Canonical and unrestricted observations in a finite checkerboard gradient model

Lluis Eriksson
THE ERIKSSON PROGRAMME

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Abstract

We compare two classes of linear observations for a finite, source-weighted, two-channel checkerboard gradient form. For orthogonal alternating weak lines in dimensions three and four, the minimum restoring unrestricted rank is exactly Dn on all fields and $(D-1)(n-2)$ on the actual coexact domain. For any fixed positive angle, an explicit transverse Fourier cutoff gives an unrestricted observation count of order n and a quadratic penalty independent of n, with constants depending on the angle and dimension. A real coexact counterexample shows that the mean-only orthogonal construction does not extend unchanged to small positive angles. In contrast, canonical original-channel rectangle means have a necessary ambient count of order n^{D-2}. This last argument does not apply to the coexact domain, because rectangular averaging need not preserve it. A further explicit coexact witness excludes partitions whose every cell in either channel leaves a coordinate uncut. For an enlarged three-dimensional cyclic product-slice class, an explicit tagged-window cutoff proves a necessary inequality involving corner matching over every number of slices. Its restriction to nonwrapping canonical product slices gives a logarithmic count loss for fixed or bounded-motion cuts. A rapid helical grid in the cyclic extension fails through a short resonance; its wrapped complements are outside the nonwrapping canonical class. A separate construction with added chart-boundary cutoff centres excludes its nonwrapping refinement with $36n - 24$ full rows. The general canonical coexact three-dimensional question, linear count versus a possible logarithmic loss, remains open. These are finite analytical results; we establish neither historical priority, formal verification nor a physical Yang–Mills spectral gap.

1 Problem and scope

Spectral coarse spaces that retain modes responsible for slow convergence are established constructions; see Spillane et al. [1, 2]. The calculations here concern a particular finite coefficient family and a restoring quadratic inequality. They do not prove a preconditioned condition number bound or a new general principle of spectral enrichment. Our purpose is to keep the coefficients, observed rows and domain explicit enough to compare their different rank and count requirements.

The manuscript consolidates the programme's analytical sources 251–254, 256–259 and261C, rather than presenting their corollaries as separate papers. A preceding working paper treats broader canonical constructions [3]. The present manuscript is a proposed substantial revision of the existing checkerboard working paper [4]. Its new PDF is ungraded at this source stage; earlier assessments apply only to their original bytes.

Fix an integer $D \geq 3$, an even integer $n \geq 2$, and $c > 0$. Write

$$X = (\mathbb{Z}/n\mathbb{Z})^D, \quad V = n^D, \quad a = \frac{c}{16D}, \quad H = cn^2.$$

We identify the original chart with $\{0,\dots,n-1\}^D$ and retain all positive periodic labels $\partial_i f(x) = f(x+e_i) - f(x)$, including at $n=2$. The source parity is $p(x) = \sum_i x_i \pmod 2$. For each parity $l = 0, 1$, let $v_l, s_l \in \mathbb{R}^2$ be an orthonormal pair. The two weak lines $\mathbb{R}v_0, \mathbb{R}v_1$ have principal angle $\theta \in (0, \pi/2]$. Put

$$B(x) = a v_{p(x)} v_{p(x)}^T + H s_{p(x)} s_{p(x)}^T.$$

An original field is $z = (z_1, \dots, z_D)$, with $z_r(x) \in \mathbb{R}^2$. All inner products and norms use counting measure, summing sites, spatial components and internal channels. Define

$$Q(z) = \sum_{x,i,r} (\partial_i z_r(x))^T B(x) \partial_i z_r(x), \qquad E(z) = \sum_{x,i,r} |\partial_i z_r(x)|^2. \tag{1}$$

The weights in (1) belong to the source x.

For a real spatial one-form u, use the forward curl $(d_1 u)_{ij} = \partial_i u_j - \partial_j u_i$, with independent indices $i < j$. Its adjoint is taken for counting measure on those independent components. Set $U_0 = \mathrm{im}(d_1^*)$ and let $U = U_0^2$ mean one copy for each original internal channel. This is the actual coexact domain, not a substitute divergence-free space or an abstract low-energy band.

For a domain A, a linear map $C : A \to F$ restores target c if

$$V^{-1} Q(z) + K \|Cz\|^2 \geq c V^{-1} \|z\|^2 \quad (z \in A) \tag{2}$$

for some finite $K \geq 0$. The lower bounds below also apply to any finite penalty $\Phi(Cz)$ with $\Phi(0) = 0$. The rank on U is always the actual restricted rank $\mathrm{rank}(C|_U)$.

A canonical map uses a nonempty integer rectangle partition $\mathcal{P}_\alpha$ of the original chart for each original channel $\alpha = 1, 2$, shared by all spatial components. Its rows and full count are

$$(Cz)_{\alpha,r,S} = V^{-1/2} |S|^{-1/2} \sum_{x \in S} z_{r,\alpha}(x), \qquad N_{\mathrm{out}} = D(|\mathcal{P}_1| + |\mathcal{P}_2|). \tag{3}$$

The rectangles are products of nonwrapping chart intervals. On the ambient space these disjoint, nonzero rows are orthogonal and have rank N_{out}. On U, that equality is not presumed. Unrestricted maps may instead use arbitrary real linear rows, including internal-channel mixtures and nonlocal Fourier coefficients.

2 Actual strong components and their spectrum

Let S be the 2×2 matrix with rows s_0^T, s_1^T and put

$$Y_{l,r} = s_l \cdot z_r, \qquad \delta = 1 - |s_0 \cdot s_1| = 1 - \cos\theta > 0. \tag{4}$$

The row Gram matrix has eigenvalues $1 \pm |s_0 \cdot s_1|$. Thus S is invertible and $\|Sz\|^2 \geq \delta \|z\|^2$. This is a proof coordinate; we will specify its resulting observations as actual original-field rows. Dropping only nonnegative weak terms gives

$$Q(z) \geq H E_{\mathrm{str}}(Y), \qquad E_{\mathrm{str}}(Y) = \sum_{l,r} \sum_{x : p(x) = l} \sum_i |Y_{l,r}(x + e_i) - Y_{l,r}(x)|^2. \tag{5}$$

For each residue s of parity l, define

$$G_{l,s} = \{x : \textstyle\sum_i x_i \in \{s, s+1\} \pmod n\}.$$

For each l these $n/2$ sets partition X. They include the wrap pair $\{n-1, 0\}$ where appropriate. Put $p = D - 1$ and, for $y \in (\mathbb{Z}/n\mathbb{Z})^p$, set

$$x_s(y) = (y_1, \dots, y_p, s - \textstyle\sum_i y_i), \quad u(y) = Y_{l,r}(x_s(y)), \quad t(y) = Y_{l,r}(x_s(y) + e_D).$$

These are bijections onto the actual source and destination layers. Since $x_s(y) + e_i = x_s(y + e_i) + e_D$ for $i < D$, the component energy is exactly

$$E_s = \sum_y |u(y) - t(y)|^2 + \sum_{y,i<D} |u(y) - t(y + e_i)|^2. \tag{6}$$

Rungs and cross edges generate every source translation, and every target has a rung. The sets $G_{l,s}$ are therefore connected actual strong components. No labelled edge has been reassigned or deduplicated.

At a counting-orthonormal Fourier frequency $k \in (\mathbb{Z}/n\mathbb{Z})^p$, put $T_k = 1 + \sum_{i=1}^p e^{2\pi i k_i/n}$. The Hermitian symbol of (6) has diagonal D and off-diagonal entries $-T_k$ and its conjugate. Its eigenvalues are $D - |T_k|$ and $D + |T_k|$. Expansion gives

$$D^2 - |T_k|^2 = \sum_{0 \le i < j \le p} |e^{i\vartheta_i} - e^{i\vartheta_j}|^2, \quad \vartheta_0 = 0, \quad \vartheta_i = 2\pi k_i/n. \tag{7}$$

Consequently

$$D - |T_k| \ge \frac{1}{2D} \sum_{i=1}^p |1 - e^{2\pi i k_i/n}|^2. \tag{8}$$

At nonzero k, at least one squared term is at least $4\sin^2(\pi/n) \ge 16/n^2$. At zero frequency the eigenvalues are $0, 2D$; the zero vector branch is precisely the common constant on both layers. The other branch is also at least $8/(Dn^2)$ for $D \ge 3, n \ge 2$. Finite Parseval therefore proves

$$E_s(f) \ge \frac{8}{Dn^2} \|f - \mathrm{mean}_{G_{l,s}} f\|^2. \tag{9}$$

Conjugate Fourier frequencies pair in the real space; real Nyquist frequencies obey the same real matrix inequality. All formulas include $n = 2$ with its original positive labels.

3 Exact unrestricted ranks at the orthogonal angle

Assume $D = 3$ or 4, and fix the original basis by $v_0 = e_1, v_1 = e_2$, so channel 1 is strong on odd starts and channel 2 on even starts. Let R_{str} average each original channel and spatial component over its correct strong components. This is an ambient counting-orthogonal projection. Define C_{str} by the normalized complete component means as in (3), replacing rectangles by the corresponding $G_{l,s}$. There are Dn full rows and

$$\|C_{\mathrm{str}} z\|^2 = V^{-1} \|R_{\mathrm{str}} z\|^2.$$

Theorem 3.1. *For the orthogonal family in $D = 3, 4$, the minimum unrestricted restoring ambient rank is exactly Dn. The minimum actual coexact restoring rank is exactly $(D-1)(n-2)$, including zero at $n = 2$. The same displayed map C_{str} attains both ranks and restores target c with $K = c$ on its fixed normalized rows.*

Proof. Sum (9) over all original components. By (5),

$$Q(z) \ge \frac{8}{D} c \|(I - R_{\mathrm{str}})z\|^2.$$

Since $8/D \ge 2 > 1$, orthogonality gives $Q(z) + c\|R_{\mathrm{str}} z\|^2 \ge c\|z\|^2$ on all original fields. This proves the claimed normalized upper without any assertion about preservation of U.

We next derive that preservation and the restricted rank for this particular map. For one internal channel, write its sum-layer averages as

$$A_r(t) = n^{-(D-1)} \sum_{\sum x_i = t} z_r(x).$$

A coexact field has zero componentwise mean and backward divergence zero, by summing adjoint differences and using $d_1 d_0 = 0$. Averaging the divergence over layer t shows that $F(t) = \sum_r A_r(t)$ satisfies $F(t-1) - F(t) = 0$. Its total sum is zero, so $F = 0$. On a strong pair $\{s, s+1\}$ the projected value is $[A_r(s) + A_r(s+1)]/2$. Hence $R_{\mathrm{str}} z$ depends only on $t = \sum x_i$, has zero mean, has spatial sum zero pointwise, and is constant on the appropriate strong pairs in each channel.

Conversely, any real mean-zero spatial vector function of t whose spatial sum is zero is actually coexact. At a nonzero diagonal frequency $(j, \dots, j)$ the forward multipliers are the common value $\xi_i = e^{2\pi i j/n} - 1$. For its coefficient u, $\sum_i u_i = 0$, and hence $\sum_i \overline{\xi_i} u_i = 0$. The independent two-form coefficients

$$\psi_{ik} = \frac{\xi_i u_k - \xi_k u_i}{\sum_i |\xi_i|^2}, \quad i < k, \tag{10}$$

have actual curl adjoint u, since expansion yields

$$(d_1^* d_1 u)_r = (\textstyle\sum_i |\xi_i|^2) u_r - \xi_r \sum_i \overline{\xi_i} u_i.$$

There is no zero frequency; conjugation and the real Nyquist mode make the inverse two-form real. Thus membership in $\mathrm{im}(d_1^*)$ has been proved.

Let W_0 be these functions, additionally constant on each channel's strong pairs. With $m = n/2$, the scalar pair-constant mean-zero space has dimension $m - 1$. Choose $D - 1$ such functions per channel and set the last to minus their sum. It follows that

$$R_{\mathrm{str}}(U) = W_0, \quad \dim_{\mathbb{R}} W_0 = 2(D-1)(m-1) = (D-1)(n-2).$$

Indeed the forward inclusion was derived above, $W_0 \subset U$ by (10), and R_{str} is the identity on W_0. The kernels of $C_{\mathrm{str}}|_U$ and $R_{\mathrm{str}}|_U$ agree, so its actual rank is this dimension. Its ambient rank is Dn, because all full rows are disjoint.

Finally the full strong kernel consists of constants on each connected component and has dimension Dn. Every field w in it has $Q(w) = aE(w) \le 4Da\|w\|^2 = c\|w\|^2/4$, by the isometry of forward shifts. Its intersection with U is exactly W_0, by the two preceding directions. Any restoring linear map, with any finite penalty vanishing at zero, must be injective on each corresponding low-energy subspace. A nonzero kernel vector there contradicts the target. These give both sharp lower ranks and complete the proof. □

For $n \ge 4$ a set $G_{l,s}$ projects onto the full chart in every coordinate, but has size $2n^{D-1} < n^D$. It cannot be a product rectangle: all its factors would then be full. Thus the attaining rows are not canonical rectangle rows. The $n = 2$ component is the whole torus and is the stated harmless exception. This special projection preserves U by proof; no such property is implied for other block-averaging maps.

4 An explicit upper at any fixed positive angle

The angle assumption in Theorem 3.1 can be removed for an order-n unrestricted upper by retaining transverse modes. We do not extend its exact minima to arbitrary angles.

Theorem 4.1. *For all $D \ge 3$ and $\theta \in (0, \pi/2]$, put*

$$R = \left\lceil \sqrt{\frac{D}{4\delta}} \right\rceil, \qquad L = \min(n, 2R+1), \qquad K = \frac{2c}{\delta}.$$

There is an unrestricted real map with $2DnL^{D-1}$ displayed rows satisfying

$$V^{-1}Q(z) + K\|Cz\|^2 \geq 2cV^{-1}\|z\|^2 \tag{11}$$

on all ambient fields and hence on U. Both unrestricted minimum restoring ranks have order n as even $n \to \infty$ at fixed $D, \theta > 0$.

Proof. On each actual two-layer normal-coordinate component from Section 2, let P retain both layer coefficients at frequencies

$$\mathcal{A}_R = \{k : |k_i|_n \leq R \text{ for every } i < D\},$$

where $|k_i|_n$ is cyclic distance to zero. There are exactly L^{D-1} such frequencies. Conjugate pairs produce sine and cosine real coordinates, and each self-conjugate character is counted once. The real dimension is therefore $2L^{D-1}$ per component. The direct sum over all l, s, r is an orthogonal projection in the Y norm.

Outside the cube, some cyclic coordinate distance is at least $R+1$ and at most $n/2$. Thus (8) and $\sin(\pi q/n) \geq 2q/n$ give the lower eigenvalue bound $\gamma = 8(R+1)^2/(Dn^2)$. The upper branch is at least $D \geq \gamma$, because a nonempty tail implies $\gamma \leq 2/D$. By Parseval,

$$E_{\text{str}}(Y) \geq \gamma\|(I-P)Y\|^2.$$

If the cube is the whole frequency torus, $P = I$ and the zero-tail statement is automatic; no nonexistent exterior mode is invoked. This includes $n = 2$.

Choose a real orthonormal character basis on each retained layer and extend by zero to the other components; denote the resulting vectors by b_j. They are explicit counting-orthonormal vectors in Y coordinates. Define actual original-field rows by

$$(Cz)_j = V^{-1/2}\langle b_j, Y_{l,r}\rangle, \qquad Y_{l,r} = s_l \cdot z_r.$$

There are $2L^{D-1}$ rows per component, $n/2$ components per parity, two parities and D spatial components. Consequently $N_{\text{out}} = 2DnL^{D-1}$ and $\|Cz\|^2 = \|PSz\|^2/V$ exactly. Our cutoff yields $H\gamma \geq 2c/\delta = K$. Therefore

$$Q(z) + VK\|Cz\|^2 \geq K\|(I-P)Sz\|^2 + K\|PSz\|^2 = K\|Sz\|^2 \geq 2c\|z\|^2.$$

The same conclusion holds for a full cube using only the observed term. Orthogonality is used in the Y norm, followed by the proved singular-value bound in (4). We have not assumed that P or $S^{-1}PS$ preserves U; the ambient inequality is restricted only after its proof.

For the lower ranks, let $\mathcal{F}_l$ consist of mean-zero functions on $\mathbb{Z}_n$ constant on the strong pairs $\{l+2j, l+2j+1\}$. Its dimension is $m-1$, $m = n/2$. Choose independently $f_{l,r} \in \mathcal{F}_l$ for $r < D$, put $f_{l,D} = -\sum_{r<D} f_{l,r}$, and define original fields

$$z_r(x) = S^{-1}\begin{pmatrix} f_{0,r}(\sum x_i) \\ f_{1,r}(\sum x_i) \end{pmatrix}.$$

This is injective and has real dimension $(D-1)(n-2)$. Every original channel has zero mean and pointwise spatial sum zero; the explicit real preimage (10) proves membership in U. All actual strong normal differences vanish. Thus $Q = aE \leq c\|z\|^2/4$ on this whole subspace. Every restoring map must inject it. Dropping the mean and spatial-sum conditions gives Dn independent ambient parameters with the same energy bound, and hence the ambient lower rank Dn. Since R, K are independent of n at fixed D, θ, these lower ranks and the constructed upper count give order n. The coexact lower is zero at $n = 2$; no positive endpoint lower is asserted. □

The count constant and penalty diverge as $\theta \to 0$. At $\theta = 0$ the matrix S is singular, and the theorem does not follow by taking a limit. Even at a fixed angle, K is a coefficient on these fixed normalized rows; it is not the condition number of the full operator. The observations are nonlocal and can mix the original internal channels. No canonical upper has been obtained from this construction.

5 Why complete component means do not suffice at small angles

Proposition 5.1. *Let $v_0 = (1, 0)$, $v_1 = (\cos\theta, \sin\theta)$ and $0 < \theta \le 1/4$ radians. For every even $n \ge 2$ and $D \ge 3$, there is a nonzero real $z \in U$ annihilated by all complete strong-component means, in any constant original-channel or normal-coordinate direction, with $Q(z)/(c\|z\|^2) < 17/24$. Any finite penalty vanishing at zero therefore fails to restore target c for the map consisting only of those means.*

Proof. Put $w = (\cos(\theta/2), \sin(\theta/2))$ and let $z_D(x) = w\cos(2\pi x_1/n)$, $z_r = 0$ for $r \ne D$. At its supported original frequencies $\pm e_1$, only the first forward multiplier is nonzero and only the Dth spatial field component is nonzero. Thus $\sum_i \overline{\xi_i} u_i = 0$, and formula (10), now with these actual multipliers, constructs its real coexact preimage. At $n = 2$ the coincident Nyquist mode and preimage are real.

On any sum layer s, parameterize $x = (y, s - \sum y)$. Its cosine sum is

$$n^{D-2} \sum_{y_1=0}^{n-1} \cos(2\pi y_1/n) = 0.$$

Hence every $G_{l,s}$ mean vanishes, in every constant internal direction and every spatial component. Normalization and linear combinations of these means do not change this conclusion.

The bisector satisfies $|v_l \cdot w|^2 = \cos^2(\theta/2)$ and $|s_l \cdot w|^2 = \sin^2(\theta/2)$ for both parities. Thus its source quadratic coefficient is the same number $\beta = a\cos^2(\theta/2) + cn^2\sin^2(\theta/2)$ at every site. The periodic cosine is an exact positive-Laplacian eigenfunction, so

$$\frac{Q(z)}{\|z\|^2} = 4\sin^2(\pi/n)\left[a\cos^2(\theta/2) + cn^2\sin^2(\theta/2)\right].$$

This retains the forward labels also at $n = 2$. Using $\sin t \le t$ and $\sin^2(\pi/n) \le 1$ gives

$$\frac{Q(z)}{c\|z\|^2} \le \frac{1}{4D} + \pi^2\theta^2 \le \frac{1}{12} + \frac{\pi^2}{16} < \frac{17}{24} < 1.$$

The last strict bound uses $\pi^2 < 10$. Since every observation is zero, the target fails for every admitted penalty. The strong energy is positive at positive angle; it was not incorrectly set to zero. □

This proposition refutes the unchanged mean-only extension for a specific angle interval. It neither locates a sharp critical angle nor excludes the enriched observations in Theorem 4.1, which retain transverse modes missed by these means.

6 A canonical ambient count lower bound

Return to the orthogonal original basis $v_0 = e_1, v_1 = e_2$. Let R be the canonical counting-orthogonal block projection, so $\ker C = \ker R$ and $\|Cz\|^2 = \|Rz\|^2/V$.

Theorem 6.1. *If a canonical map restores target c on all ambient fields, with any finite penalty $\Phi(Cz)$ satisfying $\Phi(0) = 0$, then*

$$N_{\mathrm{out}} \geq \frac{3DV}{2^{2D+4}(1+8Dn^2)} \geq \frac{3D}{2^{2D+4}(1+8D)} n^{D-2}. \tag{12}$$

The statement concerns the ambient count and rank. It does not assert this lower bound on the actual coexact domain alone.

Proof. First suppose $n \geq 4$. Set $m = n/2$, $k = \lfloor m/2 \rfloor$ and $\varphi = 2\pi k/n \in [\pi/3, \pi/2]$. On sum residues put $h(2j) = h(2j+1) = \cos(2\varphi j)$. Choose a unit spatial vector b and set $z_{r,2}(x) = b_r h(\sum_i x_i)$, $z_{r,1} = 0$. Every even-source strong difference of channel 2 vanishes. Finite character sums give

$$\|z\|^2 = 2n^{D-1} \sum_j \cos^2(2\varphi j) \geq V/2.$$

The cyclic labelled difference identity on the m pair values yields

$$Q(z) = 2aD \sin^2 \varphi \|z\|^2 \leq (c/8)\|z\|^2.$$

The period-two pair cycle retains both forward labels, so this includes $n = 4$.

Write $A = (1 + e^{-i\varphi})/2$, $B = (1 - e^{-i\varphi})/2$. The paired complex carrier is $f(t) = Ae^{i\varphi t} + Be^{i(\varphi+\pi)t}$ and $h = \mathrm{Re}\, f$. Its two frequencies are well-defined on $\mathbb{Z}_n$. Each interval character sum is at most $2/|1 - e^{i\omega}| \leq 2$ for these frequencies, and $|A| + |B| \leq \sqrt{2}$. Factoring over every original rectangle S gives

$$\left| \sum_{x \in S} h(\sum_i x_i) \right|^2 \leq 2^{2D+1}.$$

For $w = z/\|z\|$, put $t = \|Rw\|^2$. With the shared spatial-component partition and nonempty cells,

$$t \leq \frac{2^{2D+2}}{V} \sum_{S \in \mathcal{P}_2} \frac{1}{|S|} \leq \frac{2^{2D+2} N_{\mathrm{out}}}{DV}.$$

The actual field $u = (I - R)w$ has $Cu = 0$ and $\|u\|^2 = 1 - t$. Ambient restoration requires $Q(u) \geq c(1 - t)$. On all ambient fields, $B \leq HI$ and forward-shift isometry imply $Q(y) \leq 4DH\|y\|^2$. Positivity of the form then gives

$$Q(w - Rw) \leq 2Q(w) + 2Q(Rw) \leq c/4 + 8Dcn^2 t.$$

Consequently $t \geq 3/[4(1 + 8Dn^2)]$. Combining these bounds proves the first inequality in (12); the second follows from $1 + 8Dn^2 \leq (1 + 8D)n^2$.

At $n = 2$, use no nonexistent oscillatory pair mode. Each channel partition has at least one cell, so $N_{\mathrm{out}} \geq 2D$. The first right-hand side of (12), divided by $2D$, equals $3/[2^{D+5}(1 + 32D)] < 1$. This proves the endpoint directly. □

In $D = 4$ the second constant is $1/11264$. The earlier ambient canonical construction in programme source 247, included in the public source package of [3], has count at most $2^{25}n^2$. Together these give order n^2 for the ambient canonical class. Theorem 3.1 gives the same-profile ambient unrestricted minimum $4n$. These are comparisons of row classes on the same domain, not comparisons of unlike domains. The earlier upper is a cited proved input, not a result rederived here.

The indispensable domain boundary in the lower proof is $u = (I - R)w$. Even if b is chosen with spatial sum zero so that w is coexact, a rectangular block projection need not preserve U. Thus u has not been shown coexact, and applying a coexact target inequality to it would be invalid. The argument supplies no canonical coexact n^2 lower in $D = 4$. In $D = 3$ the canonical coexact question of $O(n)$ versus a logarithmic loss also remains open; Theorem 4.1 does not settle it.

7 A coexact falsifier for uncut rectangular cells

Proposition 7.1. *In the orthogonal family, let $n \geq 4$ be even. If every cell of the canonical partition of either original internal channel has at least one full coordinate interval, that map cannot restore target c on U with any finite penalty vanishing at zero. The full coordinate direction may differ from cell to cell; the other channel may even observe all sites.*

Proof. Choose the indicated channel α and its strong-source parity $p_1 = 1, p_2 = 0$. Set $m = n/2 \geq 2$ and

$$F_\alpha(t) = \cos\left(\frac{2\pi}{m}\left\lfloor\frac{(t - p_\alpha) \bmod n}{2}\right\rfloor\right).$$

It is constant on each correct strong pair, including the wrap pair, has zero sum and is nonzero. Put $z_{r,\alpha}(x) = b_r F_\alpha(\sum_i x_i)$ for $b = (1, -1, 0, \ldots, 0)$, and set the other channel to zero. The mean is zero and the spatial sum is zero. Formula (10) constructs its actual real coexact preimage. Its strong differences vanish, so $Q(z) = aE(z) \leq c\|z\|^2/4$.

For any cell in that channel, fix all coordinates except a complete one. As the complete coordinate varies, the sum residue runs through every element of $\mathbb{Z}_n$ once. Its F_α sum is zero; summing the remaining cell coordinates preserves zero. Thus every spatial mean in that cell vanishes. The same field works simultaneously for all cells, irrespective of which coordinate is full. The other channel is identically zero. Therefore $Cz = 0$ and the nonzero coexact field contradicts the target. □

A necessary condition is consequently that each original channel have at least one cell proper in every coordinate. This is not sufficient, gives no asymptotic logarithmic count lower, and does not exclude arbitrary $O(n)$ partitions with all coordinates cut. At $n = 2$ the scalar zero-mean one-pair space vanishes; the proposition deliberately excludes that case.

8 Temporal matching of product slice corners

We now restrict to $D = 3$, $4 \mid n$, $n \geq 32$, and the orthogonal internal basis. Thus $a = c/48$ and $H = cn^2$. This section concerns the actual coexact domain $U = U_0^2$ and the original mean normalization, but permits an enlarged observation class: cyclic intervals may wrap in the fixed chart. We call this the cyclic extension. Section 1's canonical class uses nonwrapping chart intervals and is not redefined here. Every nonwrapping product-slice partition is a member of this extension, so restricting the theorem to those partitions gives a canonical necessary bound. The rapid example below belongs only to the cyclic extension. This supplies no unrestricted lower bound for every canonical partition.

At each longitudinal slice $x_1 = i$, let $\{J_{i,j}\}_j$ and $\{K_{i,k}\}_k$ be nonempty cyclic interval partitions of $\mathbb{Z}_n$ whose product of factor counts is the same integer q for every i. The cells of channel 1 are $\{i\} \times J_{i,j} \times K_{i,k}$. Their full spatial-row count is $3nq$. The second channel can use any permitted partition. Write E_i for the product of the terminal endpoints of the two interval partitions, so that E_i has exactly q distinct transverse points. For $K \geq 1$, let S_K be the maximum over cyclic i of the minimum bottleneck matching distance between E_i and E_{i+K}, using the periodic transverse ℓ^1 metric. These extrema exist over finite sets. We impose no common labelling around the cycle.

Theorem 8.1. *Put $L = \log((n/4 + 1)/3)$. If for some integer $K \geq 1$,*

$$L \geq 2^{18} q A_K, \qquad A_K = \max\{16K, 9K + 6S_K + S_K^2\}, \tag{13}$$

there exists a real nonzero $z \in U$ with every original mean zero and $Q(z) < 7c\|z\|^2/16$. In particular, no finite quadratic penalty on those means can restore target c. Consequently

recovery requires

$$L < 2^{18} q \max\{16K, 9K + 6S_K + S_K^2\} \quad \textit{for every integer } K \geq 1. \tag{14}$$

The assertion permits $K \geq n$ and includes the longitudinal seam.

We give the constructive proof, including the norm and original-source energy bounds. No assumed capacity estimate, projection invariance or coexact gap is used as an input.

8.1 A finite transverse cutoff

Let $Y = \mathbb{Z}_n^2$, $R = n/4$, and let ρ_p be periodic ℓ^∞ distance from $p \in Y$. Set

$$f_p(y) = \min\{1, \max\{0, L^{-1} \log((\rho_p(y) + 1)/3)\}\}.$$

It is zero for $\rho_p \leq 2$ and one for $\rho_p \geq R$. We claim

$$\sum_{y,h=2,3} |\partial_h f_p(y)|^2 \leq \frac{64}{L}, \qquad \|1 - f_p\|_Y^2 \leq \frac{4n^2}{L^2}. \tag{15}$$

For a nonconstant edge, the smaller radius is $2 \leq r < R$. There are $8r + 4 \leq 32r$ forward-labelled edges crossing that square boundary, disregarding direction. Since $r + 1 \leq R \leq n/4$, no antipodal ambiguity enters this count. Each scalar increment is at most $1/((r+1)L)$. Thus the energy is at most

$$\frac{32}{L^2} \sum_{r=2}^{R-1} \frac{r}{(r+1)^2} \leq \frac{32 \log(R+1)}{L^2} \leq \frac{64}{L},$$

where $R \geq 8$ gives $\log(R+1) \leq 2L$. For the norm, group sites by $(R+1)/2^{j+1} < \rho_p + 1 \leq (R+1)/2^j$, $j \geq 0$. Each group has at most $4(R+1)^2/4^j$ sites and deficit at most $(j+1)\log 2/L$. The latter estimate holds also where clipping makes $f_p = 0$. Since $\sum_{j \geq 0} (j+1)^2/4^j = 80/27 < 4$ and $R + 1 \leq n/2$, the deficit norm is at most $16(R+1)^2/L^2 \leq 4n^2/L^2$. This proves (15). Telescoping a shortest lattice path also gives

$$\|f_p(\cdot + u) - f_{p'}\|_Y \leq (|u|_1 + d(p, p'))\sqrt{64/L}. \tag{16}$$

Here $|u|_1$ can be any chosen integer representative length, and all seams are periodic.

8.2 Tagged windows and their translation cost

Fix K and write $S = S_K$. At slice i take the tagged list $F_i = \{(j, p) : 0 \leq j < K,\ p \in E_{i+j}\}$. Repeated spatial points keep their tags, so its size is exactly Kq even if $K \geq n$. Match F_i to F_{i+1} by keeping the designated $(K-1)q$ shared entries at the same spatial position and matching the remaining q outgoing entries to E_{i+K} with distance at most S. Some replaced entries can also be stationary; no claim about the total number of stationary matches is needed. Define

$$\eta(i, y) = \min_{(j,p) \in F_i} f_p(y).$$

It lies in $[0, 1]$ and vanishes within radius 2 of E_i. Squared maximum is bounded by the sum of squares, so (15) gives

$$\|1 - \eta\|_X^2 \leq 4KqV/L^2. \tag{17}$$

For two equally indexed real lists, $|\min a - \min b| \leq \max |a - b|$. Using the designated matching and (16),

$$\begin{aligned} \|\eta(i+1, \cdot + u) - \eta(i, \cdot)\|_Y^2 &\leq \frac{64q}{L}\{(K-1)|u|_1^2 + (|u|_1 + S)^2\} \\ &= \frac{64q}{L}(K|u|_1^2 + 2|u|_1 S + S^2). \end{aligned} \tag{18}$$

With no longitudinal shift, identity matching of all Kq entries instead gives $64Kq|u|_1^2/L$. Summing either bound over i adds a factor n. The window is only a cutoff recipe; the observed cells remain the original q cells per slice.

8.3 Actual coexact membership and observation zeros

Put $\delta_h f(x) = f(x) - f(x - e_h)$. For a real periodic Ψ, take $\phi = \delta_2\delta_3\Psi$ and the actual two-form potential $\psi_{23} = \phi$, $\psi_{12} = \psi_{13} = 0$. Since $\partial_h^* = -\delta_h$, direct adjunction yields

$$d_1^*\psi = (0, \delta_2\delta_3^2\Psi, -\delta_2^2\delta_3\Psi) = P\Psi.$$

Every component has zero full sum by telescoping. Summing the second component on a cell telescopes to four corner values of $\delta_3\Psi$; the third gives the negative four corner values of $\delta_2\Psi$. Predecessors of initial endpoints are terminal endpoints of the preceding interval. Hence all relevant corners are in E_i, including wrap cells. A whole-cycle factor cancels identically.

Use the period-four wave $\Psi_0(x) = \sin(\pi(x_1+x_2+x_3)/2)$, and set $\Psi = \eta\Psi_0$. At every corner, η and its backward transverse neighbors vanish. Thus both required differences of Ψ vanish there. Every channel 1 mean of $P\Psi$ is exactly zero, with its original positive normalization. Take channel 2 identically zero. This constructs an actual real $z = (P\Psi, 0) \in U$ annihilating all observed rows.

8.4 Nonzero norm and original energy

Direct period-four differences give

$$z_0 = P\Psi_0 = (0, w, -w), \qquad w = (2, -2, -2, 2).$$

Thus $|z_0(x)|^2 = 8$, $\|z_0\|^2 = 8V$, and at every original odd source all $\partial_h z_0$ vanish, including across seams because $4 \mid n$. For a finite convolution $\mathcal{T} = \sum_r a_r\tau^r$, where τ^r is translation by r, subtraction gives

$$\mathcal{T}(\eta\Psi_0) - \eta\mathcal{T}\Psi_0 = \sum_r a_r(\tau^r\eta - \eta)\tau^r\Psi_0.$$

Each nonzero component of P has coefficient absolute sum at most 8, longitudinal shift 0 and transverse radius at most 3. Since $|\Psi_0| \le 1$, the no-longitudinal translation bound yields

$$\|P(\eta\Psi_0) - \eta z_0\|^2 \le 73728nKq/L. \tag{19}$$

Each component of $\partial_2 P$ or $\partial_3 P$ has coefficient absolute sum at most 16 and transverse radius at most 4. Its squared translation factor is at most $16K$. Each component of $\partial_1 P$ has coefficient absolute sum at most 16, longitudinal shifts 0 or 1 and transverse radius at most 3. Its factors are bounded by $9K$ and $9K + 6S + S^2$ respectively. Consequently each derivative commutator has norm at most $16\sqrt{64nqA_K/L}$. Restricting to original odd sources where $\eta\partial_h z_0 = 0$, and summing the six squared component bounds,

$$\sum_{x \text{ odd},h} |\partial_h(P\Psi)(x)|^2 \le 98304nqA_K/L. \tag{20}$$

These are the original strong sources; no weight has been shifted.

By (17), (19) and $n \le V$,

$$\|P\Psi - z_0\|^2 \le \left(64Kq/L^2 + 147456Kq/L\right)V.$$

The threshold and $A_K \ge 16K$ imply $L \ge 2^{22}Kq$. The coefficient above is at most $2^{-38}+9/256 < 2$. Reverse triangle therefore gives $\|P\Psi\| > \sqrt{8V} - \sqrt{2V} = \sqrt{2V}$. In particular the constructed field, not just its potential, is nonzero. Multiplying (20) by $H = cn^2$ gives

$$Q_{\text{strong}}(z) \le 98304cVqA_K/L \le 3cV/8 < 3c\|z\|^2/16.$$

Every directional forward difference has norm at most $2\|z\|$, so $Q_{\mathrm{weak}}(z) \le 12a\|z\|^2 = c\|z\|^2/4$. Adding proves Theorem 8.1. Any quadratic penalty vanishes on the identically zero observation vector.

8.5 Count consequences and a rapidly moving grid

For fixed cuts, $S_1 = 0$ and $A_1 = 16$. With bounded one-step corner motion, A_1 stays bounded as well. In either class, (14) forces q to grow at least proportionally to L, and hence requires order $n \log n$ full channel 1 rows. This conclusion is restricted to these product classes, including their nonwrapping canonical subclasses. Arbitrary canonical rectangle partitions and rapidly moving product partitions remain outside this count conclusion.

In the cyclic extension, large adjacent motion does not itself guarantee recovery. For $n = 2^m$, $m \ge 5$, let $\kappa = n/4 + 2$ and $t_i = \kappa i \bmod n$. Use the singleton $\{t_i\}$ and its cyclic complement for the first transverse factor, and $\{0\}$ and its complement for the second. Use these partitions in both original channels. There are $24n$ full rows and $E_i = \{t_i, t_i - 1\} \times \{0, n-1\}$, so $q = 4$. The identity $4\kappa = n + 8$ gives $S_4 \le 8$ and $A_4 \le 148$. For $L \ge 2^{20} \cdot 148$, the theorem gives an invisible real nonzero coexact field below target. Such dyadic sizes exist, but the threshold is enormous and is not a practical crossover estimate.

The adjacent bottleneck is exactly κ: the next corner with first transverse coordinate $t_i + \kappa$ is at least that far from each old corner, since $\kappa < n/2 - 1$; translation realizes that maximum distance. Also $\gcd(\kappa, n) = 2$, so the orbit visits $n/2$ positions. This is a rapidly moving cyclic family with a short resonance. For an interior singleton, its cyclic complement wraps and is not one of the nonwrapping chart intervals in Section 1. Splitting a wrapped cell into chart rectangles introduces separate means; invisibility of their sum does not imply invisibility of each refined mean. Thus neither this $24n$ count nor its failure is transferred to that chart refinement. The theorem does not assert that every rapidly moving $O(n)$ family fails.

8.6 A separate counterexample for nonwrapping chart cells

The previous cyclic example does not imply failure of its chart refinement. We now construct a new cutoff that annihilates every refined mean separately. This gives a counterexample inside the original nonwrapping canonical class.

Proposition 8.2. *Let $n = 2^m$, $m \ge 5$, $\kappa = n/4 + 2$ and $t_i = \kappa i \bmod n$. At slice i partition the second-coordinate chart into the nonempty members of $[0, t_i - 1]$, $\{t_i\}$ and $[t_i + 1, n - 1]$, and the third coordinate into $\{0\}$ and $[1, n - 1]$. Use these nonwrapping product cells in both original channels, with first coordinate $\{i\}$. Their exact full count is $36n - 24$. If*

$$L \ge 2^{18} \cdot 6 \cdot 148, \tag{21}$$

there is a real nonzero $z \in U$ with each of these original means zero and $Q(z) < 7c\|z\|^2/16$. This particular canonical family therefore cannot restore target c with any finite quadratic observation penalty.

Proof. Since $\gcd(\kappa, n) = 2$, the orbit visits each even position twice. Thus $t_i = 0$ occurs twice, $t_i = n - 1$ never occurs, and all other t_i are interior. There are four cells per channel at the two zero slices and six at each other slice. Counting all spatial rows in both channels gives $2 \cdot 3\{2 \cdot 4 + (n-2) \cdot 6\} = 36n - 24$.

For interior t_i, the actual terminal second-coordinate endpoints are $t_i - 1, t_i, n - 1$; for $t_i = 0$ they are $0, n - 1$. The third-coordinate endpoints are $0, n - 1$. Let G_i be the six-entry tagged list obtained from $t_i, t_i - 1 \bmod n, n - 1$, each paired with those two third-coordinate endpoints. It covers every actual corner, including the extra chart boundary. At $t_i = 0$ some

positions coincide but retain their tags. Six is a covering-list count, not an assertion of six cells at every slice.

The identity $4\kappa = n + 8$ gives a same-tag matching from G_i to G_{i+4}: four entries translate by $(8, 0)$ and two are stationary. Form the 24-entry window F_i from $G_i, G_{i+1}, G_{i+2}, G_{i+3}$ and let $\eta(i, y)$ be the minimum of the associated radial functions f_p. The designated 18 shared entries remain stationary between consecutive windows; the six replaced entries have displacement at most8. Directly using (15) and the minimum inequality gives

$$\|1 - \eta\|_X^2 \le 4 \cdot 4 \cdot 6V/L^2,$$
$$\|\eta(i+1, \cdot + u) - \eta(i, \cdot)\|_Y^2 \le \frac{64 \cdot 6}{L}\{4|u|_1^2 + 16|u|_1 + 64\}.$$

Without longitudinal shift the bound is $64 \cdot 4 \cdot 6|u|_1^2/L$. These explicit estimates remain valid with repeated centres. We have not applied the constant-cell-count theorem to this varying four/six-cell map.

Set $\Psi = \eta\Psi_0$ and use the actual potential and field $z = (P\Psi, 0)$ from the preceding proof. At every actual refined corner, η and its backward transverse neighbours vanish. For each individual chart interval, its initial-endpoint predecessor is a terminal endpoint of the preceding interval, with $-1 = n-1$ at the chart boundary. Every one of the four corners in the telescoped cell sum is therefore covered by G_i. Both needed first differences of Ψ vanish there, so each separately normalized refined mean is zero. No zero sum on a union is used to infer zero sums on its pieces. The actual curl-adjoint formula proves real membership in U.

For the convolution shifts used by P and its derivatives, the translation factors are bounded by $A = \max\{64, 36 + 48 + 64\} = 148$. The same coefficient sums, now applied to the explicit six-centre estimates above, give

$$\|P(\eta\Psi_0) - \eta z_0\|^2 \le 73728n \cdot 4 \cdot 6/L,$$
$$\sum_{x \text{ odd}, h} |\partial_h(P\Psi)(x)|^2 \le 98304n \cdot 6 \cdot 148/L,$$
$$\|P\Psi - z_0\|^2 \le (64 \cdot 24/L^2 + 147456 \cdot 24/L)V.$$

Threshold (21) implies $L \ge 2^{22} \cdot 24$. The last coefficient is at most $2^{-38} + 9/256 < 2$, so $\|P\Psi\| > \sqrt{2V}$. The field itself is nonzero. Multiplying the strong estimate by the original $H = cn^2$ bounds strong energy by $3cV/8 < 3c\|z\|^2/16$; weak energy is at most $12a\|z\|^2 = c\|z\|^2/4$. This proves the strict $7c/16$ bound. All observation penalties vanish on the zero mean vector. □

This is a newly constructed invisible field for this particular nonwrapping refinement. It does not retroactively validate a split-cell inference for the old cyclic field. The threshold is enormous and unoptimized, and no general impossibility of canonical $O(n)$ partitions or global canonical logarithmic lower bound follows.

9 Interpretation and remaining question

The sharp orthogonal ranks count the actual strong kernel and its explicitly derived coexact image. Positive-angle enrichment pays a deteriorating singular-value comparison by retaining finitely many transverse modes per component. The temporal matching theorem supplies an actual invisible coexact field for the cyclic product-slice extension, including a rapidly moving resonant example outside the nonwrapping canonical class. The separate chart-refinement construction adds the previously missing boundary corners and gives a new counterexample within that class. These restricted quantifiers do not settle the general canonical count optimum.

The small-angle counterexample explains why that enrichment cannot simply be replaced by an assertion that the old means still suffice. The canonical ambient lower instead uses a real oscillatory carrier and an actual ambient kernel projection. Its failure to transfer to U is an explicit domain obstruction, not a technical detail that can be omitted.

The remaining question is whether, for the orthogonal $D = 3$ family and actual U, canonical original-channel integer rectangle partitions admit $O(n)$ full count with some finite size-dependent quadratic penalty, or whether the known logarithmic upper loss is unavoidable. The mean supports and Fourier rows used in the unrestricted theorems are not permitted substitutes. No theorem here establishes a size-uniform canonical penalty, efficient implementation or a full-operator condition number. All results concern a finite toy gradient family. The continuum and physical Yang–Mills mass gap remain open; the programme's PR29 accounting remains 20/41, with `TermSource=0`.

10 Model assessments and review context

The analytical source texts 251–254, 256–259 and261C received complete involved AI source readings. Their exact source bytes, full reports, objections and context must remain in the reproducibility package. The temporal proof uses the designated shared-slot wording specified in the retained 259 clarification. These readings are not human peer reviews, blind readings, independent provider samples or Lean verification. The exposed source reviewer identity was Codex, based on the OpenAI GPT-6 family; exact served backend, version and reasoning effort were not exposed. Reviewers shared account, infrastructure and programme context and had prior involvement.

At this source stage the proposed revision has no exact-PDF numerical assessment, successful production claim or visual-review claim. No grade is assigned or inherited. The existing 255B version 1 PDF has SHA256

`d45a17a4142204628189441b9133aa1027b0d7577a56dc5de6c576d2af6102e7`

Its first exact-PDF assessment recorded importance 4.00/10, overall 4/10 stars, recommendation `major_revision` and three material objections open. Those are historical facts about that version, not predetermined grades or objection states for a successor. Its full original report must be retained unchanged. A new version requires its genuine first assessment of the exact new PDF; repeated samples to improve a grade are not evidence of scientific progress.

Analytical correctness is separate from originality and relevance. Spectral enrichment is established literature. The finite radial cutoff, minimum inequality and window-cover mechanisms are elementary; historical priority of this particular application remains unestablished. The huge sufficient threshold is not a demonstrated practical regime. Neither a deposit nor Working-paper status constitutes editorial acceptance or human peer review.

Reproducibility and provenance

Scientific inputs are frozen programme sources 251–254, 256–259 and261C. The source archive must retain their raw bytes, complete analytical reports and context, this manuscript's mapping, the frozen Git commit and all actual production logs. In particular the temporal theorem is derived in source 259 at the frozen commit

`2f370a7da850b8066a9afe95c523e49bd1ef725e`

Its full audit and clarification were integrated unchanged through merge

`8485cfb2b8b11bb5acd240d8a0b133457bbfe2a1`

The separate nonwrapping refinement construction is frozen at

`1a01fa019069e0f5a7392601ddb3c7a3febe222d`

The separate golden-orbit working derivation 260 is excluded from this manuscript: no unreviewed frontier result is promoted here.

Linux compilation, extraction and page rendering are production checks; they do not certify mathematics or transfer a model score. No such build or exact-PDF review is claimed at this draft-source stage. Subsequent production and review facts belong to dated sidecars linked to the exact PDF hash. The original published version remains immutable. The frozen production source is `fb3ba8d2c0ae52d4a2e909b0fb2df02b2f953bc4`.

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