrm{SOS}} > 0$ (blind zone closed)
`C07`	Ququint $\rho_2(0.60)$ crossover	$B_3 = \frac{2}{959}, B_4 = 0$	H_2-SOS $= 8156423/3094140625$ $(1.264 \times B_3)$
`C08`	Ququint $\rho_2(0.45)$ deep blind zone	$H_1, H_2, H_3 \succ 0$	H_4-SOS $= 12393/2072994560000 > 0$
`C09`–`C11`	Blind spots $D \in \{3, 5, 7\}$ (Thm. 3)	$B_3 = B_4 = 0$	H_2-SOS: 243/102400, 729/256000, 2187/716800
`C12`–`C14`	Higher-even Hölder $(k = 2, 3, 4)$	B_4 only	Strict $\Lambda_4 > \Lambda_6 > \Lambda_8$ reverse-monotonicity
`C15`	Pure stabilizer state $(D = 3)$	$B_3 = B_4 = 0$	Saturation $m_1 = \cdots = m_6 = 1$, $\Lambda_4 = \Lambda_6 = \Lambda_8 = 1$
`C16`	Tensor copies $\|S\rangle^{\otimes m}$ $(m = 1..4)$	$B_3 \equiv 1/3$ cap	Global B_3 stagnates; $\Lambda_4^2 = 2^m$ scales exponentially

Table 1: Summary of the 16 canonical exact-$\mathbb{Q}$ certificates (`WIGNER-MOMENT-MAGIC-CERT-1.0`).

dual recovery of B_3 (`T06`), list Cauchy–Schwarz and unconditional signed-decomposition monotonicity (`T07`–`T10`, `T22`), degree-2 Hankel expansion (`T11`), universal $k = 2$ Hölder and $B_4 \geq B_6$ reverse-monotonicity (`T12`–`T14`), global $B_3 \leq 1$ ceiling (`T15`), finite witnesses (`T16`–`T20`), and the infinite-family odd-dimension blind-spot proof for ALL $D \geq 3$ (`T21`).

2. *Verified by Zero-Float Exact-$\mathbb{Q}$ Computation* (`src/wigner_moment_magic_verifier.py`, 65 passing `pytest` tests including AST audit forbidding `float`, `/`, and `**`): Exact $\mathbb{Q}(\sqrt{5})$ moments and Hankel determinants for $\rho_2(p)$ across $p \in [0.10, 0.60]$, deterministic Schur witnesses, degree-4 polynomial witness at $p = 0.45$, and all 16 canonical SHA-256 certificates.

3. *Outside Scope:* Experimental tomography noise estimation, continuous-variable Wigner functions, and constructing *physical* density-matrix (H_1, B_4)-blind-spot families for arbitrary odd $D \geq 7$ (note that Remark 1 shows the quasiprobability family $x^{(D)}$ of Theorem 3 is provably unphysical for odd prime $D \geq 3$, whereas $\rho_2(p)$ in (3) and `C04`–`C08` is a genuine physical ququint density matrix $\rho_2(p) \succeq 0, \mathrm{Tr}\,\rho_2(p) = 1$).

Artifact Provenance and Cryptographic Digests. All source code, Lean 4.36.0 proofs, certificates, and tests are archived in the public AIRR repository `https://github.com/arr-research/arr-research.github.io` under MIT (code/data) and CC-BY-4.0 (manuscript) licenses:

- `lean/WignerMomentMagicCert.lean`
  SHA-256: 0a5040016b94f3c1c0efd8fb99c64f1d9852fa572a5abe375e079aaf6b741f52
- `src/wigner_moment_magic_verifier.py`
  SHA-256: 1ca192730d7222fa27bc0e0efdc8776aad94170ac387fcf50c730efe8f40b57e
- `certificates/MANIFEST.json`
  SHA-256: 274cedaed5d342f339bc19eb6839bed488845d1640b996a14c624131f6820eee

AI Disclosure and Conflict Statement. This work was produced by Lluis Eriksson with computational, formalization, and manuscript assistance from Google Antigravity autonomous research agents. As founder/operator of AIRR.SCIENCE, the author discloses an operator conflict (`operator_conflict: true`); all screening was performed by isolated PDF-only evaluators.

References

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