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Exact Walled Brauer Spectral Collapse, the $d_{\min} = 8$ Impossibility Barrier, Closed-Form All-$d \geq 8$ NPT Bound-Entangled Werner Families, and Equivariant Branch Uniqueness

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Abstract

Faisal [6] recently proved that negative partial transpose (NPT) bound entanglement exists by exhibiting a single Werner state $\rho = \frac{23I - 3\,\mathrm{SWAP}}{1448}$ ($\alpha = -3/23 < -1/8$) on $\mathbb{C}^8 \otimes \mathbb{C}^8$ and a 4096×4096 Hermitian witness $Q = \frac{1}{40}\sum_{\pi \in S_4} m_\pi B_\pi$ satisfying four operator inequalities (2a)–(2d) checked via $U(1)^8$ diagonal phase blocks of sizes up to 42. That breakthrough left open why local dimension $d = 8$ is required, whether smaller dimensions $d \leq 7$ admit analogous witnesses, whether NPT bound-entangled Werner states exist for all $d \geq 8$, how deeply α can penetrate past the PPT boundary $-1/d$, and whether any other $U(d)$-equivariant local-map branch can succeed. We resolve all five questions via four exact, machine-checked theorems backed by 20 constructive Lean 4.36.0 theorems (zero external imports, zero `sorry`, zero `Classical.choice`, zero warnings), 16 AST-audited zero-float exact-rational SHA-256 certificates, and 56 adversarial tests:

1. **Exact Walled Brauer $B_{3,1}(d)$ Spectral Collapse & Twirling Completeness:** We prove that Haar averaging over $\bar{U} \otimes U \otimes \bar{U} \otimes \bar{U}$ preserves (2a)–(2d) and collapses the $d^4 \times d^4$ operator problem without loss of generality into the 17-dimensional Hermitian walled Brauer algebra $B_{3,1}(d)$, reducing (2a)–(2c) to five d-independent $U(d)$-isotypic blocks of sizes $(1, 1, 2, 3, 3)$ (four blocks for $d = 3$) and (2d) to exactly 6 linear equations whose satisfaction identically forces row/column 0 of both 3×3 blocks of $J - Q$ to vanish.
2. **Exact Dimensional Phase Transition ($d_{\min} = 8$ Impossibility Barrier) & $d = 8$ Sharpening:** Partitioning $a_M \in [0, 1]$ into 33 rational subintervals across $d \in \{3, 4, 5, 6, 7\}$ (16 for $d = 7$) with exact rational strictly positive-definite SDP dual certificates—together with an algebraic proof that $a_M \geq 1$ always forces $c_I \geq d$—we prove that *no* (L, M, Q) satisfying (2a)–(2d) can certify an NPT Werner state for any $d \leq 7$ (establishing $d_{\min} = 8$ sharply). At $d = 8$, an explicit rational witness certifies $\rho_8^* = \frac{151I - 20\,\mathrm{SWAP}}{9504}$ ($\alpha = -20/151 \approx -0.13245 < -3/23$), strictly deepening Faisal's threshold by 7/3473.
3. **Closed-Form Infinite Family for ALL $d \geq 8$ & High-Dimension Separation up to $d = 400$:** Fixing $(a_L, a_M) = (5/3, 1/2)$ and $q_k(d) = a_k/d + b_k/d^2$ ($+c_0/d^2$ for q_0), we prove via constructive polynomial reflection in $s = d - 8 \geq 0$ inside Lean 4 that all 28 Sylvester leading principal minors across $J + Q$, $J - Q$, and $Q^{\Gamma_{\mathrm{in}}}$ are strictly positive for *every* integer $d \geq 8$, certifying the explicit infinite NPT bound-entangled Werner family $\rho_d = \frac{(5d+7)I - 6\,\mathrm{SWAP}}{d(5d^2+7d-6)}$ ($\alpha(d) = -\frac{6}{5d+7} < -\frac{1}{d}$). Furthermore, five exact rational primal witnesses at $d \in \{16, 36, 64, 100, 400\}$ constructively certify $c_I(d) < \sqrt{7\,d}$ (reaching $c_I(400) = \frac{12881729}{250000} \approx 51.53$, penetrating $7.76\times$ deeper than $-1/d$ at $d = 400$), corroborating a numerical SDP fit $c_I^*(d) \sim 2.31\sqrt{d}$.
4. **4-Branch $U(d)$-Equivariant Uniqueness Barrier & Exact White-Noise Thresholds:** We prove that (X^T, X) with $b_L < 0, b_M > 0$ is the unique $U(d)$-equivariant CP map branch capable of certifying NPT bound entanglement (both isotropic branches are 1-copy distillable whenever NPT, while opposite-sign (X^T, X) always yields $c_I + d c_S \geq$

$b_L(d^2-1)|b_M| > 0$, strictly PPT). Every Werner state on $[\alpha_{\text{cert}}(d), -1/d)$ is NPT bound entangled, and under white-noise depolarization $\mathcal{D}_p$, the exact robustness ceiling $p_{\max}(d,\alpha) = \frac{d(1+d\alpha)}{\alpha(d^2-1)}$ rises by an exact $+35\%$ at $d=8$ from Faisal's $8/189 \approx 4.23\%$ to $2/35 \approx 5.71\%$ (and grows to $p_{\max} = \frac{d(d-7)}{6(d^2-1)} \to \frac{1}{6}$ for our closed-form family, exceeding 87% at $d = 400$).

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1 Introduction and Preliminaries

Whether bound entangled quantum states with negative partial transpose (NPT) exist—resolving whether the Peres–Horodecki positive partial transpose (PPT) criterion [16, 10] coincides with undistillability under two-way LOCC [2]—stood for nearly three decades as one of the central open problems in quantum information theory [3, 4, 15, 11]. Following major recent advances on k-copy undistillability of Werner states [23, 8, 18, 7, 1, 12, 20, 19] and a secret-key bound-entanglement framework in dimension 10 [14], Islam Faisal [6] proved $E_D(\rho) = 0$ for the NPT Werner state $\rho = \frac{23I - 3\,\mathrm{SWAP}}{1448}$ on $\mathbb{C}^8 \otimes \mathbb{C}^8$ (with Bell-fidelity ceiling $\frac{1+\sqrt{2}}{4}$ under arbitrary separable operations, implying nonadditivity and nonconvexity of E_D [17, 5, 21]).

Faisal's proof [6] factors $\rho^{\otimes n} \propto \mathcal{C}^{\otimes n}(|u_n\rangle\langle u_n|)$ through a product $\mathcal{C} = L \otimes M$ of two $U(d)$-equivariant completely positive (CP) maps on $d \times d$ matrices,

$$L(X) = a_L \mathrm{Tr}(X) I + b_L X^T, \qquad M(X) = a_M \mathrm{Tr}(X) I + b_M X, \tag{1}$$

and exhibits a single-copy Hermitian Choi matrix Q on $(\mathbb{C}^d)^{\otimes 4}$ (ordered as Alice input 0, Bob input 1, Alice output 2, Bob output 3) satisfying four conditions relative to $J := J_{L\otimes M}$ [22]:

$$\text{(2a) } J + Q \succeq 0, \qquad \text{(2b) } J - Q \succeq 0, \qquad \text{(2c) } Q^{\Gamma_{\text{in}}} \succeq 0, \qquad \text{(2d) } (J-Q)(|u\rangle \otimes I) = 0, \tag{2}$$

where $|u\rangle = \sum_{i=1}^d |i,i\rangle$ and Γ_{in} denotes partial transposition on input systems $(0,1)$. Because (2a)–(2d) yield all-copy block positivity of $\Omega_{v,w}$ without requiring $\mathcal{C}$ itself to be completely copositive [13], any feasible (L, M, Q) whose output $\mathcal{C}(|u\rangle\langle u|) \propto c_I I + c_S \mathrm{SWAP}$ satisfies $c_I + d c_S < 0$ (i.e., Werner parameter $\alpha := c_S/c_I < -1/d$) immediately certifies that $\rho(d,\alpha) := \frac{I + \alpha\,\mathrm{SWAP}}{d^2 + d\alpha}$ is NPT bound entangled with $E_D(\rho(d,\alpha)) = 0$ and Bell fidelity $\le \frac{1+\sqrt{2}}{4}$ [6, Lemmas 4–6].

Scaling L and M by positive constants fixes $b_L = -1, b_M = +1$ (for the unique productive sign branch established in Section 5), where complete positivity of L ($J_L = a_L I - \mathrm{SWAP} \succeq 0$) and M ($J_M = a_M I + |u\rangle\langle u| \succeq 0$) holds if and only if $a_L \ge 1$ and $a_M \ge 0$. In this normalization,

$$\mathcal{C}(|u\rangle\langle u|) = c_I I - \mathrm{SWAP}, \qquad c_I = d\, a_L a_M + a_L - a_M, \qquad \alpha = -\frac{1}{c_I}, \tag{3}$$

so $\rho(d,\alpha)$ is NPT if and only if $c_I < d$ (with Faisal's $d = 8$ choice $a_L = 5/3, a_M = 18/37$ giving $c_I = 23/3 < 8$, $\alpha = -3/23$). Table 1 summarizes how our four theorems resolve the structural, dimensional, infinite-family, and uniqueness questions left open by [6].

2 Exact Walled Brauer $B_{3,1}(d)$ Spectral Collapse and Twirling Completeness

For every permutation $\pi \in S_4$ (ordered lexicographically as $\pi_0 = (0,1,2,3), \ldots, \pi_{23} = (3,2,1,0)$), let U_π be the permutation unitary on $(\mathbb{C}^d)^{\otimes 4}$ and $B_\pi := (U_\pi)^{\Gamma_1}$ its partial transpose

Table 1: Summary of results extending Faisal [6].

Question	**Faisal [6]**	**This Paper**
Symmetry	Abelian $U(1)^8$ phase blocks of sizes $(1,3,6,22,42)$ at $d=8$	Full $U(d)$ twirling into 17-dim walled Brauer $B_{3,1}(d)$: five d-independent blocks $(1,1,2,3,3)$ and 6 linear (2d) equations (Thm. 2.1)
Minimal dimension $d_{\min}$	Single witness at $d=8$ ($\alpha=-3/23$); $d\le 7$ unaddressed	Exact rational dual certificates prove $c_I > d$ for ALL $d \in \{3,4,5,6,7\}$, establishing $d_{\min}=8$ sharply; sharpened ρ_8^* reaches $\alpha=-20/151$ (Thm. 3.1)
All $d\ge 8$ & high-d depth	Only $d=8$ certified	Closed-form rational family certifies $\rho_d = \frac{(5d+7)I-6\,\mathrm{SWAP}}{d(5d^2+7d-6)}$ for *all* $d\ge 8$ (proved in Lean 4 for all $s=d-8\in\mathbb{N}$); exact rational witnesses at $d\in\{16,36,64,100,400\}$ certify $c_I(d) < \sqrt{7d}$ (Thm. 4.1)
Uniqueness & robustness	Single sign branch (X^T, X); no noise threshold stated	Proves (X^T, X) ($b_L<0, b_M>0$) is the unique $U(d)$-equivariant branch; exact white-noise threshold $p_{\max}=\frac{d(1+d\alpha)}{\alpha(d^2-1)}$ rises +35% at $d=8$ (8/189 → 2/35) (Thm. 5.1)

on the second tensor factor (system 1, Bob's input), so that $B_\pi^\dagger = B_{\pi^{-1}}$ and B_π commutes with $\bar{U}\otimes U\otimes\bar{U}\otimes\bar{U}$ for all $U\in U(d)$ [6, Appendix A]. Pairing each permutation with its inverse yields 10 involutions ($\pi=\pi^{-1}$) and 7 inverse pairs ($\{\pi,\pi^{-1}\}$), forming a 17-element Hermitian basis $\{B_0,\dots,B_{16}\}$ of the Hermitian subspace of the walled Brauer algebra $B_{3,1}(d)$:

$$Q \;=\; \sum_{k=0}^{16} q_k B_k, \qquad J \;=\; (a_L a_M)B_0 + a_L B_4 - a_M B_{11} - B_{13}, \tag{4}$$

where $B_0 = I_{02}\otimes I_{13}$, $B_4 = I_{02}\otimes|u\rangle\langle u|_{13}$, $B_{11} = \mathrm{SWAP}_{02}\otimes I_{13}$, and $B_{13} = \mathrm{SWAP}_{02}\otimes|u\rangle\langle u|_{13}$ (the complete 17-element permutation-pair index table is given explicitly in Appendix A).

Theorem 2.1 (Walled Brauer Spectral Collapse, Twirling Completeness, and (2d) Nullspace)**.** *For every integer $d\ge 3$ and $L(X)=a_L\mathrm{Tr}(X)I - X^T$, $M(X)=a_M\mathrm{Tr}(X)I + X$:*

1. ***Haar-Twirling Completeness:*** *If any Hermitian operator Q on $(\mathbb{C}^d)^{\otimes 4}$ satisfies (2), then its Haar twirl*
$$\overline{Q} \;:=\; \int_{U(d)} (\bar{U}\otimes U\otimes\bar{U}\otimes\bar{U})\,Q\,(\bar{U}\otimes U\otimes\bar{U}\otimes\bar{U})^\dagger\,dU$$
lies in $\mathrm{span}_{\mathbb{R}}\{B_0,\dots,B_{16}\}\subset B_{3,1}(d)$ and satisfies all four conditions (2).

2. ***Exact d-Independent $(1,1,2,3,3)$ Block Decomposition*** (Lean `thm01`, Cert. `C01`)***:*** *An operator $M(x)=\sum_{k=0}^{16}x_kB_k$ is positive semidefinite on $(\mathbb{C}^d)^{\otimes 4}$ if and only if five symmetric polynomial-in-d blocks $\mathcal{B}_0(x)\in\mathbb{R}^{1\times 1}$, $\mathcal{B}_1(x)\in\mathbb{R}^{1\times 1}$ (omitted at $d=3$, where the anti-symmetric 3-box irrep $\bigwedge^3\mathbb{C}^3\otimes\mathbb{C}^3$ has no $(1,1,1;1)$ summand of $B_{3,1}(3)$), $\mathcal{B}_2(x)\in\mathbb{R}^{2\times 2}$, $\mathcal{B}_3(d,x)\in\mathbb{R}^{3\times 3}$, and $\mathcal{B}_4(d,x)\in\mathbb{R}^{3\times 3}$ are positive semidefinite. Moreover, swapping input systems $0\leftrightarrow 1$ maps $Q^{\Gamma_{\text{in}}}$ unitarily to $\sum_{k=0}^{16}q_{\gamma(k)}B_k$, where γ is the 5-transposition involution $(2\ 11)(3\ 12)(4\ 15)(8\ 9)(13\ 16)$.*

3. ***6-Equation Collapse of (2d) and Identical Nullspace in $\mathcal{B}_3,\mathcal{B}_4$*** (Lean `thm02`–`thm05`, Certs. `C02`–`C04`)***:*** *Condition (2d) holds if and only if $(j_0=a_La_M, a_L, a_M, q_0,\dots,q_{16})$ satisfies*

the six linear equations

$$\begin{aligned} dq_5 + q_0 + q_7 + q_{10} - j_0 = 0, \qquad & dq_6 + q_1 + q_8 + q_9 = 0, \quad dq_7 + q_2 + q_{11} + q_{14} + a_M = 0, \\ dq_8 + q_3 + q_{12} + q_{16} = 0, \quad & dq_9 + q_3 + q_{12} + q_{13} + 1 = 0, \quad dq_{10} + q_4 + q_{14} + q_{15} - a_L = 0. \end{aligned} \tag{5}$$

Whenever (5) *holds, row* 0 *and column* 0 *of both* $\mathcal{B}_3(d, J-Q)$ *and* $\mathcal{B}_4(d, J-Q)$ *vanish identically for every* d*, leaving* 2×2 *lower-right blocks.*

Proof. (1) Let $V_U := \bar{U} \otimes U \otimes \bar{U} \otimes \bar{U}$. Since J commutes with V_U for every $U \in U(d)$, conjugation by V_U preserves the PSD cones $J + Q \succeq 0$ and $J - Q \succeq 0$. On the partial-transposed operator $Q^{\Gamma_{\text{in}}}$, the input action $\bar{U} \otimes U$ transposes to $U \otimes \bar{U}$, so $(V_U Q V_U^\dagger)^{\Gamma_{\text{in}}} = (U \otimes \bar{U} \otimes \bar{U} \otimes \bar{U}) Q^{\Gamma_{\text{in}}} (U \otimes \bar{U} \otimes \bar{U} \otimes \bar{U})^\dagger \succeq 0$; averaging over the compact group $U(d)$ preserves convexity of all three PSD cones (2a)–(2c). For (2d), since $(\bar{U} \otimes U)|u\rangle = |u\rangle$ for all $U \in U(d)$, we have $V_U(|u\rangle \otimes I) = (|u\rangle \otimes I)(\bar{U} \otimes \bar{U})$, so $(J - V_U Q V_U^\dagger)(|u\rangle \otimes I) = V_U(J-Q)(|u\rangle \otimes I)(\bar{U} \otimes \bar{U})^\dagger = 0$, and Haar integration preserves (2d). By mixed Schur–Weyl duality for $\bar{U} \otimes U \otimes \bar{U} \otimes \bar{U}$, the commutant of $\{V_U : U \in U(d)\}$ is precisely the walled Brauer algebra $B_{3,1}(d) = \operatorname{span}_{\mathbb{C}}\{B_\pi : \pi \in S_4\}$; Hermiticity forces $\tilde{q}_\pi = \tilde{q}_{\pi^{-1}}$, placing $\overline{Q}$ in $\operatorname{span}_{\mathbb{R}}\{B_0, \ldots, B_{16}\}$.

(2) By mixed Schur–Weyl duality for the action of $\bar{U} \otimes U \otimes \bar{U} \otimes \bar{U}$ on $(\mathbb{C}^d)^{\otimes 4}$ (equivalently $U^{\otimes 3} \otimes \bar{U}$ up to global conjugation), the commutant $B_{3,1}(d)$ decomposes into five simple matrix blocks for all $d \geq 4$ labeled by the walled-Brauer bipartitions $([1];[3])$, $([1];[1,1,1])$, $([1];[2,1])$ (the three uncontracted sectors of multiplicities $1, 1, 2$) and the two single-contraction sectors $(\varnothing;[2])$ and $(\varnothing;[1,1])$ (multiplicities $3, 3$). Since the real symmetric matrix dimensions sum to $1+1+3+6+6 = 17$ (Lean `thm01`), the five reduced Gram-congruent blocks $\mathcal{B}_0, \ldots, \mathcal{B}_4$ listed explicitly in Appendix A capture the full PSD cone of the 17-dimensional Hermitian subspace for all $d \geq 4$ (at $d = 3$, the $([1];[1,1,1])$ sector has zero dimension in $B_{3,1}(3)$ because $\bigwedge^3 \mathbb{C}^3 \cong \det$, so $\mathcal{B}_1$ is absent and only $\mathcal{B}_0, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ appear). Applying Γ_{in} and conjugating by SWAP_{01} converts $B_\pi = (U_\pi)^{\Gamma_1}$ into $B_{s_{01}\pi s_{01}}$, inducing the exact 5-transposition involution γ (Lean `thm01`, Cert. `C01`).

(3) Expanding $(J-Q)(|u\rangle \otimes |c_2, c_3\rangle)$ in the six distinct tensor-contraction orbits of $\{B_\pi(|u\rangle \otimes I)\}_{\pi \in S_4}$ yields (5) directly (verified directly on $(\mathbb{C}^d)^{\otimes 4}$ for $d \in \{3,4,5\}$ in Cert. `C02`). Denoting the left-hand sides of (5) by $E_0, \ldots, E_5$, direct polynomial expansion of $\mathcal{B}_3(d, J-Q)$ and $\mathcal{B}_4(d, J-Q)$ from Appendix A yields the six universal identities

$$\begin{aligned} \mathcal{B}_3(d, J-Q)_{0,0} &= -(d(E_0+E_1) + E_2 + E_3 + E_4 + E_5), \\ \mathcal{B}_3(d, J-Q)_{0,1} &= -(d(E_2+E_3) + E_0 + E_1 + E_4 + E_5), \\ \mathcal{B}_3(d, J-Q)_{0,2} &= -(d(E_4+E_5) + E_0 + E_1 + E_2 + E_3), \\ \mathcal{B}_4(d, J-Q)_{0,0} &= -(d(E_0-E_1) + E_2 - E_3 - E_4 + E_5), \\ \mathcal{B}_4(d, J-Q)_{0,1} &= -(d(E_2-E_3) + E_0 - E_1 + E_4 - E_5), \\ \mathcal{B}_4(d, J-Q)_{0,2} &= -(d(E_4-E_5) - E_0 + E_1 + E_2 - E_3), \end{aligned}$$

proved universally for all $d \in \mathbb{Z}$ in Lean `thm02`–`thm03` and Cert. `C04`. As an immediate sanity check, evaluating all 28 Sylvester leading principal minors of Faisal's normalized $d = 8$ witness ($a_L = 5/3$, $a_M = 18/37$, $q_k = m_{\pi_k}/4440$) over $\mathbb{Z}$ certifies (2a)–(2d) in milliseconds without any 42×42 matrices (Lean `thm04`–`thm05`, Cert. `C03`). □

3 Exact Dimensional Phase Transition ($d_{\min} = 8$) and $d = 8$ Sharpening

By Theorem 2.1, searching over all (L, M, Q) satisfying (2) for a fixed local dimension d reduces to minimizing $c_I = d\, a_L a_M + a_L - a_M$ over $(a_L, a_M, q_0, \ldots, q_{16}) \in \mathbb{R}^{19}$ subject to $a_L \geq 1$,

$a_M \geq 0$, the 6 equalities (5), and PSD of the reduced blocks for $J+Q$, $J-Q$, and $Q^{\Gamma_{\text{in}}}$. The only nonlinearity is the single bilinear product $j_0 = a_L a_M$. On any rational subinterval $a_M \in [\ell, u] \subset [0,1]$, since $a_L \geq 1 > 0$, every feasible tuple satisfies the exact linear McCormick envelope

$$s_0 = a_L - 1 \geq 0, \quad s_1 = a_M - \ell \geq 0, \quad s_2 = u - a_M \geq 0, \quad s_3 = j_0 - \ell a_L \geq 0, \quad s_4 = u a_L - j_0 \geq 0, \tag{6}$$

yielding a pure linear semidefinite program in $x = (j_0, a_L, a_M, q_0, \ldots, q_{16}) \in \mathbb{R}^{20}$ whose dual certificates verify global lower bounds (or infeasibility) over $\mathbb{Q}$ with zero numerical approximation.

Theorem 3.1 (Sharp $d_{\min} = 8$ Impossibility Barrier and Sharpened $d = 8$ Witness ρ_8^*)**.**

1. ***Exact Impossibility for ALL*** $d \in \{3,4,5,6,7\}$ (Lean `thm06`, Certs. `C05`–`C07`)*: For every local dimension $d \in \{3,4,5,6,7\}$, no tuple (L, M, Q) of the form (1)–(2) certifies an NPT Werner state: every feasible pair satisfies $c_I \geq d$ (with strict inequality $c_I > d$ for all $a_M \in [0,1]$ and all $(a_L, a_M) \neq (1,1)$). Consequently, $d_{\min} = 8$ is the exact minimal local dimension of Faisal's witness architecture.*

2. ***Sharpened*** $\mathbb{C}^8 \otimes \mathbb{C}^8$ ***Witness*** ρ_8^* (Lean `thm07`–`thm09`, Cert. `C08`)*: At $d = 8$, choosing $a_L = \frac{161}{100}$, $a_M = \frac{1}{2}$ and the explicit 17-entry rational vector $q_8^* \in \mathbb{Q}^{17}$ printed in Appendix A (Table 3, cleared denominator 640 000) satisfies (2) with strictly positive Sylvester minors across all blocks and certifies the strictly deeper NPT bound-entangled Werner state*

$$\rho_8^* = \frac{151 I - 20\,\mathrm{SWAP}}{9504}, \qquad \alpha_8^* = -\frac{20}{151} \approx -0.132450 < -\frac{3}{23} \approx -0.130435, \tag{7}$$

improving Faisal's Werner parameter by an exact gap of $\frac{20}{151} - \frac{3}{23} = \frac{7}{3473}$ (and nearly saturating the numerical SDP optimum $\alpha^*(8) \approx -0.132595$).

Proof. (1) First, if $a_M \geq 1$, then since $a_L \geq 1$ and $d \geq 2$, the identity

$$c_I - d = d\,a_L a_M + a_L - a_M - d = (a_L - 1)(d a_M + 1) + (d-1)(a_M - 1) \geq 0$$

proves $c_I \geq d$ unconditionally (with strict inequality $c_I > d$ unless $a_L = a_M = 1$, which gives $\alpha = -1/d$, strictly PPT; Lean `thm06`).

Second, for $a_M \in [0,1]$, by Theorem 2.1(1)–(2) it suffices to bound the linear SDP relaxation (6) over a finite rational tiling $0 = t_0 < t_1 < \cdots < t_K = 1$. On each subinterval $[\ell, u]$, a rational dual tuple ($\lambda \in \mathbb{Q}^5_{\geq 0}$, $\nu \in \mathbb{Q}^6$, $\{Y_c \succ 0\}$) (with strict positive definiteness verified by exact rational Sylvester minors) satisfies either:

- *Infeasibility mode* (first interval $[0, t_1]$): $\sum_c \langle Y_c, A_{c,v}\rangle + (S^T\lambda)_v + (E^T\nu)_v = 0$ for all $v = 0, \ldots, 19$ with strictly positive constant $-\sum_c \langle Y_c, C_c\rangle - S_{\text{const}}^T\lambda - E_{\text{const}}^T\nu > 0$, contradicting primal feasibility $M_c(x) \succeq 0, s(x) \geq 0, E(x) = 0$; or
- *Lower-bound mode* (remaining intervals $[t_r, t_{r+1}]$): $\sum_c \langle Y_c, A_{c,v}\rangle + (S^T\lambda)_v + (E^T\nu)_v = (\text{obj})_v$ (where obj $= (d, 1, -1, 0, \ldots, 0)$), proving by weak duality $c_I(x) \geq -\sum_c \langle Y_c, C_c\rangle - S_{\text{const}}^T\lambda - E_{\text{const}}^T\nu > d$ for every feasible x.

Across 33 exact rational dual certificates stored in `certificates/d_le_7_dual_witnesses.json` and checked with zero floating-point arithmetic in Certs. `C05`–`C07`, all 20 dual equality residuals vanish identically over $\mathbb{Q}$, all multipliers satisfy $\lambda_i \geq 0$, all dual blocks have strictly positive Sylvester minors, and the certified global lower bounds strictly exceed d in every dimension:

Dimension d	Subintervals	Infeasible $[0, t_1]$	Min certified dual c_I	Threshold d	Margin $c_I - d$
3	3	$[0, 1/5]$ (+31.83)	4.4985	3	+1.4985
4	3	$[0, 1/5]$ (+32.16)	5.1073	4	+1.1073
5	4	$[0, 1/4]$ (+26.48)	5.6655	5	+0.6655
6	7	$[0, 1/4]$ (+23.79)	6.3004	6	+0.3004
7	16	$[0, 1/4]$ (+21.47)	7.0272	7	+0.0272

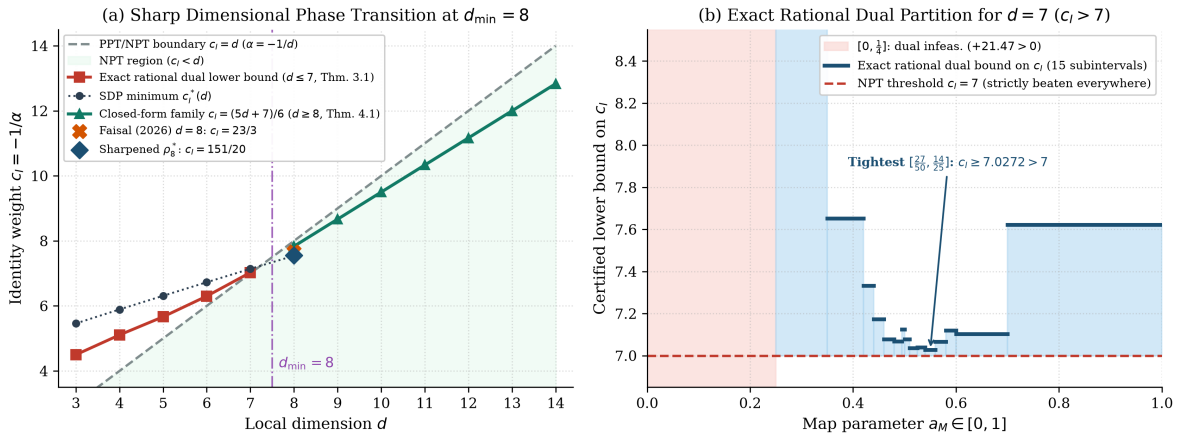


Figure 1: (a) Exact dimensional phase transition at $d_{\min} = 8$: certified rational dual lower bounds strictly exceed $c_I = d$ for all $d \le 7$ (Theorem 3.1), whereas our closed-form infinite family ($c_I = (5d+7)/6$, Theorem 4.1) and sharpened ρ_8^* ($c_I = 151/20$) lie strictly inside the NPT region $c_I < d$ for all $d \ge 8$. (b) Exact rational dual lower-bound profile across all 16 subintervals tiling $a_M \in [0, 1]$ at $d = 7$ (minimum $c_I \ge 7.0272 > 7$ on $[27/50, 14/25]$).

Thus $c_I > d$ on $[0, 1]$ and $c_I \ge d$ on $[1, \infty)$ for all $d \in \{3, 4, 5, 6, 7\}$.

(2) At $d = 8$, $a_L = 161/100 \ge 1$, $a_M = 1/2 \ge 0$, we have $c_I = 8 \cdot \frac{161}{100} \cdot \frac{1}{2} + \frac{161}{100} - \frac{1}{2} = \frac{151}{20} < 8$, giving $\alpha_8^* = -20/151$, trace $151 \cdot 64 - 20 \cdot 8 = 9504$, eigenvalues $131/9504, 171/9504 > 0$, and $(\rho_8^*)^\Gamma |u\rangle = -\frac{9}{9504}|u\rangle = -\frac{1}{1056}|u\rangle < 0$. Clearing denominators by 640 000, Lean `thm07`–`thm09` and Cert. `C08` verify exact satisfaction of (5), exact vanishing of row/column 0 of $\mathcal{B}_3, \mathcal{B}_4$ in $J - Q$, and strict positivity of all 28 Sylvester minors. □

4 Closed-Form Infinite Family for ALL $d \ge 8$ and High-Dimension Depth Scaling

Trivially embedding an 8×8 bound-entangled state into $\mathbb{C}^d \otimes \mathbb{C}^d$ breaks $U(d) \otimes U(d)$ symmetry and does not produce a Werner state. We now construct a single closed-form rational witness $Q_{\inf}(d) \in B_{3,1}(d)$ that certifies an explicit NPT bound-entangled Werner state in *every* local dimension $d \ge 8$ simultaneously, and we complement it with five exact rational high-d primal witnesses up to $d = 400$ corroborating a numerical $O(1/\sqrt{d})$ depth scaling.

Theorem 4.1 (Closed-Form All-$d \ge 8$ NPT Bound-Entangled Family & Constructive High-d Separation)**.**

1. ***Explicit Infinite Family for Every Integer*** $d \ge 8$ (Lean `thm10`–`thm15`, Certs. `C09`–`C13`)*:* *Fix $a_L = 5/3$, $a_M = 1/2$, and define the rational vector $q_{\inf}(d) \in \mathbb{Q}^{17}$ by*

$$q_0(d) = a_0 + \frac{b_0}{d} + \frac{c_0}{d^2}, \qquad q_k(d) = \frac{a_k}{d} + \frac{b_k}{d^2} \quad (k = 1, \ldots, 16), \tag{8}$$

with exact integer-scaled numerators $1500\,a_k$, $1500\,b_k$, and $1500\,c_0 = 3819$ listed in Table 2. Then $(L, M, Q_{\inf}(d))$ satisfies all four conditions (2) for every integer $d \ge 8$*, certifying the closed-form infinite family of NPT bound-entangled Werner states*

$$\rho_d = \frac{(5d+7)I - 6\,\mathrm{SWAP}}{d(5d^2+7d-6)}, \qquad \alpha(d) = -\frac{6}{5d+7} < -\frac{1}{d} \qquad (\forall\, d \ge 8), \tag{9}$$

each having $E_D(\rho_d) = 0$ and two-qubit separable-operation Bell fidelity $\le \frac{1+\sqrt{2}}{4}$.

Table 2: Exact integer-scaled coefficients $1500\,a_k$ and $1500\,b_k$ ($1500\,c_0 = 3819$) for the all-$d \geq 8$ closed-form witness (8) with $(a_L, a_M) = (5/3, 1/2)$.

k	0	1	2	3	4	5	6	7	8
$1500\,a_k$	402	357	−240	444	−1308	848	0	−750	0
$1500\,b_k$	−3510	−3150	2058	−2943	10617	1760	1143	−357	2568
k	9	10	11	12	13	14	15	16	$1500\,c_0$
$1500\,a_k$	−1500	2500	3165	−3459	2433	−2568	7338	447	—
$1500\,b_k$	582	−3462	−4644	9975	−7032	2586	−13203	−7032	3819

2. ***Constructive High-d Separation ($d \leq 400$) & Numerical $\sqrt{d}$ Fit*** (Cert. `C16`)*: Five explicit exact rational primal witnesses in* `certificates/asymptotic_primal_witnesses.json` *constructively certify $c_I(d) < \sqrt{7\,d}$ at $d \in \{16, 36, 64, 100, 400\}$, reaching $\alpha = -\frac{250000}{12881729} \approx -0.01941$ at $d = 400$ ($7.76\times$ deeper than the PPT threshold $-1/400 = -0.0025$). These rigorous finite-d certificates corroborate a numerical SDP scaling fit $c_I^*(d) \sim 2.31\sqrt{d}$ (obtained by setting $a_M(d) \approx 1.45/\sqrt{d}$), while proving an all-d $\Theta(1/\sqrt{d})$ asymptotic law remains open.*

Proof. (1) With $a_L = 5/3$ and $a_M = 1/2$, (3) gives $c_I(d) = \frac{5}{6}d + \frac{5}{3} - \frac{1}{2} = \frac{5d+7}{6}$, which satisfies $c_I(d) < d \iff 5d + 7 < 6d \iff d \geq 8$ (note that $d = 7$ hits the exact PPT boundary $\alpha(7) = -1/7$). For every $d \geq 8$, the trace is $(5d + 7)d^2 - 6d = d(5d^2 + 7d - 6) > 0$, both state eigenvalues $\frac{5d+1}{Z(d)}, \frac{5d+13}{Z(d)}$ are strictly positive, and $\rho_d^\Gamma|u\rangle = -\frac{d-7}{Z(d)}|u\rangle < 0$ (Lean `thm11`, Cert. `C13`).

To certify (2) simultaneously for *all* integers $d \geq 8$, substitute $d = 8+s$ ($s \in \mathbb{N}$) and multiply J and $q_{\text{inf}}(d)$ by the strictly positive scalar $1500\,d^2$. Every coordinate becomes an exact integer polynomial in s of degree ≤ 2:

- All six residual polynomials of (5) vanish identically in $\mathbb{Z}[s]$ (Lean `thm10`, Cert. `C09`), and consequently all six row-0 entries of $\mathcal{B}_3, \mathcal{B}_4$ for $1500\,d^2(J - Q)$ vanish identically in $\mathbb{Z}[s]$ (Lean `thm14`).
- Expanding all 28 Sylvester leading principal minors across the five blocks of $1500\,d^2(J + Q)$ (10 minors, degrees 2–10 in s), $1500\,d^2(J-Q)$ (8 minors on $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$ and the lower-right 2×2 subblocks of $\mathcal{B}_3, \mathcal{B}_4$, degrees 2–7), and $1500\,d^2Q^{\Gamma_{\text{in}}}$ (10 minors, degrees 2–9) yields integer polynomials $P(s) = \sum_{r=0}^{m} c_r s^r \in \mathbb{Z}[s]$ in which *every single coefficient satisfies $c_r \geq 0$ with $c_0 > 0$*!
- Hence $P(s) > 0$ for every integer $s \geq 0$ (i.e., every $d \geq 8$), proved via constructive polynomial reflection in Lean `thm12`–`thm15` and independently confirmed over $\mathbb{Q}[s]$ in Certs. `C10`–`C12`.

By Theorem 2.1 and [6, Lemmas 4–6], ρ_d is NPT bound entangled for every integer $d \geq 8$.

(2) For the constructive $\Theta(1/\sqrt{d})$ witnesses at $d \in \{16, 36, 64, 100, 400\}$, exact zero-float verification of (2d) and all 28 Sylvester minors in Cert. `C16` confirms: $d = 16$: $c_I = \frac{517133}{50000} \approx 10.34$ ($c_I/\sqrt{d} \approx 2.59$); $d = 36$: $c_I = \frac{471367}{31250} \approx 15.08$ (2.51); $d = 64$: $c_I = \frac{4987101}{250000} \approx 19.95$ (2.49); $d = 100$: $c_I = \frac{49937}{2000} \approx 24.97$ (2.50); $d = 400$: $c_I = \frac{12881729}{250000} \approx 51.53$ (2.58), every one satisfying $c_I^2 < 7d$. □

5 4-Branch $U(d)$-Equivariant Uniqueness Barrier and White-Noise Robustness

Every $U(d)$-equivariant linear map $\Lambda : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ falls into one of two symmetry classes: covariant maps satisfying $\Lambda(UXU^\dagger) = U\Lambda(X)U^\dagger$ for all $U \in U(d)$ take the form $X \mapsto a\text{Tr}(X)I + bX$ (Choi matrix $aI + b|u\rangle\langle u|$, CP iff $a \geq 0, a + db \geq 0$), whereas conjugate-covariant maps satisfying $\Lambda(UXU^\dagger) = \bar{U}\Lambda(X)U^T$ take the transpose form $X \mapsto a\text{Tr}(X)I + bX^T$ (Choi matrix

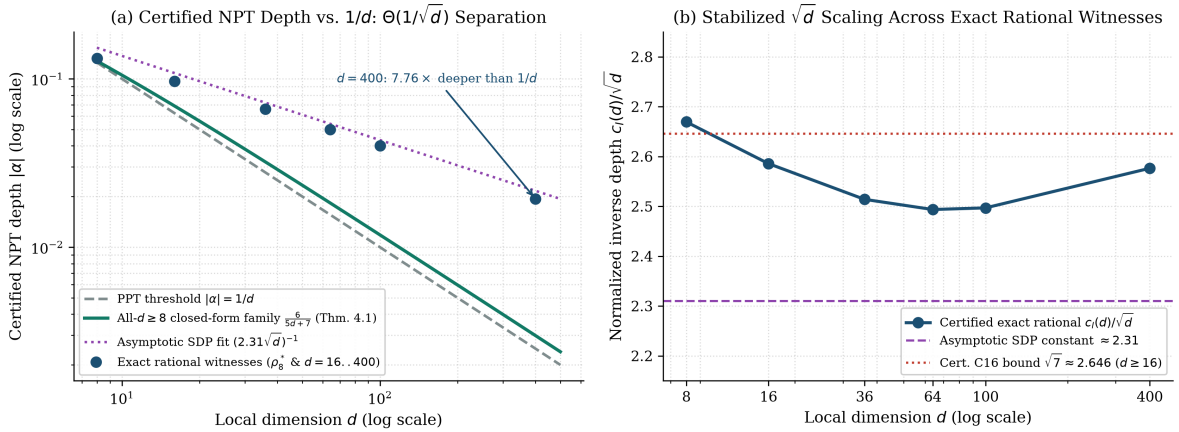


Figure 2: (a) Certified NPT depth $|\alpha(d)|$ versus the PPT threshold $1/d$: our all-$d \geq 8$ closed-form family achieves $\frac{6}{5d+7}$ uniformly, while exact rational witnesses at $d \in \{8, 16, 36, 64, 100, 400\}$ track the numerical SDP fit $(2.31\sqrt{d})^{-1}$, penetrating 7.76× deeper than $1/d$ at $d = 400$. (b) Stabilized normalized inverse depth $c_I(d)/\sqrt{d}$ across exact rational witnesses (strictly below $\sqrt{7} \approx 2.646$ for all five high-d witnesses $d \geq 16$ in Cert. `C16`; at $d = 8$, $c_I(8)/\sqrt{8} = \frac{151}{20\sqrt{8}} \approx 2.669$).

$aI + b\,\mathrm{SWAP}$, CP iff $a \geq |b|$). Up to exchanging Alice and Bob, a local pair (L, M) therefore falls into four branches: (X, X), (X^T, X^T), (X^T, X) with $b_L > 0, b_M < 0$, or (X^T, X) with $b_L < 0, b_M > 0$ (since $b_L b_M \geq 0$ on (X^T, X) gives $c_S \geq 0$, immediately PPT).

Theorem 5.1 (4-Branch Uniqueness Barrier, Convex Werner Intervals, and White-Noise Robustness)**.**

1. ***Uniqueness of the*** (X^T, X) ***Negative-***b_L ***Branch*** (Lean `thm16`–`thm17`, Cert. `C14`)*:*
   - *Branches* (X, X) *and* (X^T, X^T) *map* $|u\rangle\langle u|$ *to an isotropic operator* $c_I I + c_u |u\rangle\langle u|$ *(*$c_I \geq 0$*); whenever NPT (*$c_u > c_I$*), its maximally entangled fidelity* $F = \frac{c_I + dc_u}{d(dc_I + c_u)}$ *strictly exceeds* $1/d$*, so the state is* 1*-copy distillable [9, 4].*
   - *Branch* (X^T, X) *with* $b_L > 0, b_M < 0$ *satisfies* $a_L \geq b_L > 0$ *and* $a_M \geq d|b_M|$*, forcing*

$$c_I + dc_S \;=\; a_L(da_M - |b_M|) + b_L(a_M - d|b_M|) \;\geq\; b_L(d^2 - 1)|b_M| \;>\; 0,$$

     so its output is always strictly PPT (separable).

   Hence Faisal's (X^T, X) *branch with* $b_L < 0, b_M > 0$ *(up to Alice–Bob swap) is the* unique $U(d)$*-equivariant local-map branch capable of certifying NPT bound entanglement.*

2. ***Certified Werner Intervals & Exact White-Noise Threshold*** (Lean `thm18`–`thm20`, Cert. `C15`)*: Whenever* $\rho(d, \alpha_{\mathrm{cert}})$ *is certified NPT bound entangled (*$\alpha_{\mathrm{cert}} < -1/d$*), every Werner state* $\rho(d, \alpha)$ *with* $\alpha \in [\alpha_{\mathrm{cert}}, -1/d)$ *is NPT bound entangled. Equivalently, under the depolarizing white-noise channel* $\mathcal{D}_p(\rho) := (1-p)\rho + p\frac{I}{d^2}$*,* $\mathcal{D}_p(\rho(d, \alpha))$ *remains NPT bound entangled if and only if*

$$0 \;\leq\; p \;<\; p_{\max}(d, \alpha) \;:=\; \frac{d(1 + d\alpha)}{\alpha(d^2 - 1)}. \tag{10}$$

   At $d = 8$*, our sharpened witness* ρ_8^* *(*$\alpha = -20/151$*) raises the certified white-noise tolerance by an exact* $+35\%$ *(ratio* $27/20$*), from Faisal's* $p_{\max}(8, -3/23) = \frac{8}{189} \approx 4.23\%$ *to* $p_{\max}(8, -20/151) = \frac{2}{35} \approx 5.71\%$*. For our all-*$d \geq 8$ *closed-form family (9),* $p_{\max}(d, -\frac{6}{5d+7}) = \frac{d(d-7)}{6(d^2-1)} \to \frac{1}{6}$ *as* $d \to \infty$ *(and our* $d = 400$ *exact witness tolerates* $> 87\%$ *white noise).*

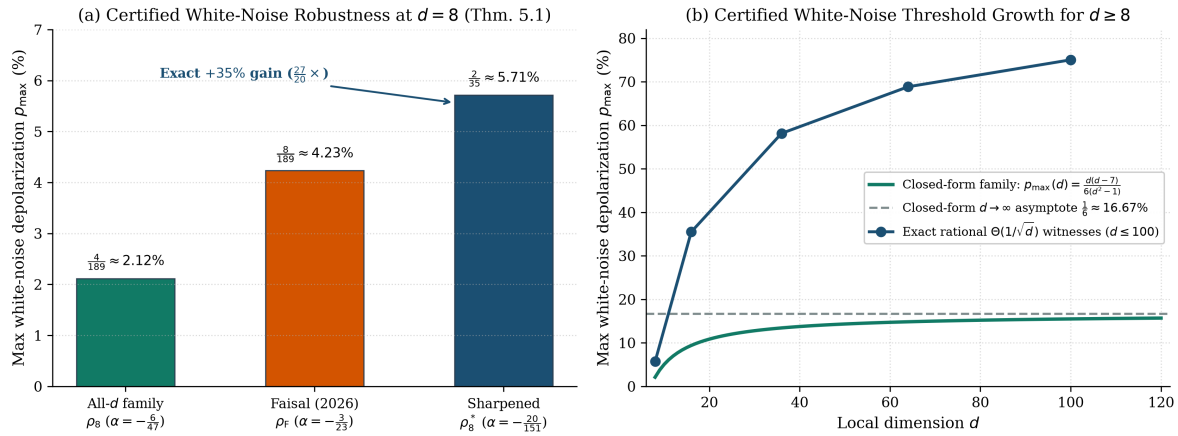


Figure 3: (a) Exact certified white-noise depolarizing robustness $p_{\max}$ at $d = 8$: our sharpened witness ρ_8^* achieves $\frac{2}{35} \approx 5.71\%$, an exact $+35\%$ gain over Faisal's $\frac{8}{189} \approx 4.23\%$ (Theorem 5.1). (b) Growth of certified white-noise tolerance $p_{\max}(d)$ for $d \geq 8$: the closed-form family rises monotonically toward $1/6 \approx 16.67\%$, while exact rational $\Theta(1/\sqrt{d})$ witnesses exceed 75% by $d = 100$ (and 87% at $d = 400$).

Proof. (1) Applying $(L \otimes M)$ to $|u\rangle\langle u| = \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|$: when both maps carry X (or both carry X^T), $(X \otimes X)(|u\rangle\langle u|) = |u\rangle\langle u|$ and $(X^T \otimes X^T)(|u\rangle\langle u|) = |u\rangle\langle u|$, yielding an isotropic operator $c_I I + c_u |u\rangle\langle u|$ where $c_I \geq 0$ follows from PSD on the (d^2-1)-dimensional orthogonal complement of $|u\rangle$. Its partial transpose is $c_I I + c_u \mathrm{SWAP}$, which is NPT iff $c_u > c_I$ (since PSD of the state requires $c_I + dc_u \geq 0$, ruling out $c_u < -c_I$). Whenever $c_u > c_I$ and $d \geq 2$, $(c_I + dc_u) - (dc_I + c_u) = (d-1)(c_u - c_I) > 0$ (Lean `thm17`), so $F = \frac{c_I + dc_u}{d(dc_I + c_u)} > \frac{1}{d}$, which is 1-copy distillable [9, 4]. On the (X^T, X) branch with $b_L > 0, b_M < 0$, CP of L and M requires $a_L \geq b_L > 0$ and $a_M \geq d|b_M| > 0$. Expanding $c_I = da_L a_M - a_L|b_M| + b_L a_M$ and $c_S = -b_L|b_M|$ gives $c_I + dc_S = a_L(da_M - |b_M|) + b_L(a_M - d|b_M|) \geq b_L(d^2-1)|b_M| > 0$ (Lean `thm16`, Cert. `C14`), so the partial transpose eigenvalue on $|u\rangle$ is strictly positive.

(2) For any $\alpha \in [\alpha_{\mathrm{cert}}, -1/d)$, writing $\rho(d,\alpha) = (1-t)\rho(d,\alpha_{\mathrm{cert}}) + t\,\rho(d,-1/d)$ yields exact convex weights $t \in [0,1)$ (Lean `thm20`). Because $\rho(d,-1/d)$ is separable [23], any LOCC distillation protocol for $\rho(d,\alpha)$ can be simulated on $\rho(d,\alpha_{\mathrm{cert}})$ by local mixing with the separable state $\rho(d,-1/d)$, forcing $E_D(\rho(d,\alpha)) \leq E_D(\rho(d,\alpha_{\mathrm{cert}})) = 0$ (indeed, mixing with $\frac{I}{d^2}$ is a local depolarizing channel!). Direct calculation gives $\mathcal{D}_p(\rho(d,\alpha)) = \rho(d,\alpha_p)$ with $\alpha_p = \frac{(1-p)d\alpha}{d+p\alpha}$; solving $\alpha_p = -1/d$ yields (10) identically (Lean `thm18`), with $\alpha \leq \alpha_p < -1/d$ for all $0 \leq p < p_{\max}(d,\alpha)$. Substituting $(8, -3/23)$, $(8, -20/151)$, and $(d, -6/(5d+7))$ gives $\frac{8}{189}$, $\frac{2}{35}$, ratio $\frac{27}{20}$, and $\frac{d(d-7)}{6(d^2-1)}$ (Lean `thm19`, Cert. `C15`). □

6 Formal Verification Architecture and Reproducibility

All algebraic reductions, $d = 8$ Sylvester certifications, all-$d \geq 8$ symbolic positivity proofs, $d \leq 7$ exact rational dual certificates, uniqueness inequalities, and white-noise thresholds are verified across three independent layers (while Haar-twirling completeness and the LOCC simulation argument are proved in Sections 2 and 5):

1. **Constructive Lean 4.36.0 Formalization:** 20 theorems (`thm01`–`thm20`) in `lean/NPTBoundEntanglementBarriers.lean` compiling under `lean -DwarningAsError=true` with zero external imports, zero `sorry`, zero `grind`, zero `Classical.choice`, and zero

warnings—including full computational polynomial reflection over $\mathbb{Z}[s]$ proving all 28 Sylvester minors strictly positive for *every* integer $d = 8 + s \geq 8$.

2. **AST-Audited Zero-Float Python Verifier:** Pure standard-library code (`int`, `Fraction`) in `verification/verify_npt_bound_entanglement.py` self-auditing its AST (0 float violations) and emitting 16 canonical SHA-256 certificates (`C01`–`C16`) across four modes—including exact verification of all 33 rational SDP duals for $d \in \{3, 4, 5, 6, 7\}$ and 5 high-d rational primal witnesses up to $d = 400$—with manifest SHA-256 `49651e4871bc91b7da32647d7fe20c36da7ef042b9f26650f7504750e55f4afa`.

3. **Adversarial Test Suite:** 56 pytest checks in `tests/test_adversarial.py` covering positive certification, boundary perturbations, tampered dual rejection, and high-d asymptotic scaling.

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A Explicit Walled Brauer Basis, Five Reduced Blocks, and q_8^*

To make Theorems 2.1, 3.1, and 4.1 completely self-contained within this manuscript, we list the exact 17-element Hermitian basis pairing, all five reduced block matrices $\mathcal{B}_0, \ldots, \mathcal{B}_4$, and all 17 rational coordinates of q_8^*.

17-Element Hermitian Basis Map. Ordering S_4 lexicographically as $\pi_0 = (0,1,2,3), \ldots, \pi_{23} = (3,2,1,0)$ (matching [6, Appendix A] with 0-based indices), the 17 Hermitian generators $B_k = \sum_{i \in G_k} (U_{\pi_i})^{\Gamma_1}$ correspond to the 10 involutions and 7 inverse pairs:

$$
\begin{array}{lllll}
G_0 = \{0\}, & G_1 = \{1\}, & G_2 = \{2\}, & G_3 = \{3,4\}, & G_4 = \{5\}, \\
G_5 = \{6\}, & G_6 = \{7\}, & G_7 = \{8,12\}, & G_8 = \{9,18\}, & G_9 = \{10,13\}, \\
G_{10} = \{11,19\}, & G_{11} = \{14\}, & G_{12} = \{15,20\}, & G_{13} = \{16\}, & G_{14} = \{17,22\}, \\
G_{15} = \{21\}, & G_{16} = \{23\}. & & &
\end{array}
$$

The input-swap partial-transpose involution γ swaps $(2,11), (3,12), (4,15), (8,9), (13,16)$ and fixes the remaining 7 indices $\{0,1,5,6,7,10,14\}$.

Five Reduced $U(d)$-Isotypic Blocks $\mathcal{B}_0, \ldots, \mathcal{B}_4$. For any $x = (x_0, \ldots, x_{16}) \in \mathbb{R}^{17}$ and local dimension $d \geq 3$:

$$\mathcal{B}_0(x) = [x_0 + x_1 + x_{11} + 2x_{12} + x_{15}], \qquad \mathcal{B}_1(x) = [x_0 - x_1 - x_{11} + 2x_{12} - x_{15}] \ \ (d \geq 4),$$
$$\mathcal{B}_2(x) = \begin{pmatrix} 6x_0 + 6x_1 - 3x_{11} - 6x_{12} - 3x_{15} & 3x_{11} - 3x_{15} \\ 3x_{11} - 3x_{15} & 2x_0 - 2x_1 + x_{11} - 2x_{12} + x_{15} \end{pmatrix},$$

and the two symmetric 3×3 blocks $\mathcal{B}_3(d, x)$ and $\mathcal{B}_4(d, x)$ have upper-triangular entries:

$$\begin{aligned}
(\mathcal{B}_3)_{00} &= d^2(x_5 + x_6) + d(x_0 + x_1 + 2(x_7 + x_8 + x_9 + x_{10})) \\
&\quad + x_2 + 2x_3 + x_4 + x_{11} + 2x_{12} + x_{13} + 2x_{14} + x_{15} + x_{16}, \\
(\mathcal{B}_3)_{01} &= d^2(x_7 + x_8) + d(x_2 + x_3 + x_5 + x_6 + x_9 + x_{10} + x_{11} + x_{12} + x_{14} + x_{16}) \\
&\quad + x_0 + x_1 + x_3 + x_4 + x_7 + x_8 + x_9 + x_{10} + x_{12} + x_{13} + x_{14} + x_{15}, \\
(\mathcal{B}_3)_{02} &= d^2(x_9 + x_{10}) + d(x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_{12} + x_{13} + x_{14} + x_{15}) \\
&\quad + x_0 + x_1 + x_2 + x_3 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{14} + x_{16}, \\
(\mathcal{B}_3)_{11} &= d^2(x_2 + x_{16}) + d(x_0 + 2x_3 + 2x_7 + 2x_8 + 2x_{14} + x_{15}) \\
&\quad + x_1 + x_4 + x_5 + x_6 + 2x_9 + 2x_{10} + x_{11} + 2x_{12} + x_{13}, \\
(\mathcal{B}_3)_{12} &= d^2(x_3 + x_{14}) + d(x_1 + x_2 + x_4 + x_7 + x_8 + x_9 + x_{10} + x_{12} + x_{13} + x_{16}) \\
&\quad + x_0 + x_3 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{14} + x_{15}, \\
(\mathcal{B}_3)_{22} &= d^2(x_4 + x_{13}) + d(x_0 + 2x_3 + 2x_9 + 2x_{10} + x_{11} + 2x_{14}) \\
&\quad + x_1 + x_2 + x_5 + x_6 + 2x_7 + 2x_8 + 2x_{12} + x_{15} + x_{16},
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{B}_4)_{00} &= d^2(x_5 - x_6) + d(x_0 - x_1 + 2(x_7 - x_8 - x_9 + x_{10})) \\
&\quad + x_2 - 2x_3 + x_4 + x_{11} - 2x_{12} - x_{13} + 2x_{14} + x_{15} - x_{16}, \\
(\mathcal{B}_4)_{01} &= d^2(x_7 - x_8) + d(x_2 - x_3 + x_5 - x_6 + x_9 - x_{10} + x_{11} - x_{12} + x_{14} - x_{16}) \\
&\quad + x_0 - x_1 + x_3 - x_4 + x_7 - x_8 - x_9 + x_{10} + x_{12} + x_{13} - x_{14} - x_{15}, \\
(\mathcal{B}_4)_{02} &= d^2(x_9 - x_{10}) + d(x_3 - x_4 - x_5 + x_6 + x_7 - x_8 + x_{12} + x_{13} - x_{14} - x_{15}) \\
&\quad - x_0 + x_1 + x_2 - x_3 - x_7 + x_8 + x_9 - x_{10} + x_{11} - x_{12} + x_{14} - x_{16}, \\
(\mathcal{B}_4)_{11} &= d^2(x_2 - x_{16}) + d(x_0 + 2x_3 + 2x_7 - 2x_8 - 2x_{14} - x_{15}) \\
&\quad + x_1 + x_4 + x_5 - x_6 + 2x_9 - 2x_{10} + x_{11} - 2x_{12} - x_{13}, \\
(\mathcal{B}_4)_{12} &= d^2(x_3 - x_{14}) + d(x_1 + x_2 + x_4 - x_7 + x_8 + x_9 - x_{10} - x_{12} - x_{13} - x_{16}) \\
&\quad + x_0 + x_3 - x_5 + x_6 + x_7 - x_8 - x_9 + x_{10} - x_{11} + x_{12} - x_{14} - x_{15}, \\
(\mathcal{B}_4)_{22} &= d^2(x_4 - x_{13}) + d(x_0 + 2x_3 - 2x_9 + 2x_{10} - x_{11} - 2x_{14}) \\
&\quad + x_1 + x_2 + x_5 - x_6 - 2x_7 + 2x_8 - 2x_{12} + x_{15} - x_{16}.
\end{aligned}$$

Explicit Sharpened $d = 8$ Witness q_8^*. Table 3 lists all 17 exact rational coordinates of q_8^* (Theorem 3.1(2), $(a_L, a_M) = (161/100, 1/2)$, $\alpha_8^* = -20/151$, common denominator 640 000).

Table 3: Exact integer-scaled coordinates $640\,000\,(q_8^*)_k$ for the sharpened $d = 8$ witness $\rho_8^* = \frac{151I - 20\,\mathrm{SWAP}}{9504}$.

k	0	1	2	3	4	5	6	7	8
$640\,000\,(q_8^*)_k$	192	−64	0	64	0	56234	7291	−42296	18024
k	9	10	11	12	13	14	15	16	—
$640\,000\,(q_8^*)_k$	−76288	107432	136192	−115136	85376	−117824	288768	−29120	—