Nilpotent noise and complete reverse data processing

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Abstract

We give a sufficient criterion for a reverse relative-entropy comparison that remains valid after adjoining any finite-dimensional quantum reference. The criterion combines a completely positive, trace-nonincreasing nilpotent noise map, complete-positive domination on the image of a channel, and an invertible positive output filter. A finite polynomial inverse replaces a small-perturbation assumption and gives an explicit dimension-independent coefficient. Applications include comparisons between qubit amplitude-damping channels and a three-level finite-height noise family. We also construct a rational sequence of separable two-qubit states showing that the complete mutual-information contraction coefficient of every nonconstant amplitude-damping channel equals one, despite its strict ordinary relative-entropy contraction. The latter obstruction prevents an automatic capacity application based on a strict complete contraction gap. The reverse coefficient is not asserted sharp, and no capacity-additivity theorem is inferred.

1 Introduction and scope

A reverse data-processing comparison asks how much distinguishability one channel preserves relative to another. Ordinary comparisons optimize over states on the input system. Retaining an arbitrary quantum reference is a different requirement. Belzig, Gao, Smith and Wu [1] introduce relative expansion coefficients, explain why complete-positive order alone does not control the operator argument of the Bogoliubov–Kubo–Mori (BKM) metric, and formulate complete comparisons using mutual information. Han and Li [2] give an exact ordinary amplitude-damping expansion coefficient; that scalar formula is not a contribution of this paper. Iyer, Wu, Belzig and Smith [3] study a wider family of ordinary Riemannian coefficients. Reverse comparisons after prior noise also relate to the robustness questions of LaRacuente and Smith [6].

Our criterion handles both the metric base point and its operator argument. The key assumptions are an image domination $\mathcal{Q}\mathcal{M} \leq_{\mathrm{cp}} c\mathcal{M}$ and finite nilpotency $\mathcal{Q}^k = 0$. The inverse of $\mathrm{id} + a\mathcal{Q}$ is then a finite polynomial, even when $a\sqrt{c} \geq 1$. The conditions are finite-dimensional map conditions: complete-positive domination can be checked on Choi operators. They are not an assumed entropy estimate on unbounded reference dimensions.

We separately distinguish the complete mutual-information coefficient from ordinary relative-entropy contraction and from other stabilized coefficients [4, 1]. Hirche, George, Nuradha and Wilde [5] identify this mutual-information coefficient with a product-reference relative-entropy coefficient; that equivalence is prior work. It differs from coefficients with merely equal reference marginals and from conditional coefficients that subtract reference distinguishability. A single-qubit reference in a separable two-qubit state suffices for the amplitude-damping obstruction below. This does not mean that a fixed nonzero amount of information is preserved perfectly: the information in our witnessing sequence tends to zero.

All Hilbert spaces are finite-dimensional. All logarithms are natural. We do not claim a solution of general nonunital-qubit additivity, a sharp complete reverse coefficient, or a nondegradable channel with additive quantum capacity.

2 Metric comparisons on the positive cone

For $S > 0$ and $Y = Y^*$ define

$$g_S(Y) = \int_0^\infty \mathrm{Tr}\big[Y(S+tI)^{-1}Y(S+tI)^{-1}\big]\, dt, \qquad \|Y\|_S = g_S(Y)^{1/2}. \tag{1}$$

Neither $\mathrm{Tr}\, S = 1$ nor $\mathrm{Tr}\, Y = 0$ is required. For a semidefinite S the form is taken on its support when $\mathrm{supp}\, Y \subseteq \mathrm{supp}\, S$. We use the associated real Hilbert norm on Hermitian operators.

Lemma 1 (Monotonicity and order)**.** *Let Φ be completely positive and trace preserving. Then $g_{\Phi(S)}(\Phi(Y)) \le g_S(Y)$ for every Hermitian Y. The same inequality holds for a completely positive trace-nonincreasing map, with the support convention above. If $U \le bV$, $b > 0$, then*

$$g_U(Y) \ge b^{-1} g_V(Y). \tag{2}$$

Proof. For a trace-one S, write $Y = tS + Y_0$ with $\mathrm{Tr}\, Y_0 = 0$. The identities $g_S(S, Y_0) = \mathrm{Tr}\, Y_0 = 0$ and $g_S(S) = 1$ separate the trace component. Differentiating relative-entropy data processing proves monotonicity on Y_0; a trace-preserving map preserves the separated trace component. Homogeneity handles arbitrary $\mathrm{Tr}\, S > 0$. For a trace-nonincreasing map append a failure flag of weight $\mathrm{Tr}[(I - \Phi^*(I))S]$ to obtain a channel. The BKM form on its direct-sum output is the desired form plus a nonnegative failure-block term. Writing $E = I - \Phi^*(I)$, the channel is $T \mapsto \Phi(T) \oplus \mathrm{Tr}(ET)$. If $\mathrm{Tr}(ES) = 0$, the bound $-dS \le Y \le dS$ forces $\mathrm{Tr}(EY) = 0$, so the zero failure block is omitted without a $0/0$ term. This construction applies on the entire joint space when a reference is present. For full-rank S, $-dS \le Y \le dS$ for some d, so $\Phi(Y)$ is supported in $\Phi(S)$.

For the order statement use $(U + tI)^{-1} \ge (bV + tI)^{-1}$ and substitute $t = bu$. The required integrand comparison does not assume commutation: for positive $A \ge B$,

$$A^{\mathsf{T}} \otimes A - B^{\mathsf{T}} \otimes B = (A^{\mathsf{T}} - B^{\mathsf{T}}) \otimes A + B^{\mathsf{T}} \otimes (A - B) \ge 0.$$

Vectorization therefore gives $\mathrm{Tr}(YAYA) \ge \mathrm{Tr}(YBYB)$. For singular U, assume Y is supported in U (hence in V). Compare $U + \delta I \le bV + \delta I$, integrate, and let $\delta \downarrow 0$. The spectral formula for the BKM form gives precisely the support-restricted limits. These are the singular cases used below. □

Lemma 2 (Positive congruence)**.** *If $F > 0$ and $F^2 \ge mI$, then*

$$g_{FSF}(FYF) \ge m\, g_S(Y). \tag{3}$$

Proof. The integrand on the left of (3) is

$$\mathrm{Tr}\big[Y(S + tF^{-2})^{-1}Y(S + tF^{-2})^{-1}\big].$$

Since $F^{-2} \le m^{-1}I$, the integrand is at least its counterpart with $S + (t/m)I$, by the same tensor-order argument as above. Integration gives the factor m. This is a comparison for an unnormalized filter; it is not an application of channel data processing to Ad_F. □

3 A complete reverse comparison

We write $\mathrm{Ad}_F(X) = FXF$ for $F > 0$. The order $\Phi \le_{\mathrm{cp}} \Psi$ means that $\Psi - \Phi$ is completely positive.

Theorem 3 (Nilpotent image criterion)**.** *Let $\mathcal{M}, \mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ be channels with $\mathcal{M}(I_A) > 0$. Let $\mathcal{Q} : \mathcal{L}(B) \to \mathcal{L}(B)$ be completely positive and trace-nonincreasing, with $\mathcal{Q}^k = 0$ for an integer $k \ge 1$. Suppose $a \ge 0$, $c > 0$, $F > 0$ and $m > 0$ satisfy*

$$\mathcal{Q}\mathcal{M} \le_{\mathrm{cp}} c\mathcal{M}, \qquad F^2 \ge mI_B, \qquad \mathcal{N} = \mathrm{Ad}_F(\mathrm{id} + a\mathcal{Q})\mathcal{M}. \tag{4}$$

Set

$$P_k = \sum_{j=0}^{k-1} (a\sqrt{c})^j, \qquad \lambda = \frac{m}{(1+ac)P_k^2}. \tag{5}$$

For every finite reference R and density operators ρ, σ on $A \otimes R$ with $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$,

$$D((\mathcal{N} \otimes \mathrm{id}_R)(\rho) \| (\mathcal{N} \otimes \mathrm{id}_R)(\sigma)) \ge \lambda D((\mathcal{M} \otimes \mathrm{id}_R)(\rho) \| (\mathcal{M} \otimes \mathrm{id}_R)(\sigma)). \tag{6}$$

Proof. First take a full-rank σ. Suppress tensor identities and put $S = \mathcal{M}(\sigma) > 0$ and $Y = \mathcal{M}(X)$. Strict positivity follows from $\sigma \ge \epsilon I_{A\otimes R}$ and $\mathcal{M}(I_A) > 0$. For any Hermitian Z, complete-positive domination and Lemma 1 give

$$g_S(\mathcal{Q}Z) \le c\, g_{\mathcal{Q}(S)}(\mathcal{Q}Z) \le c\, g_S(Z). \tag{7}$$

Thus $\|\mathcal{Q}\|_{S\to S} \le \sqrt{c}$, including on nontraceless arguments. Nilpotency supplies the exact identity

$$(\mathrm{id} + a\mathcal{Q})^{-1} = \sum_{j=0}^{k-1} (-a\mathcal{Q})^j.$$

Writing $Z = (\mathrm{id} + a\mathcal{Q})Y$, the triangle inequality yields

$$\|Y\|_S \le P_k \|Z\|_S. \tag{8}$$

No convergent infinite Neumann series is used.

At the base point,

$$\mathcal{N}(\sigma) = F(S + a\mathcal{Q}(S))F \le (1+ac)FSF, \qquad \mathcal{N}(X) = FZF.$$

Lemmas 1 and 2, then (8), imply

$$\begin{aligned} g_{\mathcal{N}(\sigma)}(\mathcal{N}(X)) &\ge \frac{1}{1+ac}\, g_{FSF}(FZF) \\ &\ge \frac{m}{1+ac}\, g_S(Z) \ge \lambda g_{\mathcal{M}(\sigma)}(\mathcal{M}(X)). \end{aligned} \tag{9}$$

All map hypotheses and constants survive tensoring with id_R. The argument does not impose a bound on $\dim R$.

For full-rank ρ, σ, let $X = \rho - \sigma$ and $\sigma_t = \sigma + tX$. The Hessian formula

$$D(\rho\|\sigma) = \int_0^1 (1-t) g_{\sigma_t}(X)\, dt$$

also holds after either channel. Integrating (9) proves (6). Finally regularize both inputs with the same full-rank state. Output relative entropies are finite under the stated input support inclusion. Joint convexity gives the matching upper limit for the regularized pairs, while lower semicontinuity gives the lower limit. This passes (6) to the original pair. □

Remark 4. The domination in (4) remains a hypothesis of this criterion. Strict positivity of $\mathcal{M}(I)$ alone is not a reverse data-processing hypothesis. Nor is complete-positive order by itself a comparison of the BKM operator arguments. The finite inverse in (8) supplies that separate step.

For channels N, M with the same input, define the complete mutual-information expansion coefficient by

$$\check{\eta}^{\mathrm{cb}}_{\mathrm{MI}}(N,M) = \inf_{\substack{R,\rho_{AR} \\ I(R;B_M)>0}} \frac{I(R;B_N)}{I(R;B_M)}. \tag{10}$$

The corresponding contraction coefficient uses the supremum. These are the mutual-information quantities of [1, Definition 7.1]; they are not all interchangeable with stabilized two-state coefficients.

Corollary 5. *Under Theorem 3,* $\check{\eta}^{\mathrm{cb}}_{\mathrm{MI}}(\mathcal{N},\mathcal{M}) \geq \lambda$.

Proof. Apply (6) to ρ_{AR} and $\sigma = \rho_A \otimes \rho_R$. The latter support contains that of ρ_{AR}. Each output relative entropy is its corresponding mutual information. □

4 Amplitude damping and a three-level example

The amplitude-damping channel has Kraus operators

$$K_0(p) = \mathrm{diag}(1, \sqrt{1-p}), \qquad K_1(p) = \sqrt{p}\,|0\rangle\langle 1|.$$

For $0 < p_1 < p_2 < 1$ write $s_i = 1 - p_i$ and $d = p_2 - p_1$.

Corollary 6. *For every finite reference and every supported input pair, Theorem 3 holds with* $\mathcal{M} = \mathcal{A}_{p_1}$, $\mathcal{N} = \mathcal{A}_{p_2}$ *and*

$$\lambda_{\mathrm{AD}} = \frac{p_1 s_2}{p_2 s_1}\left(1 + \frac{d}{\sqrt{p_1 s_1}}\right)^{-2} > 0. \tag{11}$$

The same lower bound holds in (10).

Proof. Use

$$\mathcal{Q}(X) = |0\rangle\langle 1|X|1\rangle\langle 0|, \quad F = \mathrm{diag}(1, \sqrt{s_2/s_1}), \quad a = d/s_1, \quad c = s_1/p_1, \quad m = s_2/s_1.$$

Then $\mathcal{Q}^2 = 0$ and $\mathcal{Q}$ is trace-nonincreasing. On all matrix units, $\mathcal{A}_{p_2} = \mathrm{Ad}_F(\mathrm{id} + a\mathcal{Q})\mathcal{A}_{p_1}$. Moreover,

$$c\mathcal{A}_{p_1} - \mathcal{Q}\mathcal{A}_{p_1} = c\ \mathrm{Ad}_{K_0(p_1)}$$

is completely positive. Substitution into (5) gives (11). □

For $p_1 = 1/4, p_2 = 1/2$ the coefficient is $1 - \sqrt{3}/2$. For $p_1 = 1/10, p_2 = 9/10$ it is $1/1089$, although $a\sqrt{c} = 8/3 > 1$. Neither endpoint $p_1 = 0$ nor $p_2 = 1$ is included. Han and Li's ordinary coefficient [2, Theorem 4] is $p_1 s_2/(p_2 s_1)$ without the extra denominator. We make no assertion that (11) is sharp after adjoining a reference.

For a three-level example let

$$J = |0\rangle\langle 1| + |1\rangle\langle 2|, \qquad \mathcal{Q} = \mathrm{Ad}_J, \qquad \mathcal{Q}^3 = 0.$$

For $b > 0$ set

$$D_b = \mathrm{diag}\big(1, (1+b)^{-1/2}, (1+b+b^2)^{-1/2}\big), \qquad \mathcal{M}_b = (\mathrm{id} + b\mathcal{Q} + b^2\mathcal{Q}^2)\,\mathrm{Ad}_{D_b}.$$

The Kraus operators $D_b, \sqrt{b}JD_b, bJ^2D_b$ make $\mathcal{M}_b$ a channel, and $\mathcal{M}_b(I) > 0$. The identity

$$b^{-1}\mathcal{M}_b - \mathcal{Q}\mathcal{M}_b = b^{-1}\,\mathrm{Ad}_{D_b}$$

verifies image domination. For $0 < a < 1$ define

$$F_a = \mathrm{diag}(1, \sqrt{1-a}, \sqrt{1-a+a^2}), \qquad \mathcal{N} = \mathrm{Ad}_{F_a}(\mathrm{id} + a\mathcal{Q})\mathcal{M}_b.$$

This is a channel because $F_a^2 + aJ^*F_a^2J = I$. Theorem 3 gives the complete coefficient

$$\frac{1-a}{(1+a/b)(1+a/\sqrt{b}+a^2/b)^2}. \tag{12}$$

At $b = 1/100, a = 1/2$, this is $1/98022$ and $a\sqrt{c} = 5$. This example demonstrates finite-height noise beyond the qubit case, without claiming an optimal comparison or any capacity formula.

5 No strict complete mutual-information contraction for amplitude damping

The positive reverse comparison above does not imply a strict complete contraction relative to the identity.

Theorem 7. *For every $0 < p < 1$,*

$$\eta_{\mathrm{MI}}^{\mathrm{cb}}(\mathcal{A}_p, \mathrm{id}) = 1. \tag{13}$$

A sequence of rational, full-rank separable states on two qubits attains ratios tending to one. Its reference is not classical in any basis.

Proof. Data processing gives the upper bound one. Let $X = \sigma_x \otimes \sigma_x$ and, for integers $n \geq 3$, put

$$x_n = 1/n, \qquad r_n = 2^{-n^2}, \qquad \epsilon_n = r_n/n^2,$$
$$S_n = \mathrm{diag}(1-x_n, x_n) \otimes \mathrm{diag}(1-r_n, r_n), \qquad \rho_n = S_n + \epsilon_n X.$$

All entries are rational, both marginal states are unchanged, and $\lambda_{\min}(S_n) = r_n/n$. Since $\|X\|_\infty = 1$,

$$(1-1/n)S_n \leq \rho_n \leq (1+1/n)S_n. \tag{14}$$

In particular ρ_n is a full-rank state.

For positive numbers write $\ell(a,b) = (\log a - \log b)/(a-b)$, with $\ell(a,a) = 1/a$. At the product base $S_{x,r} = \mathrm{diag}(1-x,x) \otimes \mathrm{diag}(1-r,r)$, let $s = 1-p$, $u = 1-sx$ and $v = sx$. Direct evaluation of (1) gives

$$g_{\mathrm{in}} = 2\{\ell((1-x)(1-r), xr) + \ell((1-x)r, x(1-r))\}, \tag{15}$$
$$g_{\mathrm{out}} = 2s\{\ell(u(1-r), vr) + \ell(ur, v(1-r))\}. \tag{16}$$

The factor s comes from $\mathcal{A}_p(\sigma_x) = \sqrt{s}\,\sigma_x$.

For fixed x, letting $r \downarrow 0$ in these forms gives the ratio $(1-x)/(1-sx)$. For the particular joint sequence x_n, r_n, the same limit to one is seen directly. Put $T_n = n^2 \log 2$ and, for fixed $d > 0$, $q = d/n$. The sum of the two log means at base populations $(1-q, q)$ is

$$H_n(q) = \frac{T_n + \log((1-q)/q) + \log(1-r_n)}{1-q-r_n} + \frac{T_n - \log((1-q)/q) + \log(1-r_n)}{q-r_n}.$$

Since $\log((1-q)/q) = \log n - \log d + O_d(n^{-1})$, the first term is $O(T_n)$ and the second is $(n/d)T_n(1+o(1))$. Taking $d=1$ and $d=s$ shows

$$g_{\text{in}}(x_n, r_n) \sim 2n^3 \log 2, \qquad g_{\text{out}}(x_n, r_n) \sim 2n^3 \log 2$$

for each fixed $0 < s < 1$.

It remains to control the finite mutual informations, rather than merely their Hessians. The bounds (14) hold along the segment from S_n to ρ_n and after applying the channel. Metric order and the Hessian integral imply, at input and at output,

$$\frac{\epsilon_n^2}{2(1+1/n)} g \le D(\rho_n \| S_n) \le \frac{\epsilon_n^2}{2(1-1/n)} g,$$

using the appropriate output pair and form in the output expression. These relative entropies are the mutual informations because the marginal states are unchanged. Their ratio is at least

$$\frac{1-1/n}{1+1/n} \frac{g_{\text{out}}(x_n, r_n)}{g_{\text{in}}(x_n, r_n)} \longrightarrow 1.$$

This proves (13) without interchanging an uncontrolled Taylor approximation and a boundary limit.

To prove separability, abbreviate $x = x_n, r = r_n, \epsilon = \epsilon_n$ and set

$$D_A = \operatorname{diag}(\sqrt{1-x}, \sqrt{x}), \qquad D_R = \operatorname{diag}(\sqrt{1-r}, \sqrt{r}), \qquad t = \frac{\epsilon}{\sqrt{x(1-x)r(1-r)}} \in (0,1).$$

Then

$$\rho_n = (D_A \otimes D_R)(I_4 + t\, \sigma_x \otimes \sigma_x)(D_A \otimes D_R).$$

If P_a denotes the σ_x eigenprojector with eigenvalue $a \in \{-1, 1\}$, the middle operator equals $\sum_{a,b}(1+tab) P_a \otimes P_b$. All coefficients are positive. Local filtering preserves the separable positive cone, giving a separable decomposition of the normalized state. Finally the reference marginal has distinct eigenvalues and $[\rho_n, I \otimes (\rho_n)_R] \neq 0$. Every state classical on R commutes with its reference marginal, so ρ_n is not classical on R in any basis. □

Remark 8 (Small information, not perfect transmission). The input mutual information in this sequence tends to zero; the above bounds give $I(A;R)_{\rho_n} \sim (\log 2)\, 2^{-2n^2}/n$. The theorem concerns a supremum of ratios. It is not a claim of noiseless transmission of a fixed amount of information. At $p = 1$ the output is constant and the mutual-information coefficient is zero, so that endpoint is not part of Theorem 7.

Ordinary contraction is different: the bound $\eta(\mathcal{A}_p) \le \sqrt{1-p} < 1$ is given in [1, Lemma 6.3]. For a classical auxiliary the mutual information is an average of relative entropies, so the same strict ordinary coefficient controls its contraction. Theorem 7 shows that separable nonclassical references can invalidate that strict coefficient.

For a degradable amplitude-damping main channel with damping $\gamma \in (0, 1/2)$, its degrading map has damping $(1-2\gamma)/(1-\gamma) \in (0,1)$. Its complete mutual-information contraction coefficient is therefore one. Consequently the factor $1 - \eta_{\text{MI}}^{\text{cb}}$ in the sufficient construction discussed in [1, Section 7] vanishes. Our positive relative-expansion bound alone does not establish the informationally degradable nondegradable example sought by that criterion.

6 Verification, limits and review context

The proofs use finite map identities, BKM monotonicity, operator order and relative-entropy data processing. Algebraic checks verify the amplitude and three-level map identities, positivity

of selected rational witnesses, and the stated example constants. Numerical checks at selected correlated base states examine whole finite Hermitian tangent spaces, including nontraceless arguments. Neither kind of finite check is a replacement for the universal arguments or a certificate of worldwide novelty.

The standard ordinary coefficient formulas are attributed to their sources. The image domination, nilpotency, full-output-support and positive-filter hypotheses of Theorem 3 remain restrictions. The result does not address all pairs of channels, arbitrary infinite-dimensional references, or a sharp value of the complete coefficient. It does not solve the separate non-GAD output-entropy target considered in the author's earlier working notes.

Model assessments and review context

A separate mathematical component audit (audit917, 27 October 2026) was requested with configuration GPT-6 Astra, reasoning effort xhigh, on a hash-bound version of the underlying argument and checker. Its terminal report found the structural theorem, both channel examples and the separable mutual-information obstruction valid under the stated hypotheses, with no fatal proof error found. It assigned no numerical grade. The complete report is retained with SHA-256 `e5ca2f4dc39ca5de352d72e7751bafdb159a2acf359121fb40f650c743444259`. The audit required broader prior-art attribution, explicit singular-support and failure-flag details, and careful separation of coefficient definitions. Those points are incorporated here. Its reservations remain: novelty is unresolved; ordinary amplitude-damping positivity and exact coefficients are prior work; the complete lower bound is not proved optimal; no capacity or additivity theorem follows. A local algebraic issue reported in the rendered proof of an antecedent is not used to discredit that antecedent's proposition or establish priority. The requested model identity and effort are not externally attested; the context was separate but the filesystem was shared. Independence and runtime isolation are unverified. A component audit is not an assessment of this exact PDF. Any subsequent manuscript evaluation must be read with its exact-PDF hash, full report, objections and context; no numerical grade is asserted here.

AI and responsibility disclosure

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. The requested component audit used a separate conversational context with a shared filesystem. Effective model identity, effort, runtime isolation and independence have not been independently attested. No independent human review, Lean formalization, certified worldwide originality or predicted assessment score is claimed.

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