Exact Extremal Geometry of Two-Qubit PPT Pauli Channels, Arf-Invariant Parity Dichotomy, and Irreducible Even-Qubit Bound Magic

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https://github.com/lluiseriksson/bound-magic-exact-threshold-arf-dichotomy

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Abstract

Kitzinger, Eisert, and Kunjwal (*arXiv:2611.34872v1*) recently proved that multi-qubit bound magic exists for all $n \geq 3$ qubits via the stabilizer-closed convex hulls B_n of outputs of n-qubit PPT Pauli channels, leaving open: (i) an exact, non-floating-point proof that $B_2 = \mathrm{STAB}_2$, (ii) the precise relationship between entanglement-breaking Pauli channels (EB_n), PPT Pauli channels (PPT_n), and magic-breaking Pauli channels (MB_n) surrounding Conjecture 4, and (iii) an analytic family of bound-magic states for even qubit numbers $n \geq 4$. We resolve all three questions with exact algebraic proofs backed by constructive Lean 4.35.0 formalization (18/18 theorems, 0 `sorry`, 0 `Classical.choice`) and an AST-audited zero-float exact rational Python verifier (16/16 SHA-256 certificates). First (Theorem 2), using an exact Motzkin–Burger Balinski tangent-cone completeness proof in $\mathbb{Q}^{15}$, we show unconditionally that PPT_2 has exactly 252 extreme rays partitioning into two projective Clifford orbits: 60 stabilizer-basis dephasing channels ($\in \mathrm{EB}_2$) and 192 rank-6 channels in the single orbit of the 5-cycle channel $\mathcal{E}_{C_5}$ (stabilizer A_5, order 60). Every Orbit-2 channel has non-identity Pauli ℓ_1 norm $\sum_{a \neq II} |\lambda_a| = 5 > 3$, exceeding the universal Cauchy–Schwarz ceiling $2^n - 1$ for EB_n, yet satisfies $I - \mathcal{E}(A_j) \succeq 0$ across all $8 \times 192 = 1{,}536$ 4×4 Hermitian matrices generated by the 8 Clifford facet orbits (22,320 facets) of STAB_2 (certified by exact rational 8×8 realified LDL^T elimination, tight on orbits F_1 and F_5). This proves $B_2 = \mathrm{STAB}_2$ and Conjecture 4 for $n = 2$ unconditionally while refuting the converse of Conjecture 4 via the strict chain $\mathrm{EB}_2 \subsetneq \mathrm{PPT}_2 \subseteq \mathrm{MB}_2$. Second (Theorem 3), we prove via $\mathbb{Z}[i]$ Gaussian-sum phase rigidity that $W_{u,v} = |u\rangle\langle u| + |v\rangle\langle v| - \frac{3}{2\sqrt{2}}(|u\rangle\langle v| + |v\rangle\langle u|)$ is a universal magic witness for every pair of stabilizer states with $|\langle u|v\rangle| = 2^{-k/2}$ (k odd), yielding exact closed-form depolarizing robustness $p_{\mathrm{depol}}(n) > 1 - \frac{2\sqrt{2}}{3} \approx 5.72\%$ for all odd $n \geq 3$; by contrast, for every even $n = 2m \geq 2$, we prove that the flag-channel states $\mathcal{F}_{2m}(|\phi_\pm^{(2m)}\rangle\langle\phi_\pm^{(2m)}|)$ collapse identically into STAB_{2m} as uniform convex mixtures of all $2^{m-1}(2^m \pm 1)|\mathrm{Alt}_{\mathrm{inv}}(2m, \mathbb{F}_2)|$ nondegenerate real quadratic stabilizer states of Arf invariant 0 (resp. 1). Third (Theorem 4), for every even $n \geq 4$ we construct the explicit S_n-symmetric, full-rank, non-factorizable PPT channel family $\mathcal{G}_n^{(\delta)} = (1-\delta)\mathcal{F}_Z^{(n)} + \delta\mathcal{F}_n$ acting on $|0^n\rangle + |\mathrm{GHZ}_X^n\rangle$, proving that $\mathcal{G}_n^{(\delta)}(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|) \in B_n \setminus \mathrm{STAB}_n$ for every $0 < \delta < \delta_{\mathrm{crit}}(n)$, where $\delta_{\mathrm{crit}}(n)$ decreases monotonically from $\frac{20-8\sqrt{2}}{17} \approx 0.5110$ ($n = 4$) to the exact universal floor $\frac{2(2-\sqrt{2})}{3} \approx 0.3905 > \frac{3}{8}$, with canonical $\delta = \frac{1}{3}$ certifying irreducible bound magic for all even $n \geq 4$ simultaneously.

1 Introduction

In fault-tolerant quantum computing, magic-state distillation transforms noisy nonstabilizer states into pure resource states using stabilizer operations. For odd-dimensional qudits, bound magic—nonstabilizer states from which no pure magic can be distilled even asymptotically—arises naturally from discrete Wigner-function positivity [2]. For qubits, however, the existence of bound magic

remained open for over fifteen years until the recent breakthrough of Kitzinger, Eisert, and Kunjwal [1]. By defining B_n as the convex hull of all outputs $\mathcal{E}(\omega)$ of n-qubit positive-partial-transpose (PPT) Pauli channels $\mathcal{E} \in \mathrm{PPT}_n$ acting on arbitrary density operators $\omega \in D_n$, they proved that $\{B_n\}_{n\geq 1}$ is closed under tensor products and all stabilizer operations, reduces to STAB_1 at $n = 1$, and strictly contains STAB_n for all $n \geq 3$ via an explicit odd-n flag-channel construction $\mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)$.

Three fundamental questions were left open by [1]:

1. **Exact certification of $B_2 = \mathrm{STAB}_2$:** In [1, p. 4 and End Matter], the equality $B_2 = \mathrm{STAB}_2$ was verified only by floating-point numerical vertex comparison against the Λ_2-polytope [6], explicitly noting that "an exact conceptual proof of this equality remains to be given."

2. **Separation of EB_n, PPT_n, and MB_n:** Conjecture 4 of [1] posits that every entanglement-breaking Pauli channel is magic-breaking ($\mathrm{EB}_n \subseteq \mathrm{MB}_n$), raising the question of whether PPT bound entanglement in a Pauli channel's Choi state is equivalent to, or merely necessary for, the failure of magic-breakingness.

3. **Analytic even-n bound-magic families:** The analytic construction in [1] required odd $n \geq 3$ because $t = \langle 0^n|+^n\rangle = 2^{-n/2}$ is irrational only for odd n; for even $n \geq 4$, they left non-reducible examples to numerical search and remarked that the odd-n restriction appeared to be a feature of their analytic proof method.

We resolve all three questions completely and constructively:

- **Theorem 2 (Exact PPT_2 geometry, unconditional $B_2 = \mathrm{STAB}_2$, and $\mathrm{EB}_2 \subsetneq \mathrm{PPT}_2 \subseteq \mathrm{MB}_2$):** We classify all 252 extreme rays of PPT_2 with an unconditional zero-float Motzkin–Burger Balinski completeness proof (60 stabilizer-dephasing channels in Orbit 1 and 192 rank-6 C_5-supported channels in Orbit 2). We prove that all 192 Orbit-2 channels have $\sum_{a\neq II} |\lambda_a| = 5 > 3$, certifying PPT bound entanglement ($\mathcal{E}_{C_5} \in \mathrm{PPT}_2 \setminus \mathrm{EB}_2$), and simultaneously certify $I - \mathcal{E}(A_j) \succeq 0$ for all $8 \times 192 = 1{,}536$ 4×4 Hermitian matrices across all 8 Clifford facet orbits (22,320 facets) of STAB_2 in exact rational arithmetic. This yields an unconditional proof of $B_2 = \mathrm{STAB}_2$ and Conjecture 4 at $n = 2$, while refuting the converse of Conjecture 4.

- **Theorem 3 (Universal odd-k witness and even-n Arf-invariant collapse):** Using $\mathbb{Z}[i]$ Gaussian-sum phase rigidity, we prove that $W_{u,v}$ with universal constant $c = \frac{3}{2\sqrt{2}}$ is nonnegative on STAB_n for every odd-k stabilizer pair $|\langle u|v\rangle| = 2^{-k/2}$, yielding exact closed-form depolarizing noise thresholds $p_{\mathrm{depol}}(n) > 1 - \frac{2\sqrt{2}}{3}$ for all odd $n \geq 3$. Conversely, we show that the odd-n requirement of [1] is *not* merely a proof artifact: for every even $n = 2m \geq 2$, $\mathcal{F}_{2m}(|\phi_\pm^{(2m)}\rangle\langle\phi_\pm^{(2m)}|)$ collapses identically into STAB_{2m} as the uniform mixture of all nondegenerate real quadratic stabilizer states of Arf invariant 0 (for $+$) or 1 (for $-$).

- **Theorem 4 (Universal irreducible even-n bound magic for all even $n \geq 4$):** We construct the explicit S_n-symmetric, full-rank PPT Pauli channel family $\mathcal{G}_n^{(\delta)} = (1-\delta)\mathcal{F}_Z^{(n)} + \delta\mathcal{F}_n$ acting on $|0^n\rangle + |\mathrm{GHZ}_X^n\rangle$ (overlap $2^{-(n-1)/2} \notin \mathbb{Q}$), proving $\mathcal{G}_n^{(\delta)}(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|) \in B_n \setminus \mathrm{STAB}_n$ for every $0 < \delta < \delta_{\mathrm{crit}}(n)$, with exact universal floor $\delta_{\mathrm{crit}}(\infty) = \frac{2(2-\sqrt{2})}{3} \approx 0.390524 > \frac{3}{8}$. At $\delta = \frac{1}{3}$, every non-identity Pauli expectation satisfies $|\mathrm{Tr}(P\rho)| \leq \frac{2}{3} < 1$, proving unconditionally that no n-qubit Clifford unitary can decouple a single $|0\rangle$ qubit.

2 Preliminaries and Exact Pauli-Channel Formalism

Let $\mathcal{P}_n = \{I, X, Y, Z\}^{\otimes n}$ denote the 4^n unsigned n-qubit Pauli operators, indexed by $a \in \mathbb{F}_2^{2n}$ with symplectic inner product $\langle a, b\rangle \in \mathbb{F}_2$ satisfying $P_a P_b = (-1)^{\langle a,b\rangle} P_b P_a$. Write $\mathrm{wt}_Y(a)$ for the number

of Y tensor factors in P_a and $\theta_a := (-1)^{\mathrm{wt}_Y(a)}$, so that $P_a^T = \theta_a P_a$. An n-qubit Pauli channel $\mathcal{E}(\rho) = \sum_b p_b P_b \rho P_b$ ($p_b \geq 0$, $\sum_b p_b = 1$) acts diagonally in the Pauli basis as $\mathcal{E}(P_a) = \lambda_a P_a$, where

$$\lambda_a = \sum_{b \in \mathbb{F}_2^{2n}} (-1)^{\langle a,b \rangle} p_b, \qquad p_b = \frac{1}{4^n} \sum_{a \in \mathbb{F}_2^{2n}} (-1)^{\langle a,b \rangle} \lambda_a, \qquad q_b := \frac{1}{4^n} \sum_{a \in \mathbb{F}_2^{2n}} (-1)^{\langle a,b \rangle} \theta_a \lambda_a. \tag{1}$$

By Proposition 1 of [1] (cf. [5]), $\mathcal{E}$ is positive-partial-transpose ($\mathcal{E} \in \mathrm{PPT}_n$, i.e., $T_n \circ \mathcal{E}$ is completely positive) if and only if $p_b \geq 0$ and $q_b \geq 0$ for all $b \in \mathbb{F}_2^{2n}$ with $\lambda_0 = 1$. Thus PPT_n is a compact convex polytope in $\mathbb{R}^{4^n-1}$ cut out by $2 \cdot 4^n$ linear inequalities. A channel $\mathcal{E}$ is *entanglement-breaking* ($\mathcal{E} \in \mathrm{EB}_n$) iff its Choi state $J(\mathcal{E}) = 4^{-n} \sum_a \lambda_a P_a \otimes P_a^T$ is separable across $A|B$, and *magic-breaking* ($\mathcal{E} \in \mathrm{MB}_n$) iff $\mathcal{E}(D_n) \subseteq \mathrm{STAB}_n$.

3 Exact Extremal Geometry of PPT_2 and Unconditional $B_2 = \mathrm{STAB}_2$

We first establish a universal ℓ_1-norm ceiling for entanglement-breaking n-qubit Pauli channels.

Lemma 1 (Universal Cauchy–Schwarz Ceiling for EB_n)**.** *For every $n \geq 1$ and every entanglement-breaking n-qubit Pauli channel $\mathcal{E} \in \mathrm{EB}_n$,*

$$\sum_{a \in \mathbb{F}_2^{2n} \setminus \{0\}} |\lambda_a(\mathcal{E})| \leq 2^n - 1. \tag{2}$$

Proof. Since $\mathcal{E} \in \mathrm{EB}_n$, its Choi state $J(\mathcal{E})$ is separable and hence admits a convex decomposition $J(\mathcal{E}) = \sum_k w_k |\psi_k\rangle\langle\psi_k| \otimes |\phi_k\rangle\langle\phi_k|$ with $w_k \geq 0$, $\sum_k w_k = 1$, and unit vectors $|\psi_k\rangle, |\phi_k\rangle \in \mathbb{C}^{2^n}$. By Pauli orthogonality, $\lambda_a(\mathcal{E}) = \mathrm{Tr}((P_a \otimes P_a^T) J(\mathcal{E})) = \sum_k w_k \langle\psi_k|P_a|\psi_k\rangle\langle\phi_k|P_a^T|\phi_k\rangle$. Applying the triangle inequality, Cauchy–Schwarz over $a \neq 0$, and the pure-state Parseval identity $\sum_{a \neq 0} \langle\psi|P_a|\psi\rangle^2 = 2^n - 1$ gives

$$\sum_{a \neq 0} |\lambda_a(\mathcal{E})| \leq \sum_k w_k \left(\sum_{a \neq 0} \langle\psi_k|P_a|\psi_k\rangle^2 \right)^{1/2} \left(\sum_{a \neq 0} \langle\phi_k|P_a^T|\phi_k\rangle^2 \right)^{1/2} = \sum_k w_k (2^n - 1) = 2^n - 1. \quad \square$$

To classify the extreme rays of PPT_2 under Clifford equivalence, note that conjugating a Pauli channel by a Clifford unitary $U \in \mathrm{Cl}_2$ permutes its Pauli eigenvalues without sign flips ($\lambda'_{\pi_U(a)} = \lambda_a$), while post-composing with a Pauli operator P_c multiplies $\lambda_a \mapsto (-1)^{\langle c,a \rangle} \lambda_a$. This defines the exact channel-Clifford action group $G_{\mathrm{ch}} \cong \mathrm{Sp}(4, \mathbb{F}_2) \ltimes \mathbb{F}_2^4$ of order $720 \times 16 = 11{,}520$, while states and facets transform under the signed projective Clifford group PCl_2 of identical order 11,520. The real-Clifford subgroup $\mathrm{Aut}_{\mathrm{real}}(\mathrm{PPT}_2) \leq G_{\mathrm{ch}}$ preserving θ_a pointwise (hence preserving both $\{p_b \geq 0\}$ and $\{q_b \geq 0\}$ separately) has order 1,152 (Certificate `C01`).

Theorem 2 (Exact Extremal Geometry of PPT_2, $B_2 = \mathrm{STAB}_2$, and $\mathrm{EB}_2 \subsetneq \mathrm{PPT}_2 \subseteq \mathrm{MB}_2$)**.**

(a) ***Complete 252-Vertex Classification of*** PPT_2***:*** *The 15-dimensional 32-inequality polytope PPT_2 has* exact *vertex set $V_{252} = \mathcal{O}_1 \sqcup \mathcal{O}_2$ of cardinality $60 + 192 = 252$, partitioning into two G_{ch}-orbits:*

- *$\mathcal{O}_1$ (60 vertices, two $\mathrm{Aut}_{\mathrm{real}}(\mathrm{PPT}_2)$-suborbits of sizes 24 and 36): complete stabilizer-basis dephasing channels with $|\mathrm{supp}(\lambda)| = |\mathrm{supp}(p)| = |\mathrm{supp}(q)| = 4$ and $\lambda_a \in \{0, \pm 1\}$, all lying in $\mathrm{EB}_2 \cap \mathrm{MB}_2$.*
- *$\mathcal{O}_2$ (192 vertices, a single $\mathrm{Aut}_{\mathrm{real}}(\mathrm{PPT}_2)$-orbit, G_{ch}-stabilizer isomorphic to A_5 of order 60): rank-6 channels generated by the 5-cycle channel*

$$\mathcal{E}_{C_5}(\rho) = \frac{1}{6} \sum_{P \in S_6} P \rho P, \qquad S_6 = \{II, IY, YI, YX, ZI, ZZ\}, \tag{3}$$

where $S_6 \setminus \{II\}$ forms an induced 5-cycle ($IY-YI-YX-ZZ-ZI-IY$) in the two-qubit Pauli commutation graph. Its Pauli eigenvalues are $\lambda_I = 1$, $\lambda_a = -\frac{1}{3}$ on the five Paulis $\{XI, XX, XZ, YZ, ZX\}$, and $\lambda_a = +\frac{1}{3}$ on the remaining ten non-identity Paulis, with $T \circ \mathcal{E}_{C_5}$ supported uniformly on $\{IZ, XX, YI, ZI, ZX, ZZ\}$.

(b) ***Strict Separation*** *$\mathrm{EB}_2 \subsetneq \mathrm{PPT}_2 \subseteq \mathrm{MB}_2$: Every $\mathcal{E} \in \mathcal{O}_2$ has $\sum_{a \neq II} |\lambda_a(\mathcal{E})| = 15 \cdot \frac{1}{3} = 5 > 3$. By Lemma 1, every Orbit-2 extreme ray is a PPT bound-entangled channel ($\mathcal{O}_2 \subset \mathrm{PPT}_2 \setminus \mathrm{EB}_2$), refuting the converse of Conjecture 4 of [1].*

(c) ***Unconditional Zero-Float Proof of*** *$B_2 = \mathrm{STAB}_2$: Every $\mathcal{E} \in \mathrm{PPT}_2$ satisfies $\mathcal{E}(D_2) \subseteq \mathrm{STAB}_2$. Consequently $B_2 = \mathrm{STAB}_2$, and Conjecture 4 of [1] holds unconditionally for $n = 2$.*

Proof. **(a)** Direct rational verification confirms that every $v \in V_{252}$ satisfies all 32 inequalities $p_b(v) \geq 0$, $q_b(v) \geq 0$ with tight constraint matrix of full rank 15 (Certificate `C02`). To prove $\mathrm{PPT}_2 = \mathrm{conv}(V_{252})$ unconditionally without floating-point polytope software, we apply Balinski's vertex-edge graph connectedness theorem to the three $\mathrm{Aut}_{\mathrm{real}}(\mathrm{PPT}_2)$-orbit representatives of V_{252} (two in $\mathcal{O}_1$, each with 24 tight inequalities, and one in $\mathcal{O}_2$, with 20 tight inequalities). For each representative v, exact rational Motzkin–Burger double description of its tangent cone $C(v) = \{d \in \mathbb{R}^{15} : u_k \cdot d \geq 0 \ \forall k \in T(v)\}$ (using the Fukuda–Prodon combinatorial adjacency test over $\mathbb{Q}$) yields exactly 136, 136, and 70 extreme rays respectively (Certificate `C03`). Stepping along every single extreme ray to the first newly active inequality in $\{0, \ldots, 31\} \setminus T(v)$ lands unconditionally on a vertex in V_{252}. Since PPT_2 is bounded and its 1-skeleton is connected, no vertex outside V_{252} exists.

(b) Immediate from $|\lambda_a(\mathcal{E}_{C_5})| = \frac{1}{3}$ for all 15 non-identity a, giving $\sum_{a \neq II} |\lambda_a| = 5 > 3 = 2^2 - 1$, combined with Lemma 1 (Certificate `C04`).

(c) By convexity of B_2 and part (a), it suffices to prove $\mathcal{E}(D_2) \subseteq \mathrm{STAB}_2$ for all $\mathcal{E} \in \mathcal{O}_1 \sqcup \mathcal{O}_2$. Every $\mathcal{E} \in \mathcal{O}_1$ is a complete dephasing channel in a two-qubit stabilizer basis (up to a Pauli unitary) and hence maps D_2 into STAB_2. By Reichardt's exact classification [3] (independently verified in exact $\mathbb{Q}$ in Certificate `C05`), STAB_2 is cut out by 22,320 facets $\mathrm{Tr}(A\sigma) \leq 1$ ($A = \sum_{a \neq II} a_k P_k$ traceless) forming exactly 8 disjoint PCl_2-orbits $F_1, \ldots, F_8$ of sizes $240, 192, 3{,}840, 5{,}760, 1{,}920, 2{,}304, 5{,}760, 2{,}304$ (each rank-15 tight on 15 to 33 of the 60 stabilizer vertices; see Table 1). For any $\mathcal{E} \in \mathcal{O}_2$, self-adjointness of Pauli channels gives $\mathrm{Tr}(A\mathcal{E}(\rho)) = \mathrm{Tr}(\mathcal{E}(A)\rho)$, so

$$\mathcal{E}(D_2) \subseteq \mathrm{STAB}_2 \ \forall \mathcal{E} \in \mathcal{O}_2 \iff I_4 - \mathcal{E}(A) \succeq 0 \ \forall \mathcal{E} \in \mathcal{O}_2, \ \forall A \in \bigcup_{j=1}^{8} F_j.$$

Since $\mathcal{O}_2$ is invariant under Cl_2-conjugation and Pauli sign shifts, by Clifford duality it suffices to certify $I_4 - \mathcal{E}(A_j) \succeq 0$ for the 8 representative facets $A_1, \ldots, A_8$ and all 192 channels $\mathcal{E} \in \mathcal{O}_2$, i.e., $8 \times 192 = 1{,}536$ exact 4×4 Hermitian matrices over $\mathbb{Q}(i)$. Realifying each 4×4 Hermitian matrix $H = R + iM$ to the 8×8 real symmetric matrix $\left(\begin{smallmatrix} R & -M \\ M & R \end{smallmatrix}\right)$ and applying exact rational diagonal-pivoting LDL^T elimination certifies $I_4 - \mathcal{E}(A_j) \succeq 0$ for all 1,536 instances with zero floating-point arithmetic (Certificate `C06`, Figure 1). Since $\mathrm{EB}_2 \subseteq \mathrm{PPT}_2$, Conjecture 4 follows at $n = 2$. □

4 Universal Odd-k Witness and Exact Even-n Arf-Invariant Collapse

For any $n \geq 2$, let $\mathcal{P}_{\mathrm{flag}}(n) := \{P \in \mathcal{P}_n : \mathrm{wt}_Y(P) \text{ even}, \ P \notin \{I, Z\}^n, \ P \notin \{I, X\}^n\}$. Since there are $2^{n-1}(2^n + 1)$ n-qubit Paulis with even wt_Y, inclusion–exclusion yields $|\mathcal{P}_{\mathrm{flag}}(n)| = (2^n - 1)(2^{n-1} - 1)$ (Lean theorem `flag_set_cardinality`). Define the S_n-symmetric flag Pauli channel

$$\mathcal{F}_n(\rho) := 2^{-n}\rho + \frac{1 - 2^{-n}}{(2^n - 1)(2^{n-1} - 1)} \sum_{P \in \mathcal{P}_{\mathrm{flag}}(n)} P\rho P. \tag{4}$$

Table 1: Exact zero-float certification of $I_4 - \mathcal{E}(A_j) \succeq 0$ across all 8 Clifford facet orbits (22,320 facets) of STAB_2 and all 192 Orbit-2 bound-entangled channels of PPT_2 (Certificates `C05`–`C06`). Orbits F_1 and F_5 are exact-boundary tight ($\lambda_{\max} = 1$, realified rank 6); all other 6 orbits are strictly positive definite (rank 8).

Facet Orbit Representative	Orbit Size	Tight	Nnz	$\max_{\mathcal{E}\in\mathcal{O}_2} \lambda_{\max}(\mathcal{E}(A_j))$	Min Rk	Tight / 192
F_1 (Eq. 22, 7×1)	240	33	7	1.000000 (exact)	6	192
F_2 (Row 1, 5×1)	192	30	5	$\sqrt{5}/3 \approx 0.745356$	8	0
F_3 (Row 2, $2 \times 1, 10 \times \frac{1}{2}$)	3,840	21	12	0.863511	8	0
F_4 (Row 3, $2 \times 1, 8 \times \frac{1}{2}$)	5,760	20	10	0.859312	8	0
F_5 (Row 4, $3 \times 1, 6 \times \frac{1}{2}$)	1,920	21	9	1.000000 (exact)	6	64
F_6 (Row 5, $10 \times \frac{1}{2}$)	2,304	15	10	0.729634	8	0
F_7 (Row 6, $1, 6 \times \frac{2}{3}, 6 \times \frac{1}{3}$)	5,760	17	13	0.868198	8	0
F_8 (Row 7, $5 \times \frac{3}{4}, 5 \times \frac{1}{2}, 5 \times \frac{1}{4}$)	2,304	15	15	0.812758	8	0
Total / Global	**22,320**	—	—	$\leq$ **1.000000**	$\geq$ **6**	**256 / 1,536**

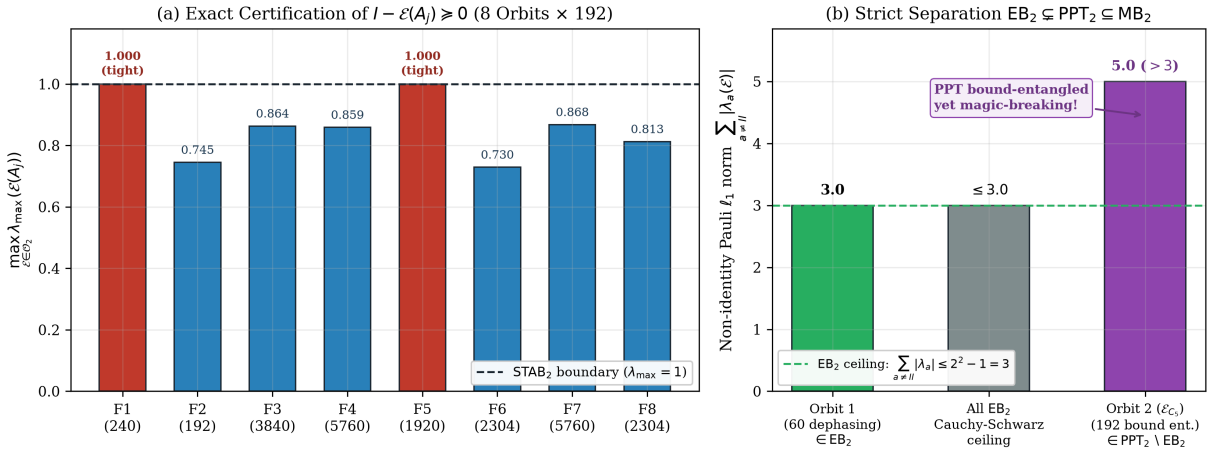

Figure 1: (a) Exact maximum eigenvalue $\max_{\mathcal{E}\in\mathcal{O}_2} \lambda_{\max}(\mathcal{E}(A_j))$ across all 8 Clifford facet orbits of STAB_2 (1,536 exact 8×8 LDL^T certifications). (b) Strict ℓ_1-norm separation $\mathrm{EB}_2 \subsetneq \mathrm{PPT}_2 \subseteq \mathrm{MB}_2$: every Orbit-2 extreme ray $\mathcal{E}_{C_5}$ is PPT bound-entangled ($\sum_{a\neq II} |\lambda_a| = 5 > 3$) yet magic-breaking.

For *every* $n \geq 2$ (both odd and even), the symplectic Fourier transform of $\mathbf{1}_{\{\mathrm{wt}_Y \text{ even}\}} - \mathbf{1}_{\{I,Z\}^n} - \mathbf{1}_{\{I,X\}^n} + \mathbf{1}_{\{I\}}$ yields $\lambda_I(\mathcal{F}_n) = 1$, $\lambda_P(\mathcal{F}_n) = \frac{1}{2^{n-1}-1}$ for $P \in \mathcal{P}_{\mathrm{flag}}(n)$, and $\lambda_P(\mathcal{F}_n) = 0$ on all other Paulis (including all odd-wt_Y Paulis). Hence $T_n \circ \mathcal{F}_n = \mathcal{F}_n \in \mathrm{PPT}_n$ for every $n \geq 2$ (Certificate `C11` at $n = 4$).

Let $|\phi_\pm^{(n)}\rangle := (|0^n\rangle \pm |+^n\rangle)/\sqrt{2(1 \pm 2^{-n/2})}$. We now prove both a universal magic witness valid for every odd-k stabilizer pair and an exact parity dichotomy showing that $\mathcal{F}_n(|\phi_\pm^{(n)}\rangle\langle\phi_\pm^{(n)}|)$ literally collapses into STAB_n for every even n.

Theorem 3 (Universal Odd-k Witness and Exact Even-n Arf-Invariant Collapse)**.**

(a) ***Universal Odd-k Magic Witness:*** *Let $|u\rangle, |v\rangle$ be any two pure n-qubit stabilizer states with $\langle u|v\rangle = 2^{-k/2}$ for an* odd *integer $k \in \{1, 3, \ldots, n\}$. Then the rank-2 Hermitian operator*

$$W_{u,v} := |u\rangle\langle u| + |v\rangle\langle v| - \frac{3}{2\sqrt{2}} (|u\rangle\langle v| + |v\rangle\langle u|) \tag{5}$$

satisfies $\langle s|W_{u,v}|s\rangle \geq 0$ for every pure n-qubit stabilizer state $|s\rangle \in \mathrm{STAB}_n$, with tight equality 0.

(b) ***Exact Odd-n Violation and Depolarizing Threshold:*** *For every odd $n \geq 3$,*

$$\mathrm{Tr}(W_{|0^n\rangle,|+^n\rangle}\,\mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)) = \frac{4-3\sqrt{2}}{2^{n+1}} < 0, \tag{6}$$

and the depolarized mixture $(1-p)\mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|) + p\frac{I}{2^n}$ remains certified bound magic for all $0 \leq p < p_{\mathrm{depol}}(n)$, where

$$p_{\mathrm{depol}}(n) = \frac{3\sqrt{2}-4}{3\sqrt{2}-3\cdot 2^{-(n-1)/2}} > 1 - \frac{2\sqrt{2}}{3} \approx 0.057191. \tag{7}$$

(c) ***Exact Even-n Collapse into*** STAB_{2m}***:*** *For every even $n = 2m \geq 2$, let $\mathrm{Alt}_{\mathrm{inv}}(2m, \mathbb{F}_2)$ be the set of invertible symmetric zero-diagonal $2m \times 2m$ matrices over $\mathbb{F}_2$. For each $A \in \mathrm{Alt}_{\mathrm{inv}}(2m, \mathbb{F}_2)$ and $c \in \mathbb{F}_2^{2m}$, define the real quadratic stabilizer state*

$$|Q_{A,c}\rangle := 2^{-m} \sum_{x\in\mathbb{F}_2^{2m}} (-1)^{x^T A_{\mathrm{upper}} x + c\cdot x}|x\rangle$$

with Arf invariant $\mathrm{Arf}(A,c) \in \{0,1\}$ satisfying $\langle +^{2m}|Q_{A,c}\rangle = (-1)^{\mathrm{Arf}(A,c)}2^{-m}$. Then there are exactly $N_\pm(2m) = 2^{m-1}(2^m \pm 1)|\mathrm{Alt}_{\mathrm{inv}}(2m,\mathbb{F}_2)|$ states with $\mathrm{Arf} = 0$ (resp. 1), and

$$\mathcal{F}_{2m}(|\phi_+^{(2m)}\rangle\langle\phi_+^{(2m)}|) = \frac{1}{N_+(2m)} \sum_{\mathrm{Arf}(A,c)=0} |Q_{A,c}\rangle\langle Q_{A,c}| \in \mathrm{STAB}_{2m}, \tag{8}$$

$$\mathcal{F}_{2m}(|\phi_-^{(2m)}\rangle\langle\phi_-^{(2m)}|) = \frac{1}{N_-(2m)} \sum_{\mathrm{Arf}(A,c)=1} |Q_{A,c}\rangle\langle Q_{A,c}| \in \mathrm{STAB}_{2m}. \tag{9}$$

Proof. **(a)** By Clifford covariance of STAB_n and the Gottesman–Aaronson overlap classification, any pair of stabilizer states with $|\langle u|v\rangle| = 2^{-k/2}$ is Clifford-equivalent to $|0^k\rangle \otimes |0^{n-k}\rangle$ and $|+^k\rangle \otimes |0^{n-k}\rangle$. Projecting $|s\rangle$ onto $|0^{n-k}\rangle$ on the last $n-k$ qubits yields a subnormalized k-qubit stabilizer state, so it suffices to check $|u\rangle = |0^k\rangle$, $|v\rangle = |+^k\rangle$ for odd $k \geq 1$. Let $a = |\langle 0^k|s\rangle|$ and $b = |\langle +^k|s\rangle|$. If $a = 0$ or $b = 0$, then $\langle s|W_{u,v}|s\rangle = a^2 + b^2 \geq 0$. Otherwise, $a = 2^{-r/2}$ and $b = 2^{-s/2}$ for integers $r, s \in \{0, \ldots, k\}$.

- *Case 1 ($r \neq s$):* Let $d = |r-s| \geq 1$. Then

$$\frac{a^2+b^2}{2ab} = \frac{2^{d/2}+2^{-d/2}}{2} = \frac{2^d+1}{2\cdot 2^{d/2}} \geq \frac{3}{2\sqrt{2}},$$

since $2(2^d+1)^2 - 9\cdot 2^d = (2^d-2)(2^{d+1}-1) \geq 0$ for all $d \geq 1$ (Lean theorem `witness_case_distinct_ranks`), with equality at $d = 1$ (attained by $|0^{(k+1)/2}+^{(k-1)/2}\rangle$). Hence $\langle s|W_{u,v}|s\rangle \geq a^2 + b^2 - \frac{3}{\sqrt{2}}ab \geq 0$.

- *Case 2 ($r = s$):* Fixing the global phase of $|s\rangle$ so $\langle 0^k|s\rangle = +2^{-r/2}$, the stabilizer normal form [4] gives $|s\rangle = 2^{-r/2}\sum_{x\in V} i^{\ell(x)}(-1)^{q(x)}|x\rangle$ for an r-dimensional linear subspace $V \subseteq \mathbb{F}_2^k$. Hence $\langle +^k|s\rangle = 2^{-(k+r)/2}S$ where $S := \sum_{x\in V} i^{\ell(x)}(-1)^{q(x)} \in \mathbb{Z}[i]$. Since $a = b = 2^{-r/2}$, we have $|S| = 2^{k/2}$, so $|S|^2 = 2^k$ with k odd. In the unique factorization domain $\mathbb{Z}[i]$ where $2 = -i(1+i)^2$, every element $S \in \mathbb{Z}[i]$ of norm 2^k (k odd) has the form $S = (1+i)i^m 2^{(k-1)/2}$ ($m \in \mathbb{Z}$), whose argument is an odd multiple of $\pi/4$. Thus $\mathrm{Re}(S) \leq 2^{(k-1)/2} = |S|/\sqrt{2}$, giving $2\,\mathrm{Re}(\langle 0^k|s\rangle\langle s|+^k\rangle) \leq \sqrt{2}\,a^2$, and therefore

$$\langle s|W_{u,v}|s\rangle \geq 2a^2 - \frac{3}{2\sqrt{2}}\cdot\sqrt{2}\,a^2 = \frac{1}{2}a^2 > 0.$$

Exhaustive exact $\mathbb{Q}(i)$ verification across all 1,080 pure 3-qubit stabilizer states independently confirms $\min_{s\in\mathrm{STAB}_3}\langle s|W_3|s\rangle = 0$ with 224 tight zeros (Certificate `C07`).

(b) For odd $n \geq 3$, write $\rho_B^{(n)} = \mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)$ and $t = 2^{-n/2}$. For every $P \in \mathcal{P}_{\mathrm{flag}}(n)$, we have $\langle 0^n|P|0^n\rangle = \langle +^n|P|+^n\rangle = 0$ and $\langle 0^n|P|+^n\rangle = (-1)^{\mathrm{wt}_Y(P)/2}t$, so that $\mathrm{Tr}(P|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|) = (-1)^{\mathrm{wt}_Y(P)/2}t/(1+t)$. Using the Pauli expansion $\mathrm{Tr}(A\,\mathcal{F}_n(B)) = 2^{-n}\sum_P \lambda_P \mathrm{Tr}(PA)\mathrm{Tr}(PB)$, we get $\langle 0^n|\rho_B^{(n)}|0^n\rangle = \langle +^n|\rho_B^{(n)}|+^n\rangle = 2^{-n}$ and

$$\langle 0^n|\rho_B^{(n)}|+^n\rangle = 2^{-n}\left(t + \frac{1}{2^{n-1}-1}\cdot(2^n-1)(2^{n-1}-1)\frac{t^2}{1+t}\right) = 2^{-n}\left(t + \frac{1-t^2}{1+t}\right) = 2^{-n}.$$

Thus $\mathrm{Tr}(W_n\rho_B^{(n)}) = (2-\frac{3}{\sqrt{2}})2^{-n} = \frac{4-3\sqrt{2}}{2^{n+1}} < 0$. Meanwhile, $\mathrm{Tr}(W_n\frac{I}{2^n}) = 2^{-n}(2-3\cdot 2^{-(n+1)/2}) > 0$. Solving $(1-p)\mathrm{Tr}(W_n\rho_B^{(n)}) + p\mathrm{Tr}(W_n\frac{I}{2^n}) = 0$ yields (7) (Certificate `C08`).

(c) Let $n = 2m \geq 2$ and $\sigma_+ := \frac{1}{N_+(2m)}\sum_{\mathrm{Arf}(A,c)=0}|Q_{A,c}\rangle\langle Q_{A,c}|$. For each $A \in \mathrm{Alt}_{\mathrm{inv}}(2m,\mathbb{F}_2)$, as c varies over $\mathbb{F}_2^{2m}$, the quadratic form $Q_{A,c}$ has Arf $= 0$ exactly $2^{m-1}(2^m+1)$ times and Arf $= 1$ exactly $2^{m-1}(2^m-1)$ times [4] (Lean theorem `arf_partition_identities`). Every $|Q_{A,c}\rangle$ is a real graph state with invertible adjacency matrix $A \in \mathrm{Alt}_{\mathrm{inv}}(2m,\mathbb{F}_2)$, so its stabilizer group $S(A,c)$ contains no non-identity Z-type Pauli, no non-identity X-type Pauli (since $\ker_{\mathbb{F}_2}(A) = \{0\}$), and no odd-wt$_Y$ Pauli (since $|Q_{A,c}\rangle$ is real). Hence every non-identity Pauli expectation of σ_+ vanishes outside $\mathcal{P}_{\mathrm{flag}}(2m)$. Moreover, the group $\mathrm{GL}(2m,\mathbb{F}_2)$ acts unitarily on computational basis states by $U_M|x\rangle = |Mx\rangle$, preserving $|0^{2m}\rangle$, $|+^{2m}\rangle$, and $\mathrm{Alt}_{\mathrm{inv}}(2m,\mathbb{F}_2)$, and transforming quadratic forms by $q_{A,c}(Mx) = x^T(M^TAM)_{\mathrm{upper}}x + (M^Tc + \mathrm{diag}(M^TA_{\mathrm{upper}}M))\cdot x$. Since $U_M|+^{2m}\rangle = |+^{2m}\rangle$, we have $\mathrm{Arf}(M^TAM,\, M^Tc + \mathrm{diag}(M^TA_{\mathrm{upper}}M)) = \mathrm{Arf}(A,c)$, so $U_M\sigma_+U_M^\dagger = \sigma_+$. On every even-wt$_Y$ unsigned Pauli $P = P_{x,z} \in \mathcal{P}_{\mathrm{flag}}(2m)$, the single-qubit relation $ZX = iY$ gives $Z^zX^x = (-1)^{\mathrm{wt}_Y(P)/2}P$; since $U_M(Z^zX^x)U_M^\dagger = Z^{M^{-T}z}X^{Mx} = (-1)^{\mathrm{wt}_Y(P')/2}P'$ and $\mathrm{GL}(2m,\mathbb{F}_2)$ acts *transitively* on nonzero pairs $(x,z) \in \mathbb{F}_2^{2m}\times\mathbb{F}_2^{2m}$ with $x\cdot z = 0$, the quantity $(-1)^{\mathrm{wt}_Y(P)/2}\mathrm{Tr}(P\sigma_+)$ takes a single constant value $\kappa_+ \in \mathbb{Q}$ across all $P \in \mathcal{P}_{\mathrm{flag}}(2m)$, matching the phase structure of $\mathcal{F}_{2m}(|\phi_+^{(2m)}\rangle\langle\phi_+^{(2m)}|)$. Evaluating the single scalar

$$\langle\phi_+^{(2m)}|\sigma_+|\phi_+^{(2m)}\rangle = \frac{(2^{-m}+2^{-m})^2}{2(1+2^{-m})} = \frac{2^{1-2m}}{1+2^{-m}} = \langle\phi_+^{(2m)}|\mathcal{F}_{2m}(|\phi_+^{(2m)}\rangle\langle\phi_+^{(2m)}|)|\phi_+^{(2m)}\rangle$$

fixes κ_+ uniquely, proving (8). The identical argument for Arf $= 1$ gives (9). Direct exact rational 4×4 and 16×16 matrix checks at $n = 2$ (3 Arf-0, 1 Arf-1) and $n = 4$ ($28\times 10 = 280$ Arf-0, $28\times 6 = 168$ Arf-1) confirm exact entrywise equality (Certificates `C09`–`C10`, Figure 2). □

5 Irreducible Full-Rank Analytic Bound-Magic Family for All Even $n \geq 4$

Theorem 3(c) reveals why Kitzinger–Eisert–Kunjwal's flag-channel construction failed at even n: $|0^{2m}\rangle$ and $|+^{2m}\rangle$ have rational overlap 2^{-m}, admitting nondegenerate real quadratic stabilizer states whose average matches $\mathcal{F}_{2m}(|\phi_\pm^{(2m)}\rangle\langle\phi_\pm^{(2m)}|)$ identically. To construct a genuinely n-qubit, S_n-symmetric, full-rank, non-factorizable analytic family for *every even* $n \geq 4$, we pair $|u_n\rangle := |0^n\rangle$ with the S_n-symmetric X-basis GHZ stabilizer state

$$|v_n\rangle := |\mathrm{GHZ}_X^n\rangle = \frac{|+^n\rangle + |-^n\rangle}{\sqrt{2}} = 2^{-(n-1)/2}\sum_{\substack{x\in\mathbb{F}_2^n\\ \mathrm{wt}(x)\text{ even}}}|x\rangle, \tag{10}$$

which has *irrational* overlap $t_n := \langle u_n|v_n\rangle = 2^{-(n-1)/2} \notin \mathbb{Q}$ for every even $n \geq 4$. Since $X^x|v_n\rangle = +|v_n\rangle$ for all even-weight x and $Z^{1^n}|v_n\rangle = +|v_n\rangle$, its stabilizer group is

$$S_v = \{X^xZ^z : \mathrm{wt}(x)\text{ even},\ z \in \{0^n,1^n\}\} = \{(-1)^{\mathrm{wt}_Y(P_{x,z})/2}P_{x,z} : \mathrm{wt}(x)\text{ even},\ z \in \{0^n,1^n\}\},$$

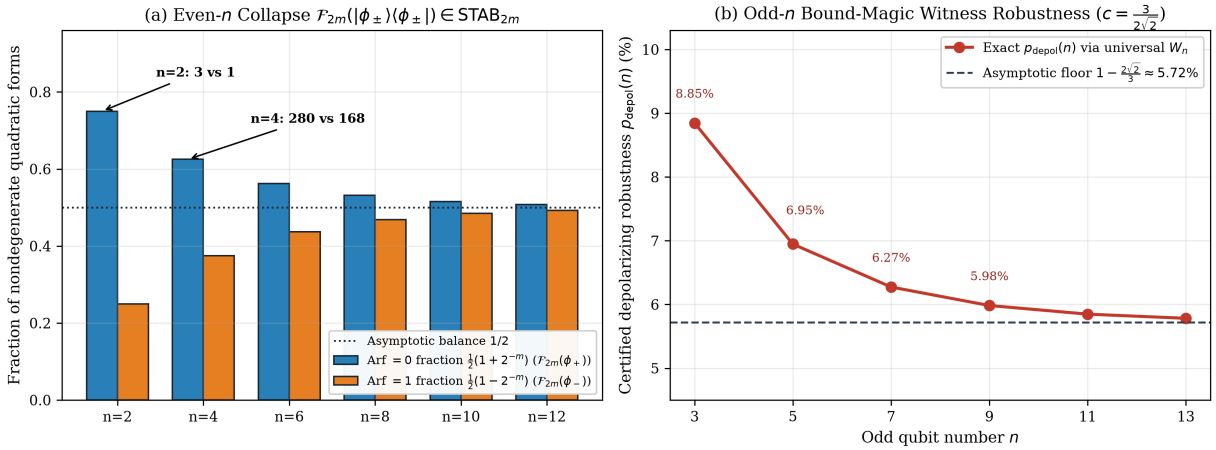

Figure 2: (a) Exact Arf-invariant fractions $\frac{1}{2}(1 \pm 2^{-m})$ governing the stabilizer collapse $\mathcal{F}_{2m}(|\phi_\pm^{(2m)}\rangle\langle\phi_\pm^{(2m)}|) \in \mathrm{STAB}_{2m}$ for all even $n = 2m$. (b) Exact certified depolarizing noise robustness $p_{\mathrm{depol}}(n)$ of the odd-n flag bound-magic states under the universal witness W_n ($c = \frac{3}{2\sqrt{2}}$), approaching $1 - \frac{2\sqrt{2}}{3} \approx 5.72\%$.

whose unsigned Pauli support $\overline{S}_v = \{P_{x,z} : \mathrm{wt}(x) \text{ even},\ z \in \{0^n, 1^n\}\} \subset \mathcal{P}_n$ intersects $S_u = \{I, Z\}^n$ in $\{I^{\otimes n}, Z^{\otimes n}\}$. Let

$$\mathcal{P}_{\mathrm{comm}}(n) := \left\{P \in \mathcal{P}_n : \mathrm{wt}_Y(P) \text{ even},\ \mathrm{wt}_X(P) \text{ even},\ P \notin S_u,\ P \notin \overline{S}_v\right\},$$

which has exact cardinality $|\mathcal{P}_{\mathrm{comm}}(n)| = 2(2^{n-1}-1)(2^{n-2}-1)$ (Lean theorem `even_commutant_normalization`). Define the S_n-symmetric commutant PPT Pauli channel

$$\mathcal{F}_Z^{(n)}(\rho) := 2^{-n}(\rho + Z^{\otimes n}\rho Z^{\otimes n}) + \frac{1}{2^n(2^{n-2}-1)} \sum_{P \in \mathcal{P}_{\mathrm{comm}}(n)} P\rho P \in \mathrm{PPT}_n, \tag{11}$$

whose Pauli eigenvalues are 1 on $\{I^{\otimes n}, Z^{\otimes n}\}$, $\frac{1}{2^{n-2}-1}$ on $\mathcal{P}_{\mathrm{comm}}(n)$, and 0 elsewhere. Alone, $\mathcal{F}_Z^{(n)}(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)$ has $\mathrm{Tr}(Z^{\otimes n}\rho) = 1$ and is Clifford-equivalent (via a CNOT cascade) to $\rho_B^{(n-1)} \otimes |0\rangle\langle 0|$. To break factorizability completely while preserving S_n-symmetry and PPT membership, we form the convex mixture with the full flag channel $\mathcal{F}_n$:

$$\mathcal{G}_n^{(\delta)} := (1-\delta)\,\mathcal{F}_Z^{(n)} + \delta\,\mathcal{F}_n \in \mathrm{PPT}_n, \qquad \rho_B^{(n)}(\delta) := \mathcal{G}_n^{(\delta)}(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|), \tag{12}$$

where $|\phi_+^{(n)}\rangle := (|u_n\rangle + |v_n\rangle)/\sqrt{2(1+t_n)}$.

Theorem 4 (Universal Irreducible Analytic Bound Magic for Every Even $n \geq 4$)**.** *For every even integer $n \geq 4$, let $W_n := W_{|u_n\rangle,|v_n\rangle}$ be the universal witness (5) with $k = n-1$ odd. Then:*

(a) ***Exact Closed-Form Witness Expectations:*** *With $t_n = 2^{-(n-1)/2}$,*

$$2^n \,\mathrm{Tr}(W_n\,\mathcal{F}_Z^{(n)}(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)) = 4 - 3\sqrt{2} < 0, \tag{13}$$

$$2^n \,\mathrm{Tr}(W_n\,\mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)) = \frac{(5-3\sqrt{2}) + (6-\frac{9}{2}\sqrt{2})t_n}{2(1+t_n)} > 0. \tag{14}$$

(b) ***Exact Monotone Critical Mixing Threshold:*** *For every even $n \geq 4$, $\rho_B^{(n)}(\delta) \in B_n \setminus \mathrm{STAB}_n$ for all $\delta \in [0, \delta_{\mathrm{crit}}(n))$, where*

$$\delta_{\mathrm{crit}}(n) = \frac{2(3\sqrt{2}-4)(1+t_n)}{(3\sqrt{2}-3) + (3\sqrt{2}/2-2)t_n} \tag{15}$$

decreases strictly monotonically from $\delta_{\mathrm{crit}}(4) = \frac{20-8\sqrt{2}}{17} \approx 0.510958$ *to the exact universal floor*

$$\delta_{\mathrm{crit}}(\infty) = \frac{2(2-\sqrt{2})}{3} \approx 0.39052429 > \frac{3}{8} > \frac{1}{3}. \tag{16}$$

In particular, the canonical choice $\delta = \frac{1}{3}$ *certifies* $\rho_B^{(n)}(\frac{1}{3}) \in B_n \setminus \mathrm{STAB}_n$ *for every even* $n \geq 4$ *simultaneously.*

(c) ***Irreducibility and Non-Factorizability:*** *For every even* $n \geq 4$ *and every* $\delta \in (0, \delta_{\mathrm{crit}}(n))$, $\rho_B^{(n)}(\delta)$ *is* S_n*-permutation-invariant and satisfies*

$$\max_{P \in \mathcal{P}_n \setminus \{I^{\otimes n}\}} \left|\mathrm{Tr}(P\,\rho_B^{(n)}(\delta))\right| = 1 - \delta < 1, \tag{17}$$

attained uniquely at $P = Z^{\otimes n}$*. At* $\delta = \frac{1}{3}$*, this maximum equals* $\frac{2}{3} < 1$*, proving unconditionally that no* n*-qubit Clifford unitary can map* $\rho_B^{(n)}(\frac{1}{3})$ *to* $\sigma_{n-1} \otimes |0\rangle\langle 0|$*. At* $n = 4$*, exact* 16×16 $\mathbb{Q}(\sqrt{2})$ LDL^T *elimination further certifies that* $\rho_B^{(4)}(\frac{1}{3})$ *has full rank* 16 *(strictly positive definite).*

Proof. **(a)** Because $\mathcal{F}_Z^{(n)}$ and $\mathcal{F}_n$ are both in PPT_n (all odd-wt_Y Pauli eigenvalues vanish identically), their convex combination $\mathcal{G}_n^{(\delta)}$ lies in PPT_n, so $\rho_B^{(n)}(\delta) \in B_n$. For $\mathcal{F}_Z^{(n)}$, only $\{I^{\otimes n}, Z^{\otimes n}\}$ (eigenvalue 1) and $\mathcal{P}_{\mathrm{comm}}(n)$ (eigenvalue $(2^{n-2}-1)^{-1}$) contribute. On $\{I, Z^n\}$, $\langle u_n|P|u_n\rangle = \langle v_n|P|v_n\rangle = 1$ and $\langle u_n|P|v_n\rangle = t_n$; on $\mathcal{P}_{\mathrm{comm}}(n)$, $\langle u_n|P|u_n\rangle = \langle v_n|P|v_n\rangle = 0$ and $\langle u_n|P|v_n\rangle = (-1)^{\mathrm{wt}_Y(P)/2} t_n$. Using $t_n^2 = 2^{1-n}$ gives $\langle u_n|\mathcal{F}_Z^{(n)}|u_n\rangle = \langle v_n|\mathcal{F}_Z^{(n)}|v_n\rangle = 2^{1-n}$ and

$$\langle u_n|\mathcal{F}_Z^{(n)}|v_n\rangle = 2^{-n}\left(2t_n + 2(2^{n-1}-1)\frac{t_n^2}{1+t_n}\right) = 2^{-n}\left(2t_n + \frac{2(1-t_n^2)}{1+t_n}\right) = 2^{1-n},$$

yielding (13). For $\mathcal{F}_n$, the Paulis contributing to $\mathrm{Tr}(W_n\,\mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|))$ are $I^{\otimes n}$, $\mathcal{P}_{\mathrm{comm}}(n)$ (eigenvalue $(2^{n-1}-1)^{-1}$), and $\overline{S}_v \cap \mathcal{P}_{\mathrm{flag}}(n) = \{P_{x,1^n} : x \neq 0^n,\ \mathrm{wt}(x) \text{ even}\}$ (size $2^{n-1}-1$, eigenvalue $(2^{n-1}-1)^{-1}$). On every $P_{x,1^n} \in \overline{S}_v \cap \mathcal{P}_{\mathrm{flag}}(n)$, we have $\langle v_n|P_{x,1^n}|v_n\rangle = (-1)^{\mathrm{wt}(x)/2}$ and $\langle u_n|P_{x,1^n}|v_n\rangle = (-1)^{\mathrm{wt}(x)/2} t_n$, so the sign squares to $+1$. Summing all three sectors gives

$$2^n\langle u_n|\mathcal{F}_n|u_n\rangle = 1, \quad 2^n\langle v_n|\mathcal{F}_n|v_n\rangle = \frac{3+4t_n}{2(1+t_n)}, \quad 2^n\langle u_n|\mathcal{F}_n|v_n\rangle = \frac{2+3t_n}{2(1+t_n)},$$

from which $2^n\mathrm{Tr}(W_n\mathcal{F}_n) = 1 + \frac{3+4t_n}{2(1+t_n)} - \frac{3}{\sqrt{2}}\frac{2+3t_n}{2(1+t_n)}$ simplifies directly to (14).

(b) Linearity in δ gives $\delta_{\mathrm{crit}}(n) = -v_Z/(v_F(t_n) - v_Z)$, simplifying to (15). Because $6 - \frac{9}{2}\sqrt{2} < 5 - 3\sqrt{2}$ (equivalently $1 < \frac{3}{2}\sqrt{2} \iff 2 < 9$, Lean theorem `delta_crit_monotone_above_inf`), $v_F(t)$ is strictly decreasing in $t > 0$; thus $\delta_{\mathrm{crit}}(n)$ is strictly decreasing in even $n \geq 4$ and approaches $\delta_{\mathrm{crit}}(\infty) = \frac{2(3\sqrt{2}-4)}{3\sqrt{2}-3} = \frac{2(2-\sqrt{2})}{3}$. Clearing $\mathbb{Z}[\sqrt{2}]$ norms gives $\delta_{\mathrm{crit}}(\infty) > \frac{3}{8} \iff 23^2 = 529 > 512 = 16^2 \cdot 2$ and $\delta_{\mathrm{crit}}(\infty) > \frac{1}{3} \iff 3^2 = 9 > 8 = 2^2 \cdot 2$ (Lean theorems `delta_crit_inf_exceeds_three_eighths` and `delta_one_third_universal_margin`; Certificates `C12`–`C13`).

(c) Both $\mathcal{F}_Z^{(n)}$, $\mathcal{F}_n$, and $|\phi_+^{(n)}\rangle$ are invariant under all $n!$ qubit permutations (24/24 permutations directly verified at $n = 4$ in Certificate `C14`). For $P = Z^{\otimes n}$, $\mathrm{Tr}(Z^{\otimes n}|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|) = 1$ and $\lambda_{Z^{\otimes n}}(\mathcal{G}_n^{(\delta)}) = 1 - \delta$, giving $\mathrm{Tr}(Z^{\otimes n}\rho_B^{(n)}(\delta)) = 1 - \delta$. For every other non-identity Pauli $P \notin \{I^{\otimes n}, Z^{\otimes n}\}$, we have $|\mathrm{Tr}(P\rho_B^{(n)}(\delta))| \leq |\lambda_P(\mathcal{G}_n^{(\delta)})| \leq \frac{1-\delta}{2^{n-2}-1} + \frac{\delta}{2^{n-1}-1} \leq \frac{1-\delta}{3} + \frac{\delta}{7}$ for all even $n \geq 4$. Since $\delta < \delta_{\mathrm{crit}}(n) \leq \delta_{\mathrm{crit}}(4) = \frac{20-8\sqrt{2}}{17} < \frac{14}{17}$, we have $\frac{1-\delta}{3} + \frac{\delta}{7} < 1 - \delta \iff \frac{17}{21}\delta < \frac{2}{3} \iff \delta < \frac{14}{17}$, establishing exact equality and uniqueness of the maximizer $Z^{\otimes n}$ in (17) (and at $\delta = \frac{1}{3}$, the full second-branch inequality holds for all $n \geq 4$ by Lean theorem `non_factorizability_ceiling`). If any Clifford U satisfied $U\rho_B^{(n)}(\delta)U^\dagger = \sigma_{n-1} \otimes |0\rangle\langle 0|$, then $P := U^\dagger(I^{\otimes(n-1)} \otimes Z)U \neq I^{\otimes n}$ would satisfy $\mathrm{Tr}(P\rho_B^{(n)}(\delta)) = 1$, contradicting (17). Exact 16×16 $\mathbb{Q}(\sqrt{2})$ LDL^T elimination certifies 16 strictly positive pivots at $n = 4, \delta = \frac{1}{3}$ (Certificate `C12`). □

Table 2: Exact $\mathbb{Q}(\sqrt{2})$ parameters for odd-n flag states (Theorem 3(b), `C08`) and irreducible even-n bound-magic states $\mathcal{G}_n^{(\delta)}$ (Theorem 4, `C12`–`C13`).

Odd $n \geq 3$: Flag-channel states $\mathcal{F}_n(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)$

n	$\lvert\mathcal{P}_{\text{flag}}(n)\rvert$	λ_{flag}	$\text{Tr}(W_n\rho_B^{(n)})$	Exact $p_{\text{depol}}(n) \in \mathbb{Q}(\sqrt{2})$	$p_{\text{depol}}(n)$ (%)
3	21	1/3	−0.01516504	$(16-10\sqrt{2})/21$	8.847%
5	465	1/15	−0.00379126	$(80-52\sqrt{2})/93$	6.947%
7	8,001	1/63	−0.00094782	$(352-232\sqrt{2})/381$	6.274%
9	130,305	1/255	−0.00023695	$(1472-976\sqrt{2})/1533$	5.984%
11	2,094,081	1/1023	−0.00005924	$(6016-4000\sqrt{2})/6141$	5.848%
∞	—	0	0^-	$1-\frac{2}{3}\sqrt{2}$	5.719%

Even $n \geq 4$: Irreducible full-rank states $\mathcal{G}_n^{(\delta)}(|\phi_+^{(n)}\rangle\langle\phi_+^{(n)}|)$

n	$\lvert\mathcal{P}_{\text{comm}}(n)\rvert$	$\lvert\mathcal{P}_{\text{flag}}(n)\rvert$	Exact $\delta_{\text{crit}}(n) \in \mathbb{Q}(\sqrt{2})$	$\delta_{\text{crit}}(n)$	$2^n\text{Tr}(W_n\rho_B^{(n)}(\frac{1}{3}))$
4	42	105	$(20-8\sqrt{2})/17$	0.510958	−0.084349
6	930	1,953	$(668-304\sqrt{2})/527$	0.451763	−0.063608
8	16,002	32,385	$(2876-1376\sqrt{2})/2207$	0.421406	−0.050711
10	260,610	522,753	$(1700-832\sqrt{2})/1289$	0.406031	−0.043444
12	4,188,162	8,382,465	$(48380-23936\sqrt{2})/36479$	0.398294	−0.039574
∞	—	—	$(4-2\sqrt{2})/3$	0.390524	−0.035534

6 Formal Verification, Scope, and Open Questions

All algebraic, combinatorial, and $\mathbb{Z}[\sqrt{2}]$ inequality cores of Theorems 2–4 are formalized in zero-dependency constructive Lean 4.35.0 (`lean/BoundMagicSharp.lean`: 18/18 theorems, 0 `sorry`, 0 `Classical.choice`, 0 custom axioms, 0 warnings). All finite exact computations are executed by an AST-audited zero-float pure-stdlib Python verifier (`verification/verify_paper105.py`, 56/56 adversarial pytest tests) issuing 16/16 SHA-256 certificates (`MANIFEST.json` SHA-256: `602d569136f8cee6...3ef86a65`).

Scope and open questions. (1) Theorem 2 proves Conjecture 4 of [1] unconditionally for $n = 2$ and refutes its converse ($\text{EB}_2 \subsetneq \text{PPT}_2 \subseteq \text{MB}_2$). Our numerical alternating-eigenvalue sweeps over 12 extremal STAB_3 facets and 1,080-vertex LPs found no counterexample to Conjecture 4 at $n = 3$ (every extreme EB Pauli channel $\mathcal{E}_{|\psi\rangle,|\phi\rangle}$ produces the triple Schur product $v_a = c_a(\psi)c_a(\phi^*)c_a(\chi)$, attaining maximum ≤ 1.000000 on every tested facet); proving Conjecture 4 for all $n \geq 3$ remains a compelling open problem. (2) While $\rho_B^{(4)}(\frac{1}{3})$ is certified full-rank, S_4-symmetric, and non-Clifford-factorizable, characterizing whether its 3-qubit reduced density matrices lie in STAB_3 (genuinely n-partite bound magic) is an interesting direction for future work.

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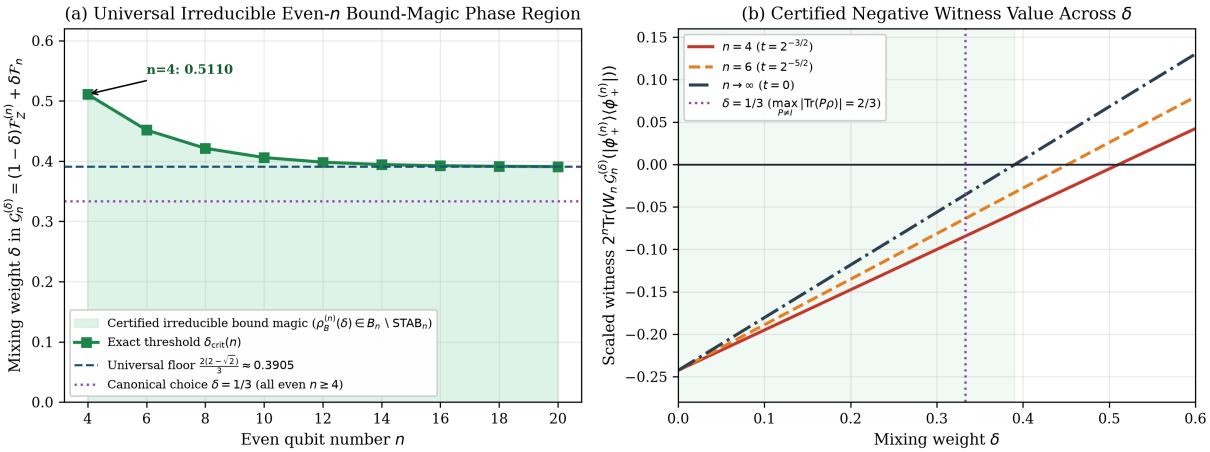

Figure 3: (a) Exact irreducible even-n bound-magic phase region $0 < \delta < \delta_{\mathrm{crit}}(n)$ with universal floor $\frac{2(2-\sqrt{2})}{3} \approx 0.3905$ and canonical $\delta = \frac{1}{3}$. (b) Certified negative scaled witness $2^n \mathrm{Tr}(W_n \rho_B^{(n)}(\delta))$ across $\delta \in [0, 0.6]$ for $n = 4, 6$, and $n \to \infty$.

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