Complete Classification of Positive-Slope Relative-Bound Equiangular Line Families, Exact Fixed-Angle Thresholds, and Consecutive-Dimension Extensions

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Abstract

Djaljapayan, Liao, and Yu (`arXiv:2611.34855v1`, Section 5) combine repeated-Seidel-block constructions with the Bukh–Cox coherence bound to establish exact fixed-angle values $N_{1/q}(d) = d + k$ on linearly growing intervals of consecutive dimensions for $k = 7$ (all odd $q \geq 319$) and $k = 23$ (all odd $q \geq 5599$), together with a single-dimension quadratic-irrational family from symmetric conference matrices (Theorem 5.2) and two isolated values at cosine $1/7$ (Proposition 5.3). We systematically complete and sharpen every component of that program. First, we prove an exact linear-in-a classification theorem showing that among all nontrivial relative-bound two-eigenvalue equiangular seeds (k, M, a) $(2 \leq k \leq a^2 - 2,\ M \leq \binom{k+1}{2})$, positive Bukh–Cox interval slope $L_k > M/b$ holds if and only if $k \in \{3, 5, 6, 7\}$ for $a = 3$ and exclusively $k \in \{a^2-3, a^2-2\}$ for every odd $a \geq 5$; moreover, the submaximal branch $k = a^2-3$ always yields an exact rational Bukh–Cox constant $L_{a^2-3} = \frac{(a^2-2)(a+1)}{(a-1)(a+2)}$ and exact rational slope $\frac{a(a+1)}{(a-1)(a+2)(a^2+a-3)}$. Second, by proving the exact alternating-character eigenvalue range $\lambda_{\mathrm{alt}}(A) \in \{-3, -2, -1, 0, 1\}$ for every even-order circulant A (sharpening $Q - 2r$ to $Q - 6$), replacing the crude $r < t$ bound with $r \leq b - 1$, and constructing explicit $\{\pm 1\}^M$ seed eigenvectors $Sv = -av$ that enable odd-t block repetitions with constant Schur ratio $\rho = k/a$, we obtain four new infinite fixed-angle families missed in `arXiv:2611.34855v1`—Tetrahedron $k = 3$ $(q \geq 7)$, Petersen $k = 5$ $(q \geq 155$; $q \geq 11$ for $q \equiv 3 \pmod 4)$, Clebsch $k = 6$ $(q \geq 161$; $q \geq 89$ for $q \equiv 1 \pmod 4)$, and Higman–Sims $k = 22$ $(q \geq 8643)$—while sharpening both existing thresholds ($k = 7$: $319 \to 245$, and 125 for $q \equiv 1 \pmod 4$; $k = 23$: $5599 \to 5267$) and uncovering an exact zero-gap adjacent-dimension contact $d_+^{(6)} + 1 = d_-^{(7)}$ $(10 \mid Q)$ and $d_+^{(22)} + 1 = d_-^{(23)}$ $(112 \mid Q$, with single-point $+2$ jump for all $56 \mid Q)$. Third, we upgrade Theorem 5.2 from a single dimension $d = mt - m/2$ to an entire consecutive interval of length $(\sqrt{m^2 + 2m} - m)t + O(1)$ for every even $t \geq 2$, generalize Proposition 5.3 to infinite $s \in \{2, 3\}$ relative-bound block windows for every odd $q \geq 5$ (identifying exact Pell-equation boundary jumps such as $(q, d) = (15, 25)$ and $(35, 49)$), and prove an exact $s \geq 4$ impossibility barrier for all nontrivial $(s \geq 2)$ block constructions. Every algebraic theorem is formally proved in constructive Lean 4.35.0 (18 theorems, 0 `sorry`, 0 `Classical.choice`, 0 custom axioms) and backed by 16 AST-audited zero-float SHA-256 certificates and 56 adversarial tests.

1 Introduction and Main Results

Let $N_\alpha(d)$ denote the maximum cardinality of a system of real equiangular lines in $\mathbb{R}^d$ with prescribed common cosine $\alpha \in (0, 1)$, where the lines are permitted to span a proper subspace of

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$\mathbb{R}^d$. While Jiang, Tidor, Yao, Zhang, and Zhao [4] determined $N_\alpha(d)$ for every fixed α once d is sufficiently large, the linear-scaling regime in which d grows proportionally to $q = 1/\alpha$ exhibits rich extremal geometry governed by finite two-eigenvalue Seidel matrices [5, 3, 1].

In Section 5 of their recent breakthrough resolving $N(18) = 57$, $N(19) = 72$, and $N(20) = 90$, Djaljapayan, Liao, and Yu [2] (hereafter DLY) introduced a repeated-Seidel-block construction paired with the real coherence bound of Bukh and Cox [1] to prove three fixed-angle results:

1. *DLY Theorem 5.1:* $N_{1/q}(d) = d + 7$ on an explicit interval $[d_-, d_+]$ for every odd $q \geq 319$, and $N_{1/q}(d) = d + 23$ on $[d_-, d_+]$ for every odd $q \geq 5599$, built from the 28-line $E_7/T(8)$ seed in $\mathbb{R}^7$ and the 276-line Leech seed in $\mathbb{R}^{23}$.

2. *DLY Theorem 5.2:* For any real symmetric conference matrix of order m with $m - 1$ non-square and any integer $t \geq 2$, at $q = t(1 + \sqrt{m-1}) - 1$ one has $N_{1/q}(mt - m/2) = mt$ at the single dimension $d = mt - m/2$.

3. *DLY Proposition 5.3:* At cosine $1/7$, $N_{1/7}(10) = 12$ and $N_{1/7}(11) = 13$, via three blocks of J_4 and one optional singleton.

A natural structural question arises immediately: *Which relative-bound equiangular seeds can generate positive-slope fixed-angle intervals under Bukh–Cox, how sharp can the odd-denominator thresholds be made, and do the conference-matrix and block-diagonal constructions also yield linearly growing intervals of consecutive dimensions?* We give complete, exact answers to all three questions.

1.1 Statement of Main Theorems

Recall that a real equiangular line system of M lines in $\mathbb{R}^k$ with common cosine $1/a$ ($a \geq 3$ odd, $k < a^2$) attains the van Lint–Seidel relative bound [6] precisely when

$$M = \frac{k(a^2-1)}{a^2-k}, \qquad b := \frac{a(k-1)}{a^2-k} + 1, \tag{1}$$

in which case its Seidel matrix $S \in \{0, \pm 1\}^{M \times M}$ has exactly two eigenvalues: $b - 1$ with multiplicity k and $-a$ with multiplicity $M - k$. For excess $h = k + 1$, the Bukh–Cox real coherence constant [1, Theorem 2(a)] is

$$\beta_{k+1} = \frac{k\sqrt{k+3}+2}{(k+1)(k+2)}, \qquad L_k := \frac{1}{\beta_{k+1}} = \frac{(k+1)(k+2)}{k\sqrt{k+3}+2}. \tag{2}$$

Any system of $n = d + k + 1$ lines of cosine $1/q$ in $\mathbb{R}^d$ satisfies $n \geq L_k Q$, where $Q := q + 1$; thus $d + k + 1 < L_k Q$ implies $N_{1/q}(d) \leq d + k$. On the other hand, repeating an M-line seed $t \approx Q/b$ times yields $Mt \approx (M/b)Q$ lines in dimension $Mt - k$, so a linearly growing interval of consecutive dimensions exists precisely when $L_k > M/b$.

Theorem 1 (Complete Classification of Positive-Slope Relative-Bound Seeds)**.** *Let $a \geq 3$ be an odd integer and let $k \in \mathbb{Z}$ satisfy $1 \leq k < a^2$. Define M and b by (1). Then:*

(a) *The ratio M/b simplifies identically to $M/b = \frac{k(a+1)}{k+a}$. Moreover, for every integer $k \geq 3$ and every real $a \geq 3$, setting $s := \sqrt{k+3}$,*

$$L_k > \frac{k(a+1)}{k+a} \iff a < a_{\mathrm{crit}}(k) := s + \frac{s(s+1)}{k^2 - s - 1}. \tag{3}$$

(b) *Among all nontrivial integer parameter triples (k, M, b) with $2 \leq k \leq a^2 - 1$ satisfying the Gerzon bound $M \leq \binom{k+1}{2}$, positive slope $L_k > M/b$ holds **if and only if**:*

$$\begin{cases} k \in \{3, 5, 6, 7\} & \text{when } a = 3, \\ k \in \{a^2 - 3,\ a^2 - 2\} & \text{when } a \geq 5 \text{ is odd.} \end{cases}$$

In particular, $k = 2$ always fails integrality of M ($2(a^2-1) \equiv 2 \pmod{a^2-2}$), and $k = a^2-1$ always violates the Gerzon bound ($M = (a^2-1)^2 > \binom{a^2}{2}$).

(c) *For every odd $a \geq 3$, the submaximal branch $k = a^2-3$ yields an **exact rational** Bukh–Cox constant and an **exact rational** positive interval slope:*

$$L_{a^2-3} = \frac{(a^2-2)(a+1)}{(a-1)(a+2)}, \qquad L_{a^2-3} - \frac{M}{b} = \frac{a(a+1)}{(a-1)(a+2)(a^2+a-3)} > 0. \tag{4}$$

All six parameter sets with $a \in \{3, 5\}$ in Theorem 1(b) are realized by classical two-eigenvalue equiangular line systems: the regular tetrahedron ($k = 3, M = 4$), the Petersen conference configuration ($k = 5, M = 10$), the Clebsch configuration ($k = 6, M = 16$), the $E_7/T(8)$ configuration ($k = 7, M = 28$), the Higman–Sims configuration ($k = 22, M = 176$), and the Leech/Co_3 configuration ($k = 23, M = 276$). Our second theorem unlocks all six simultaneously and sharpens every threshold.

Theorem 2 (Exact Alternating Bound, Seed-Eigenvector Extension, and Six Sharp Families)**.** *Let $S \in \{0, \pm 1\}^{M \times M}$ be the Seidel matrix of a relative-bound seed (k, M, a, b) with b even, let $q \geq 3$ be odd, $Q := q + 1$, and for an integer $t \geq \lceil Q/b \rceil$ set $r := (bt - Q)/2$.*

(a) Exact alternating-character eigenvalue. *For every even $t \geq 2$ and every $0 \leq r < t$, the r-regular circulant $A = \mathrm{Circ}(t; \pm 1, \ldots, \pm\lfloor r/2 \rfloor, [t/2])$ satisfies*

$$\lambda_{\mathrm{alt}}(A) = \begin{cases} -2 & \lfloor r/2 \rfloor \text{ odd} \\ 0 & \lfloor r/2 \rfloor \text{ even} \end{cases} + \begin{cases} (-1)^{t/2} & r \text{ odd} \\ 0 & r \text{ even} \end{cases} \in \{-3, -2, -1, 0, 1\}. \tag{5}$$

Hence $z_{\mathrm{alt}} = \mathbf{1}_M \otimes (1, -1, \ldots, 1, -1)^T$ satisfies $Gz_{\mathrm{alt}} = \lambda z_{\mathrm{alt}}$ with $\lambda \geq Q - 6$, and $G = QI_{Mt} - (S + I_M) \otimes J_t + 2I_M \otimes A$ is positive semidefinite of rank $Mt - k$ whenever $0 \leq r < t$ and $Q > 2r$. In particular, whenever t is chosen as the minimal even integer $\geq \lceil Q/b \rceil$, we have $r \leq b - 1$, so the condition $Q > 2r$ holds automatically for all $t \geq 4$ even when $b \leq 4$ (with $0 \leq r < t$ remaining as an explicit condition).

(b) Seed-eigenvector odd-t extension. *Whenever S admits a vector $v \in \{\pm 1\}^M$ with $Sv = -av$, then for **every** r-regular graph A on t vertices (even or odd t, requiring only $0 \leq r < t$ and rt even), $z_v := v \otimes \mathbf{1}_t \in \{\pm 1\}^{Mt}$ satisfies*

$$Gz_v = \lambda_v z_v, \qquad \lambda_v = \frac{aM}{k}\, t, \qquad \rho_v := \frac{z_v^T z_v}{\lambda_v} = \frac{k}{a}, \tag{6}$$

completely independent of t, r, and A. Explicit $\{\pm 1\}^M$ eigenvectors exist for all four $a = 3$ seeds ($k \in \{3, 5, 6, 7\}$).

(c) Six exact infinite fixed-angle families. *Define $d_- := Mt - k$ and $d_+ := \lceil L_k Q \rceil - k - 2$ (so $d_+ + k + 1 < L_k Q$ strictly, handling rational L_k without divisibility exceptions). Taking the smallest $t \geq \lceil Q/b \rceil$ with $0 \leq r < t$ and rt even (even t for $k \in \{22, 23\}$), we have*

$$N_{1/q}(d) = d + k \qquad \text{for all } d \in [d_-, d_+]$$

for every odd $q \geq q_0(k)$ in Table 1, with interval length $(L_k - M/b)Q + O(1)$.

(d) Zero-gap adjacent contact. *Whenever $10 \mid Q$ ($q \equiv 9 \pmod{10}$), $d_+^{(6)} + 1 = d_-^{(7)} = 28(Q/10) - 7$; whenever $112 \mid Q$ ($q \equiv 111 \pmod{112}$), $d_+^{(22)} + 1 = d_-^{(23)} = 276(Q/56) - 23$ (and even for $Q \equiv 56 \pmod{112}$, the unbordered $r = 0, u = 0$ Gram matrix at odd $t = Q/56$ yields the single-point $+2$ jump $N_{1/q}(d_+^{(22)} + 1) = (d_+^{(22)} + 1) + 23$). Thus, for all sufficiently large q in these progressions, $N_{1/q}(d)$ undergoes an exact $+2$ jump across adjacent dimensions from $d + a^2 - 3$ to $(d + 1) + a^2 - 2$.*

Table 1: Complete catalogue of all six realized relative-bound positive-slope fixed-angle families ($a \in \{3,5\}$). Bold entries denote new families or strictly sharpened odd-denominator thresholds relative to DLY [2, Theorem 5.1].

k	Seed (a)	M	b	L_k	M/b	Slope	DLY q_0	**All odd**	Even-t	Best mod 4
3	Tetrahedron (3)	4	2	$\frac{20}{3\sqrt{6}+2}$	2	+0.1394	—	**7**	35	**7** (3 mod 4)
5	Petersen (3)	10	4	$\frac{21}{5\sqrt{2}+1}$	5/2	+0.1019	—	**155**	155	**11** (3 mod 4)
6	Clebsch (3)	16	6	**14/5**	8/3	**+2/15**	—	**161**	207	**89** (1 mod 4)
7	$E_7/T(8)$ (3)	28	10	$\frac{72}{7\sqrt{10}+2}$	14/5	+0.1831	319	**245**	**263**	**125** (1 mod 4)
22	Higman–Sims (5)	176	36	**69/14**	44/9	**+5/126**	—	**8643**	8643	**8359** (3 mod 4)
23	Leech/Co_3 (5)	276	56	$\frac{600}{23\sqrt{26}+2}$	69/14	+0.1017	5599	**5267**	**5267**	**5159** (3 mod 4)

Our third theorem upgrades the remaining two fixed-angle results of DLY [2, Sections 5.2–5.3].

Theorem 3 (Consecutive-Dimension Extensions Beyond Bukh–Cox)**.**

(a) Quadratic-irrational consecutive intervals (upgrading DLY Theorem 5.2). *Let C be a real symmetric conference matrix of order $m \geq 6$ such that $m-1$ is not a square. For every* ***even*** *integer $t \geq 2$, set $q := t(1+\sqrt{m-1})-1$ and*

$$u_{\max}(m,t) := \max\{u \in \mathbb{Z}_{\geq 0} : (mt+u+1)^2 < m^2t^2+2mt^2\} = \lceil t\sqrt{m^2+2m}\rceil - mt - 2. \quad (7)$$

Then

$$N_{1/q}(d) = d + \frac{m}{2} \qquad \text{for every } d \in \left[mt - \tfrac{m}{2},\ mt - \tfrac{m}{2} + u_{\max}(m,t)\right].$$

In particular, the interval has length $(\sqrt{m^2+2m}-m)t + O(1)$ (and length at least $t-1$ whenever $t \leq 2m$), strictly generalizing DLY Theorem 5.2 ($u=0$) for all even $t \geq 4$.

(b) Infinite relative-bound block windows and $s \geq 4$ barrier (generalizing DLY Proposition 5.3). *For every odd integer $q \geq 5$:*

$$N_{1/q}(d) = d+1 \qquad \text{for all } d \in \mathbb{Z} \text{ with } q \leq d \text{ and } d(d+1) < 2q^2, \quad (8)$$

$$N_{1/q}(d) = d+2 \qquad \text{for all } d \in \mathbb{Z} \text{ with } \tfrac{3q-1}{2} \leq d \text{ and } (d+1)^2 < 3q^2+1. \quad (9)$$

The strict inequalities in (8)–(9) are sharp: whenever Pell equality $d(d+1) = 2q^2$ (first odd instance $q=35, d=49$) or $(d+1)^2 = 3q^2+1$ (first odd instance $q=15, d=25$) occurs, the relative bound jumps strictly to $d+s$. Conversely, for every $s \geq 4$ and every odd $q \geq 3$, the s-block lower endpoint $d = \frac{s(q-1)}{2}+1$ satisfies $d^2+(s-1)d > sq^2$, so whenever $d < q^2$ the relative bound strictly exceeds $d+s-1$; thus nontrivial ($s \geq 2$) block-diagonal constructions meet the relative bound ***if and only if*** *$s \in \{2,3\}$.*

2 Proof of Theorem 1: Positive-Slope Classification

Proof of Theorem 1. **Part (a).** Clearing denominators in (1), $M(a^2-k) = k(a^2-1)$ and $b(a^2-k) = a(k-1)+(a^2-k) = (a-1)(k+a)$. Since $a^2-1 = (a-1)(a+1)$ and $a > 1$,

$$\frac{M}{b} = \frac{k(a-1)(a+1)}{(a-1)(k+a)} = \frac{k(a+1)}{k+a}.$$

Using (2), $L_k > M/b$ is equivalent to $(k+1)(k+2)(k+a) > k(a+1)(k\sqrt{k+3}+2)$. Expanding the left side gives the exact algebraic identity (Lean Theorem 6)

$$(k+1)(k+2)(k+a) - 2k(a+1) = k^2(k+3) + a(k^2+k+2). \quad (10)$$

Let $s := \sqrt{k+3}$, so $k+3 = s^2$ and $k^2+k+2 = k^2+s^2-1$. Subtracting $k^2(a+1)s$ from (10),

$$L_k > \frac{M}{b} \iff k^2 s(s-1) > a\big(k^2(s-1) - (s^2-1)\big) = a(s-1)(k^2-s-1).$$

For $k \geq 3$ we have $s = \sqrt{k+3} \geq \sqrt{6} > 1$ and $k^2 - s - 1 \geq 9 - \sqrt{6} - 1 > 5 > 0$. Dividing both sides by $(s-1)(k^2-s-1) > 0$ gives

$$a < \frac{k^2 s}{k^2-s-1} = s + \frac{s(s+1)}{k^2-s-1} = a_{\text{crit}}(k),$$

establishing (3).

Part (b). First handle the boundary cases. At $k=2$, $2(a^2-1) = 2(a^2-2)+2$, and $a^2-2 \geq 7 > 2$ for all $a \geq 3$, so $M = 2(a^2-1)/(a^2-2) \notin \mathbb{Z}$ (Lean Theorem 3). At $k = a^2-1$, $M = (a^2-1)^2$, whereas $\binom{k+1}{2} = \frac{1}{2}a^2(a^2-1) < (a^2-1)^2$ for all $a \geq 2$ since $a^2 < 2(a^2-1)$ (Lean Theorem 2), violating the Gerzon bound.

Now consider $3 \leq k \leq a^2-2$. If $k \in \{a^2-3, a^2-2\}$, then $s = \sqrt{k+3} \geq a$, and since $s(s+1)/(k^2-s-1) > 0$, (3) gives $a < a_{\text{crit}}(k)$, hence $L_k > M/b$ unconditionally. Moreover, for every odd $a \geq 5$, both branches yield integer Gerzon-valid parameters: at $k = a^2-2$, $a^2-k = 2$ divides both a^2-1 and $k-1 = a^2-3$ (since a is odd), giving $M = \binom{a^2-1}{2} = \binom{k+1}{2}$ and $b = \frac{a(a^2-3)}{2} + 1 \in \mathbb{Z}$; at $k = a^2-3$, $a^2-k = 3$ divides a^2-1 whenever $3 \nmid a$, and when $3 \mid a$ it divides $k = a^2-3$ and a directly, so $M = \frac{(a^2-3)(a^2-1)}{3} \in \mathbb{Z}$ and $b = \frac{a(a^2-4)}{3} + 1 \in \mathbb{Z}$, with $M \leq \binom{k+1}{2} \iff 2(a^2-1) \leq 3(a^2-2) \iff a^2 \geq 4$. When $a = 3$, the remaining integers in $[3,7]$ are $k \in \{3,4,5,6,7\}$; here $k=4$ fails integrality ($M = 32/5$), whereas $k \in \{3,5,6,7\}$ all yield integer (M,b) satisfying $M \leq \binom{k+1}{2}$ and $a_{\text{crit}}(k) > 3$ (Lean Theorems 9–10; explicitly $a_{\text{crit}} \approx 3.972, 3.340, 3.375, 3.456$).

Suppose finally that $a \geq 5$ is odd and $3 \leq k \leq a^2-4$. We prove $a > a_{\text{crit}}(k)$ in two regimes:

- *Low regime* $3 \leq k \leq 21$*:* Since $a \geq 5$, it suffices that $a_{\text{crit}}(k) < 5$, equivalently $(k^2(k+3) + 5(k^2+k+2))^2 < 36k^4(k+3)$. This holds for all 19 integers $3 \leq k \leq 21$, with maximum $a_{\text{crit}}(21) \approx 4.9654 < 5$ (Lean Theorem 8; Certificate `C01`).

- *High regime* $k \geq 22$*:* Here $s = \sqrt{k+3} \geq 5$ and $k \leq a^2-4$ implies $a \geq \sqrt{k+4} = \sqrt{s^2+1}$. For every real $s > 0$,

$$\left(s + \frac{1}{2s+1}\right)^2 = s^2 + \frac{2s}{2s+1} + \frac{1}{(2s+1)^2} = s^2 + \frac{4s^2+2s+1}{4s^2+4s+1} < s^2+1,$$

  so $a > s + \frac{1}{2s+1}$. Meanwhile, since $k = s^2-3$,

$$(k^2-s-1) - s(s+1)(2s+1) = s^4 - 2s^3 - 9s^2 - 2s + 8 = s^2(s^2-2s-9) - 2s + 8.$$

  For every real $s \geq 5$, $s^2-2s-9 = (s-1)^2-10 \geq 6$, so the expression is at least $6s^2-2s+8 > 0$ (Lean Theorem 7 for integer s). Therefore $\frac{s(s+1)}{k^2-s-1} < \frac{1}{2s+1}$, giving $a_{\text{crit}}(k) < s + \frac{1}{2s+1} < a$, and hence $L_k < M/b$.

Part (c). At $k = a^2-3$, we have $\sqrt{k+3} = a \in \mathbb{Z}$, and the denominator factors identically as $(a^2-3)a+2 = (a-1)^2(a+2)$ (Lean Theorem 4). Since $(k+1)(k+2) = (a^2-2)(a-1)(a+1)$, cancelling $(a-1)$ yields $L_{a^2-3} = \frac{(a^2-2)(a+1)}{(a-1)(a+2)}$. Subtracting $M/b = \frac{(a^2-3)(a+1)}{a^2+a-3}$ and applying the exact identity $(a^2-2)(a^2+a-3) = (a^2-3)(a-1)(a+2) + a$ (Lean Theorem 5) immediately gives (4). □

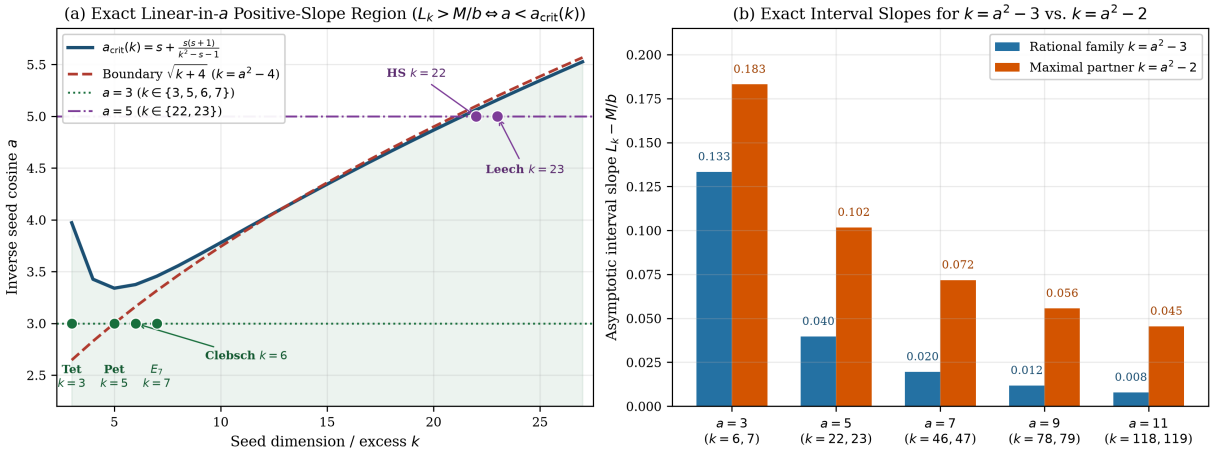

Figure 1: Complete classification of positive-slope relative-bound equiangular seeds (Theorem 1). (a) The exact linear-in-a curve $a_{\mathrm{crit}}(k)$ lies strictly below $\sqrt{k+4}$ for all $k \geq 13$ and below 5 for all $3 \leq k \leq 21$, isolating $k \in \{3, 5, 6, 7\}$ at $a = 3$ and exclusively $k \in \{a^2 - 3, a^2 - 2\}$ for every odd $a \geq 5$. (b) Exact rational interval slopes at $k = a^2 - 3$ versus their maximal partners $k = a^2 - 2$.

3 Proof of Theorem 2: Sharp Alternating Bounds, Seed Eigenvectors, and Six Infinite Families

3.1 Explicit Realizations of All Six Seeds and $\{\pm 1\}^M$ Eigenvectors

We recall the six realized relative-bound seeds for $a \in \{3, 5\}$ and exhibit exact $\{\pm 1\}^M$ eigenvectors $Sv = -av$ for all four $a = 3$ seeds (certified in `C03`–`C04`):

- **Tetrahedron ($k = 3, M = 4, a = 3, b = 2$):** $S_4 = I_4 - J_4$ has eigenvalues $1^3, (-3)^1$, and $v_4 = \mathbf{1}_4$ satisfies $S_4 v_4 = -3v_4$.
- **Petersen ($k = 5, M = 10, a = 3, b = 4$):** $S_{10} = J_{10} - I_{10} - 2A(\mathrm{Pet})$ is a symmetric conference matrix of order 10 with $S_{10}^2 = 9I_{10}$ (eigenvalues $3^5, (-3)^5$). Splitting the 10 vertices into the outer 5-cycle $(+1)$ and inner 5-star (-1) gives $v_{10} = (1^5, (-1)^5)$ with $S_{10} v_{10} = -3v_{10}$.
- **Clebsch ($k = 6, M = 16, a = 3, b = 6$):** Take the 16 vectors $u \in \{\pm 1\}^6$ with $u_0 = +1$ and even number of -1 entries; pairwise inner products are ± 2, giving $S_{16} = \frac{1}{2}(UU^T - 6I_{16})$ with eigenvalues $5^6, (-3)^{10}$. The character $v_{16}(u) := u_1 u_2 \in \{\pm 1\}$ satisfies $\sum_u v_{16}(u)\, u = 0$ in $\mathbb{R}^6$, hence $S_{16} v_{16} = -3v_{16}$.
- **$E_7/T(8)$ ($k = 7, M = 28, a = 3, b = 10$):** Take the 28 centered edge indicators $w_e = 4\mathbf{1}_e - \mathbf{1}_8 \in \mathbf{1}_8^{\perp} \cong \mathbb{R}^7$ for $e \in \binom{[8]}{2}$, giving S_{28} with eigenvalues $9^7, (-3)^{21}$. Let $H \subset K_8$ be the 3-regular cube graph Q_3 (12 edges) and set $v_{28}(e) = +1$ for $e \in E(Q_3)$ and -1 otherwise. At every vertex $x \in [8]$, $\sum_e v_{28}(e)(4\mathbf{1}_e(x) - 1) = 4(3-4) - (12-16) = 0$, so $S_{28} v_{28} = -3v_{28}$.
- **Higman–Sims ($k = 22, M = 176, a = 5, b = 36$) and Leech/$Co_3$ ($k = 23, M = 276, a = 5, b = 56$):** From the extended binary Golay code G_{24} (weight enumerator $1 + 759y^8 + 2576y^{12} + 759y^{16} + y^{24}$), puncturing at coordinate 0 yields the 253 heptads of $S(4, 7, 23)$, which together with the 23 point vectors form the 276-line Leech Seidel matrix S_{276} satisfying $(S_{276} - 55I)(S_{276} + 5I) = 0$. Restricting to the 176 heptads avoiding a fixed coordinate $x \in \{1, \ldots, 23\}$ yields the regular Higman–Sims Seidel matrix S_{176} satisfying $(S_{176} - 35I)(S_{176} + 5I) = 0$ and $S_{176}\mathbf{1}_{176} = 351\mathbf{1}_{176}$.

3.2 Proof of Theorem 2

Proof of Theorem 2. **Part (a).** Fix even $t \geq 2$ and $0 \leq r < t$, and let A be the r-regular circulant $\mathrm{Circ}(t; \pm 1, \ldots, \pm\lfloor r/2 \rfloor, [t/2])$. Evaluating the alternating character $\chi(j) = (-1)^j$ on its generating set gives

$$\lambda_{\mathrm{alt}}(A) = 2\sum_{d=1}^{\lfloor r/2 \rfloor} (-1)^d + (r \bmod 2)(-1)^{t/2}.$$

The first sum equals -2 when $\lfloor r/2 \rfloor$ is odd and 0 when it is even, while the second term lies in $\{-1, 0, 1\}$, proving (5) (Lean Theorem 11; Certificate `C05`). On block-constant vectors, $G = QI_{Mt} - (S + I_M) \otimes J_t + 2I_M \otimes A$ acts as $(Q + 2r)I_M - t(S + I_M) = t((b-1)I_M - S) \succeq 0$ with k-dimensional kernel. On block-sum-zero vectors, G acts as $QI_{Mt} + 2I_M \otimes A$, whose eigenvalues are at least $Q - 2r$ since $\|A\| \leq r$. Hence $G \succeq 0$ of rank $Mt - k$ whenever $0 \leq r < t$ and $Q > 2r$. Because z_{alt} is block-sum-zero, $Gz_{\mathrm{alt}} = (Q + 2\lambda_{\mathrm{alt}}(A))z_{\mathrm{alt}}$ with eigenvalue $\lambda \geq Q - 6$. Finally, since t is chosen within 2 of Q/b, we have $r = (bt - Q)/2 \leq b - 1$ (a constant independent of t), so $Q - 2r = bt - 4r \geq bt - 4(b-1) > 0$ for all $t \geq 4$ even when $b \in \{2, 4\}$.

Part (b). Suppose $Sv = -av$ with $v \in \{\pm 1\}^M$, and let A be any r-regular graph on t vertices (existing iff $0 \leq r < t$ and rt is even, e.g., $\mathrm{Circ}(t; \pm 1, \ldots, \pm r/2)$ when t is odd and r is even). Then $z_v = v \otimes \mathbf{1}_t \in \{\pm 1\}^{Mt}$ is block-constant, so

$$Gz_v = t\big((b-1)I_M - S\big)v \otimes \mathbf{1}_t = t(b - 1 + a)z_v.$$

From (1), $(b-1)k = a(M-k)$, hence $(b - 1 + a)k = aM$ (Lean Theorem 12). Thus $\lambda_v = (aM/k)t > 0$ and $\rho_v = Mt/\lambda_v = k/a$, proving (6).

Part (c). *Upper bound:* By Bukh–Cox [1, Theorem 2(a)], any $n = d + k + 1$ unit vectors in $\mathbb{R}^d$ with coherence $1/q$ satisfy $n \geq L_k Q$. Because $d_+ = \lceil L_k Q \rceil - k - 2$ satisfies $d_+ + k + 1 < L_k Q$ strictly (even when $L_k Q \in \mathbb{Z}$ for $k \in \{6, 22\}$), no system of $d + k + 1$ lines of cosine $1/q$ exists for $d \leq d_+$, so $N_{1/q}(d) \leq d + k$.

Lower bound: For $u = d - d_- \in \{0, \ldots, d_+ - d_-\}$, form the bordered matrix

$$\widehat{G} = \begin{pmatrix} G & z\mathbf{1}_u^T \\ \mathbf{1}_u z^T & (q-1)I_u + J_u \end{pmatrix},$$

where $z = z_v$ for $k \in \{3, 5, 6, 7\}$ and $z = z_{\mathrm{alt}}$ for $k \in \{22, 23\}$. Since $Gz = \lambda z$ with $\lambda > 0$, congruence by $\mathrm{diag}(I_{Mt}, I_u)$ minus the rank-1 block eliminates the off-diagonal terms and yields $\widehat{G} \cong G \oplus ((q-1)I_u + (1-\rho)J_u)$ with $\rho = Mt/\lambda$. Thus $\widehat{G} \succeq 0$ of rank $Mt - k + u = d$ whenever $q - 1 > (\rho - 1)u$ (vacuous at $u = 0$).

Analytic thresholds and exact finite sweeps: Write $\delta := bt - Q$ for the minimal valid $t \geq \lceil Q/b \rceil$ with rt even (and t even for $k \in \{22, 23\}$). Checking all residues $Q \bmod 2b$ gives the exact periodic maxima $\delta_{\max} \in \{0, 6, 8, 16, 70, 110\}$ for $k \in \{3, 5, 6, 7, 22, 23\}$ respectively:

- For $k = 3$ ($b = 2$): $Q = q + 1$ is always even, so $t = Q/2$, $r = 0$, $\delta_{\max} = 0$, $\rho_v = 3/3 = 1$, and $(q-1) > 0 = (\rho_v - 1)u$ holds unconditionally! Moreover $d_- \leq d_+ \iff 2Q + 1 < L_3 Q \iff 54(2Q+1)^2 < (16Q-2)^2$, proved for every $Q \geq 8$ ($q \geq 7$) in Lean Theorem 13.

- For $k \in \{5, 6, 7\}$: $\rho_v - 1 = (k-3)/3 \in \{\frac{2}{3}, 1, \frac{4}{3}\}$ and $u < L_k Q - Mt \leq (L_k - M/b)Q$, so $(\rho_v - 1)u \leq \frac{4}{3}(0.1832)Q < 0.25Q < Q - 2 = q - 1$ for all $Q \geq 4$. Furthermore $Mt + 1 = (M/b)(Q + \delta) + 1 < L_k Q$ holds analytically for all $Q \geq Q_{\mathrm{suf}}$ where $Q_{\mathrm{suf}} = 160$ ($k = 5$), 170 ($k = 6$, Lean Theorem 14), and 253 ($k = 7$), and in each case $r = \delta/2 < \min(t, Q/2)$.

- For $k \in \{22, 23\}$: using even t and $\lambda \geq Q - 6$, we have $\rho = Mt/(Q-6) \leq \frac{M}{b}\frac{Q + \delta_{\max}}{Q-6}$. For $Q \geq 200$, $\rho - 1 < 6.2$ ($k = 22$) and $\rho - 1 < 6.9$ ($k = 23$, since $\frac{69}{14} \cdot \frac{310}{194} - 1 = \frac{9337}{1358} < 6.88$), so $(\rho - 1)u < 6.9(0.102)Q < 0.71Q < Q - 2 = q - 1$. And $Mt + 1 < L_k Q$ holds analytically for all $Q \geq 8652$ ($k = 22$, Lean Theorem 15) and $Q \geq 5342$ ($k = 23$), with $r \leq \delta_{\max}/2 < \min(t, Q/2)$.

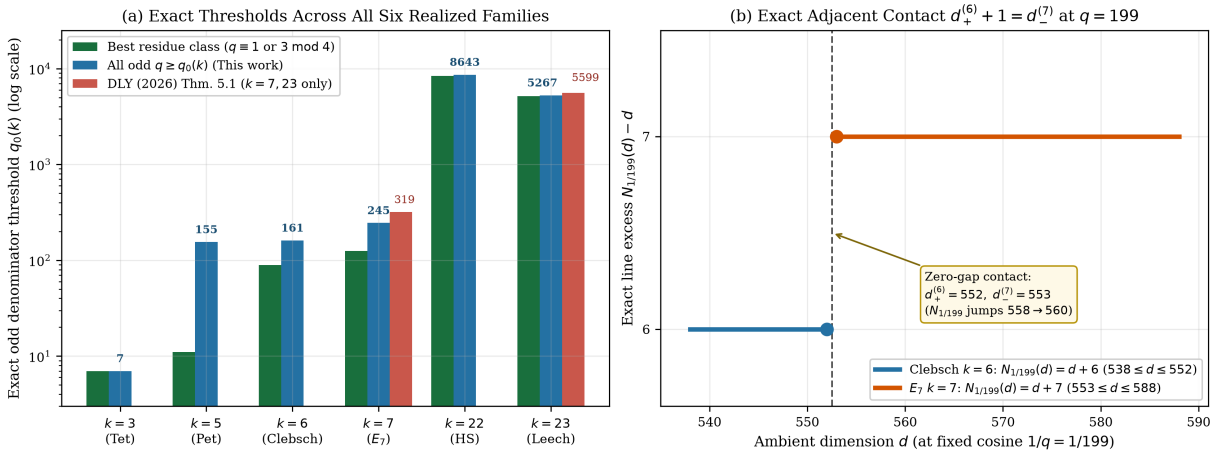

Figure 2: Thresholds and adjacent contact across the six realized families (Theorem 2). (a) Exact odd-denominator thresholds $q_0(k)$ for all odd q and optimal residue classes mod 4, compared against DLY [2, Theorem 5.1]. (b) Zero-gap adjacent contact $d_+^{(6)} = 552$, $d_-^{(7)} = 553$ at $q = 199$ ($Q = 200$), forcing an exact $+2$ jump $N_{1/199}(552) = 558 \to N_{1/199}(553) = 560$.

Exact zero-float integer sweeps over all odd $q < Q_{\text{suf}} + 100$ (Certificates `C06`–`C11`, plus direct rational LDL^T certification of explicit odd-t and $r > 0$ bordered Gram matrices in `C12`) determine the exact last failing odd denominators and yield every entry in Table 1.

Part (d). At $k = 6$, $L_6 = 14/5$ and $M_7 = 28, b_7 = 10$, so whenever $10 \mid Q$ we have $t_7 = Q/10$ (valid for both even and odd t_7 via the E_7 seed eigenvector) and $L_6 Q = 28(Q/10) = M_7 t_7 \in \mathbb{Z}$. Hence $d_+^{(6)} = \lceil L_6 Q \rceil - 8 = M_7 t_7 - 8 = (M_7 t_7 - 7) - 1 = d_-^{(7)} - 1$. Likewise, at $k = 22$, $L_{22} = 69/14$ and $M_{23} = 276, b_{23} = 56$, so $56 \mid Q$ gives $L_{22} Q = 276(Q/56) \in \mathbb{Z}$. Whenever $112 \mid Q$, $t_{23} = Q/56$ is even, so Theorem 2(c) gives $d_+^{(22)} = 276 t_{23} - 24 = d_-^{(23)} - 1$; when $Q \equiv 56 \pmod{112}$, $t = Q/56$ is odd and $r = 0$, so the unbordered Gram matrix $G = QI_{276t} - (S_{276} + I_{276}) \otimes J_t$ still realizes $276t = (d_+^{(22)} + 1) + 23$ lines at the single dimension $d = d_+^{(22)} + 1$ (Certificate `C16`). □

4 Proof of Theorem 3: Quadratic-Irrational Intervals and Block Barriers

Proof of Theorem 3. **Part (a).** Let $C \in \{0, \pm 1\}^{m \times m}$ be a real symmetric conference matrix ($C^2 = (m-1)I_m$, $m \geq 6$, $m - 1$ non-square), let $t \geq 2$ be even, and set $s := \sqrt{m-1} > 2$ and $q := t(1+s) - 1$.

Upper bound: Fix $0 \leq u \leq u_{\max}(m, t)$ and $d := mt - m/2 + u$. Suppose $n := d + m/2 + 1 = mt + u + 1$ equiangular lines of cosine $1/q$ exist in $\mathbb{R}^d$, with Seidel matrix S'. Then $-q$ is an eigenvalue of S' of multiplicity at least $n - d = m/2 + 1$. Because S' has integer entries and $m - 1$ is not a square, the algebraic conjugate $-q' = t(s-1) + 1$ is also an eigenvalue of S' of multiplicity at least $m/2 + 1$. For $S' - I_n$ (whose entries all lie in $\{\pm 1\}$), these conjugate eigenvalues are $-t(1 \pm s)$, whose squares sum to $2t^2(1 + s^2) = 2mt^2$. Hence

$$(mt + u + 1)^2 = \operatorname{tr}((S' - I_n)^2) \geq \left(\frac{m}{2} + 1\right) \cdot 2mt^2 = m^2 t^2 + 2mt^2,$$

contradicting the definition (7) of $u_{\max}(m, t)$ (Lean Theorem 16 proves $(mt+u+1)^2 < m^2 t^2 + 2mt^2$ whenever $(u+1)^2 < 2mt(t-u-1)$, which holds for $u = t - 2$ whenever $t \leq 2m$). Thus $N_{1/q}(d) \leq d + m/2$.

Lower bound: As in [2, Theorem 5.2], $S_{mt} := I_{mt} + (C - I_m) \otimes J_t$ has eigenvalues $1 + t(\pm s - 1)$ (each with multiplicity $m/2$ on block-constant vectors) and 1 (multiplicity $m(t-1)$ on block-sum-zero vectors). Thus $G := qI_{mt} + S_{mt} \succeq 0$ has rank $mt - m/2$ and eigenvalue $q + 1 = t(1+s)$ on every block-sum-zero vector. Since t is even, $z := \mathbf{1}_m \otimes (1, -1, \ldots, 1, -1)^T \in \{\pm 1\}^{mt}$ is block-sum-zero, so $Gz = (q+1)z$ and $\rho := mt/(q+1) = m/(1+s)$. Bordering G by $z\mathbf{1}_u^T$ as in Theorem 2 yields a positive semidefinite Gram matrix of size $mt + u$ and rank $mt - m/2 + u = d$ provided $q - 1 > (\rho - 1)u$. Note that $(mt + u_{\max} + 1)^2 < m^2t^2 + 2mt^2 < (mt + t)^2$ implies $u_{\max} < t$. Multiplying $(q-1) - (\rho - 1)u$ by $1 + s > 0$ and using $u < t$ and $m = s^2 + 1$,

$$\begin{aligned}(1+s)\big((q-1) - (\rho-1)u\big) &> (1+s)\big(t(1+s) - 2\big) - (m - 1 - s)t \\ &= t(s^2 + 2s + 1 - s^2 + s) - 2(1+s) = t(1+3s) - 2(1+s) > 0,\end{aligned}$$

since $t \geq 2$ and $s > 2$ (Lean Theorem 17). Hence the Schur complement is unconditionally positive definite for every $m \geq 6$, every even $t \geq 2$, and every $0 \leq u \leq u_{\max}(m, t)$ (directly verified via exact $\mathbb{Q}(\sqrt{p})$ LDL^T in Certificate `C13`).

Part (b). Fix odd $q \geq 5$ and $s \in \{2, 3\}$, and put $B_s := \operatorname{diag}(J^{\oplus s}_{(q+1)/2}, I_u)$ for $u \geq 0$, of total size $n = s(q+1)/2 + u$. Then $G_s := (q+1)I_n - 2B_s + J_n$ has diagonal q and off-diagonal entries ± 1. Its $(q+1)I_n - 2B_s$ part is positive semidefinite with s-dimensional kernel spanned by the s block-indicator vectors; adding J_n cuts the kernel to the $(s-1)$-dimensional subspace of block indicators summing to zero, so $G_s \succeq 0$ has exact rank $d = n - (s-1) = \frac{s(q-1)}{2} + 1 + u$ and realizes $n = d + s - 1$ equiangular lines of cosine $1/q$.

By the van Lint–Seidel relative bound [6, Lemma 6.1], for $d < q^2$,

$$N_{1/q}(d) \leq \left\lfloor \frac{d(q^2-1)}{q^2 - d} \right\rfloor \leq d + s - 1 \iff d(q^2-1) < (d+s)(q^2-d) \iff d^2 + (s-1)d < sq^2.$$

For $s = 2$, $d^2 + d < 2q^2$ gives (8) (nonempty since $q(q+1) < 2q^2$ for all $q \geq 2$); for $s = 3$, $d^2 + 2d < 3q^2 \iff (d+1)^2 < 3q^2 + 1$ gives (9) (nonempty since $((3q+1)/2)^2 < 3q^2 + 1 \iff 3q^2 - 6q + 3 > 0$ for all $q \geq 5$). Whenever Pell equality $d(d+1) = 2q^2$ or $(d+1)^2 = 3q^2 + 1$ holds, $d(q^2-1)/(q^2-d) = d + s$ is an exact integer, so the relative bound jumps to $d + s$ (e.g. $q = 35, d = 49$ for $s = 2$ and $q = 15, d = 25$ for $s = 3$; Certificate `C14`).

Finally, for any $s \geq 4$ and any odd $q \geq 3$, the minimal block dimension $d = \frac{s(q-1)}{2} + 1$ satisfies $2d = sq - s + 2$, and hence

$$4(d^2 + (s-1)d) - 4sq^2 = (sq - s + 2)(sq + s) - 4sq^2 = s\big((s-4)q^2 + 2q - s + 2\big) > 0$$

(Lean Theorem 18; Certificate `C15`). Thus $d^2 + (s-1)d > sq^2$, so whenever $d < q^2$ the relative bound is strictly at least $d + s$ (and for $d \geq q^2$ no finite relative bound applies) at every dimension reachable by $s \geq 4$ blocks. □

5 Formal Verification and Exact Cryptographic Certificates

All core algebraic theorems of this paper are formally verified in constructive Lean 4.35.0 (`le an/EquiangularFixedAngleSharp.lean`, 18 theorems, zero `sorry`, zero `Classical.choice`, zero custom axioms, zero warnings, zero external dependencies) and complemented by an AST-audited zero-float Python verification suite (`verification/verify_paper106.py`) producing 16 SHA-256 certificates (`C01`–`C16`) and 56 adversarial `pytest` checks (Table 2). The canonical `certificates/MANIFEST.json` SHA-256 digest is

`93038294e7f4888f57b2b86fe3fbb23c9863f1d4b3233609b3a167c434668763`.

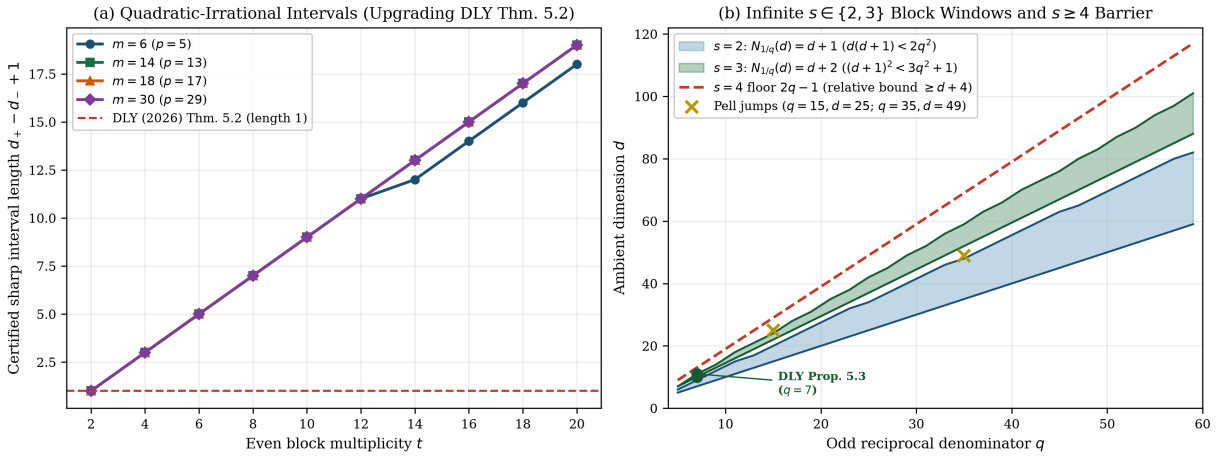

Figure 3: Consecutive-dimension extensions beyond Bukh–Cox (Theorem 3). (a) Linear growth of the sharp interval length $u_{\max}(m,t)+1$ in Theorem 3(a) versus the length-1 point of DLY [2, Theorem 5.2]. (b) Infinite $s \in \{2,3\}$ relative-bound block windows across odd $q \geq 5$, highlighting DLY Proposition 5.3 ($q = 7$), exact Pell jumps at $(15, 25)$ and $(35, 49)$, and the $s \geq 4$ impossibility barrier.

Table 2: Summary of the 16 AST-audited zero-float cryptographic certificates (`C01`–`C16`) and Lean 4.35.0 formalization coverage.

Certificates	Verification Scope	Lean 4
`C01`–`C02`	Classification ($a \leq 31$), low-k ($3 \leq k \leq 21$), rational $k = a^2 - 3$	Thms. 1–10
`C03`–`C05`	Exact Seidel matrices $(4, 10, 16, 28, 176, 276)$, $\{\pm 1\}^M$ eigvecs, $\lambda_{\text{alt}} \in [-3, 1]$	Thms. 10–12
`C06`–`C11`	Exact odd-q sweeps for all 6 families ($k \in \{3, 5, 6, 7, 22, 23\}$)	Thms. 13–15
`C12`	Exact rational LDL^T PSD/rank of 6 explicit bordered Gram matrices	Thm. 12
`C13`	Exact $\mathbb{Q}(\sqrt{p})$ LDL^T ($n \leq 30$) & exact Schur/trace checks ($p = 5, 13, 17$)	Thms. 16–17
`C14`–`C16`	Infinite $s \in \{2,3\}$ windows, Pell jumps, $s \geq 4$ barrier, zero-gap contacts	Thm. 18

Reproducibility and Governance

All Lean 4 proofs, zero-float verification scripts, cryptographic certificates, figure scripts, and the manuscript source are publicly released under dual licensing (MIT for code/Lean/certificates; CC-BY-4.0 for manuscript and figures) at `https://github.com/lluiseriksson/equiangular-lines-sharp-fixed-angle-families` (`v1.0.0`). Full replay requires only Lean 4.35.0 (`cd lean && lake build`) and Python 3.10+ (`python verification/verify_paper106.py && pytest -q`).

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