Support and quantum-reference activation of classical channels

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Abstract

For an arbitrary finite classical channel K, we characterize when quantum side information prevents strict relative-entropy contraction. The complete equal-reference Umegaki coefficient equals one if and only if the Doeblin common-component coefficient $d(K) = \sum_y \min_i K(y \mid i)$ vanishes. Positive $d(K)$ gives the known upper $1 - d(K)$. For the converse, one channel-independent family of faithful separable classical–quantum hypotheses, with reference dimension equal to the input alphabet size, admits channel-dependent binary readouts whose input, output and measured relative entropies tend to $\log 2$. Explicit scalar bounds make this convergence uniform over all zero-Doeblin channels with a fixed input alphabet, even as supports and output cardinalities vary. For symmetric channels we additionally determine exact SLD, minimal chi-square, triangular, BKM and Umegaki coefficients: $(n-1)\lambda^2/[1+(n-2)\lambda]$. Every two-symbol restriction retains its ordinary classical coefficient, exposing the failure of classical binary-support reduction. Conditioning, rare events and parameter-dependent effects remain essential costs. Certified asymmetric examples, failed numerical selectors and regularizer counterexamples are retained. No capacity advantage, robust implementation, minimal reference dimension or worldwide priority is claimed.

1 Problem and relation to prior work

A classical noisy channel acts on probability distributions, but its input may be correlated with a quantum memory that remains available after transmission. The relevant comparison is then between two joint states with the *same* memory marginal. This constraint rules out the trivial strategy of putting all distinguishing information in an untouched memory. It still permits noncommuting conditional memory states, even when the input and memory are separable.

For ordinary classical strong data processing, optimizing over a large input alphabet can be reduced to a pair of input symbols, by Ordentlich and Polyanskiy [3]. We first quantify the failure of that reduction with a quantum reference for a full symmetric family. We then characterize strict complete Umegaki contraction for every finite classical channel: Theorem 6.1 is a necessary and sufficient common-support criterion. Its converse uses a channel-independent input family and explicit binary measurements. The input alphabet and reference dimension are fixed; the common reference marginal and measurement may vary with the preparation parameter.

The definitions of complete and conditional contraction coefficients are prior; we use the equal-reference complete convention described by Hirche, George, Nuradha and Wilde [5]. Their characterization using a binary hypothesis label is distinct from restricting the *input alphabet* to two symbols. We do not claim a new correspondence between divergence and reference-assisted information measures. Hiai and Ruskai [1] provide the operator-convex divergence and metric framework. Classical equivalence of global contraction coefficients is treated by Makur and Zheng [2]. The classical symmetric-channel formula below is a benchmark, not a novelty claim; Lee and

Makur [4] is an additional relevant classical reference whose full proof has not been inspected in our directed audit.

Complete entropy decay toward a conditional expectation [6, 7] uses a different restriction on the second state. Those estimates do not automatically bound two arbitrary correlated hypotheses with equal reference. Conversely, our pair constructions do not contradict decay toward a fixed-point state. Recent asymptotic contraction bounds for primitive channels with a fixed faithful stationary state [9] concern that different quantifier regime. The source-fixed matrix-Φ framework of Wang and Gohari [8] is also distinct; we do not claim to resolve its open questions. Generic failure of strict one-step contraction, even for primitive classical channels, already appears in Cohen et al. [10]. The issue studied here is the exact complete coefficient and the quantum-reference enhancement of a classical channel whose ordinary coefficient is strictly below one. Our literature search is directed and does not certify worldwide priority.

The classical Doeblin decomposition and complete contraction definitions are prior [11, 5]. Classical scrambling uses pairwise column overlap [10]; the complete criterion below instead requires a positive row common to all columns. Nonlinear classical multi-way Doeblin curves [12] address another contraction question. Selected primary passages, rather than entire source corpora, were inspected. The general converse here extends the same simplex mechanism used for the symmetric examples; it is not a new data-processing principle or an independently invented construction.

2 Definitions and a harmonic upper bound

Let $A = \mathbb{C}^n$ be equipped with its stated classical basis, and define

$$K_\lambda(y \mid x) = \lambda\delta_{xy} + \frac{1-\lambda}{n}, \qquad -\frac{1}{n-1} \le \lambda \le 1. \tag{1}$$

The quantum channel $\mathcal{N}_\lambda$ first dephases A in this basis and then applies K_λ to its diagonal. It is not the quantum depolarizing channel on arbitrary coherences. We suppress id_R from notation.

For a divergence Q, write

$$\eta_Q^c(\mathcal{N}) = \sup_{\substack{\dim R<\infty,\ \rho,\sigma>0 \\ \operatorname{Tr}\rho=\operatorname{Tr}\sigma=1,\ \rho_R=\sigma_R \\ 0<Q(\rho\|\sigma)<\infty}} \frac{Q(\mathcal{N}\rho\|\mathcal{N}\sigma)}{Q(\rho\|\sigma)}. \tag{2}$$

The supremum is over faithful normalized states; no faithful maximizer is asserted. The ordinary coefficient omits R. All logarithms are natural.

For operator-convex $f : (0,\infty) \to \mathbb{R}$ with $f(1) = 0$, the Petz divergence is

$$Q_f(\rho\|\sigma) = \langle \sigma^{1/2}, f(L_\rho R_\sigma^{-1})\sigma^{1/2}\rangle_{\mathrm{HS}}. \tag{3}$$

Here $L_a X = aX$, $R_b X = Xb$ and $\langle X, Y\rangle_{\mathrm{HS}} = \operatorname{Tr} X^*Y$. The relevant generators include $f(t) = t\log t$ and $f_\triangle(t) = (t-1)^2/(t+1)$. We distinguish $Q_{f_\triangle}$ from the finite minimal chi-square divergence

$$\chi^2_{\min}(\rho,\sigma) = \langle \rho-\sigma, J_\sigma^{-1}(\rho-\sigma)\rangle_{\mathrm{HS}}, \qquad J_\sigma = (L_\sigma + R_\sigma)/2.$$

The SLD metric uses the same quadratic form on a Hermitian tangent D. The BKM metric uses the derivative of the matrix logarithm. Complete metric coefficients optimize over faithful base states and nonzero tangents with $\operatorname{Tr}_A D = 0$; normalization factors common to input and output cancel.

Lemma 2.1 (Cyclic harmonic cancellation)**.** *Let T_j, $0 \le j < n$, be the cyclic permutations of a classical register, amplified by an identity on a finite reference. Suppose $\sum_j T_j(\rho-\sigma) = 0$ and $w_j > 0$, $\sum_j w_j = 1$. The channel $\sum_j w_j T_j$ contracts every normalized operator-convex Petz divergence by at most*

$$h(w) = 1 - \frac{n^2}{\sum_j w_j^{-1}}.$$

The same bound holds for the SLD and BKM metrics on tangents annihilated by the uniform twirl.

Proof. For the resolvent generator $g_u(t) = (t-1)^2/(t+u)$, $u \ge 0$, put $Z = \rho - \sigma$ and $\Omega = L_\rho + uR_\sigma$. Set $Z_j = T_j Z$ and $\Omega_j = T_j \Omega T_j^*$. They satisfy $\sum_j Z_j = 0$ and $\langle Z_j, \Omega_j^{-1} Z_j\rangle = q$, independent of j. For any real c, the output difference is $\sum_j (w_j - c)Z_j$. The inverse quadratic-form inequality gives

$$\left\langle \sum_j (w_j - c)Z_j, \left(\sum_j w_j \Omega_j\right)^{-1} \sum_j (w_j - c)Z_j \right\rangle \le \sum_j \frac{(w_j - c)^2}{w_j}\, q.$$

One proof minimizes $\sum_j \langle Y_j, (w_j\Omega_j)^{-1} Y_j\rangle$ subject to a fixed sum $\sum_j Y_j$, and substitutes $Y_j = (w_j - c)Z_j$. Minimizing the scalar coefficient over c gives $c = n/\sum_j w_j^{-1}$ and $h(w)$.

All operators act on the full complex Hilbert–Schmidt space. In column vectorization R_σ contains σ^T, with no assumption that the states are real or commuting. The positive integral representation [1]

$$f(t) = a(t-1) + b(t-1)^2 + \int_{[0,\infty)} \frac{(t-1)^2}{t+u}\, d\mu(u), \qquad b \ge 0,$$

transfers this upper bound to every such f, with μ positive and $\int (1+u)^{-1}\, d\mu(u) < \infty$. The affine part cancels by equal traces. The quadratic endpoint is proved by the same argument with $\Omega = R_\sigma$, so it is not discarded. Positivity of the measure justifies the integration. Local expansion of the triangular and Umegaki cases gives the metric bounds. □

For CQ states, $\sum_j T_j(\rho - \sigma) = 0$ is exactly the equal-reference constraint. The channel (1) is a mixture of cyclic shifts with

$$w_0 = \frac{1 + (n-1)\lambda}{n}, \qquad w_j = \frac{1-\lambda}{n} \quad (j \ne 0).$$

The lemma therefore gives

$$H_n(\lambda) = \frac{(n-1)\lambda^2}{1 + (n-2)\lambda}. \tag{4}$$

At the two stochastic endpoints a weight may vanish. We use ordinary data processing to obtain the upper bound one, not a singular evaluation of $1/w_j$. At $\lambda = 0$ the output difference vanishes exactly. Dephasing before the cyclic mixture preserves the common reference and cannot increase the input divergence, so the upper applies to general quantum inputs as well.

3 Exact complete coefficients

Theorem 3.1. *For every $n \ge 2$ and physical λ in (1), the complete coefficients for SLD, $\chi^2_{\min}$, $Q_{f_\triangle}$, BKM and Umegaki relative entropy are all $H_n(\lambda)$. A reference of dimension n suffices for their suprema. For every normalized nonaffine operator-convex Petz generator, H_n is an upper bound; equality for every generator is not asserted.*

Proof. Only matching lower bounds remain. In $R = \mathbb{C}^n$ put

$$u = (1, \dots, 1)^T, \quad P = I - uu^*/n, \quad v_i = e_i - u/n.$$

Then $\sum_i v_i = 0$, $\|v_i\|^2 = (n-1)/n$ and $\sum_i v_i v_i^* = P$. Define the CQ blocks

$$\sigma_i = \frac{v_i v_i^*}{n-1}, \qquad D_i = uv_i^* + v_i u^*, \qquad \sigma = \operatorname{diag}_i \sigma_i, \quad D = \operatorname{diag}_i D_i.$$

They give $\operatorname{Tr}\sigma = 1$, $\operatorname{Tr}_A \sigma = P/(n-1)$ and $\operatorname{Tr}_A D = 0$. Use the common faithful regularization

$$\sigma_\xi = (1-\xi)\sigma + \xi I_{n^2}/n^2, \qquad 0 < \xi < 1.$$

Its actual reference marginal is $(1-\xi)P/(n-1) + \xi I_n/n$. In the plane spanned by v_i and u, the block eigenvalues and tangent coherence are

$$\alpha = \frac{1-\xi}{n} + \frac{\xi}{n^2}, \quad \beta = \frac{\xi}{n^2}, \quad |(D_i)_{12}| = \sqrt{n-1}.$$

The output preserves β, replaces α by

$$\alpha' = \frac{(1-\xi)[1+(n-2)\lambda]}{n(n-1)} + \frac{\xi}{n^2}, \tag{5}$$

and replaces D by λD. Therefore the SLD ratio is

$$R_\xi = \lambda^2 \frac{\alpha+\beta}{\alpha'+\beta} \longrightarrow H_n(\lambda) \quad (\xi \downarrow 0). \tag{6}$$

Indeed $g_{\sigma_\xi}(D) = 4n(n-1)/(\alpha+\beta)$. The finite pair $\rho = \sigma_\xi + tD$, $\sigma = \sigma_\xi$ realizes the same ratio for minimal chi-square whenever $t \neq 0$ is sufficiently small.

For the triangular divergence choose $\rho_\pm = \sigma_\xi \pm tD$, where $t^2(n-1) < \alpha\beta$. These are faithful normalized states with identical reference. To evaluate a subnormalized active block, write $W_\pm = \operatorname{diag}(a,b) \pm cS$, where $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $W_+ S + S W_- = (a+b)S$, so

$$(L_{W_+} + R_{W_-})^{-1}(W_+ - W_-) = \frac{2c}{a+b} S, \qquad Q_{f_\triangle}(W_+ \| W_-) = \frac{8c^2}{a+b}.$$

With $c = t\sqrt{n-1}$ this gives $8t^2(n-1)/(\alpha+\beta)$ per input block. At output the coherence is multiplied by λ and the denominator is $\alpha' + \beta$, so the exact finite ratio is again (6). Positivity also follows from

$$\alpha' - \lambda^2 \alpha = \frac{(1-\xi)(1-\lambda)[1+(n-1)\lambda]}{n(n-1)} + \beta(1-\lambda^2) \geq 0.$$

No active plane is renormalized or postselected.

For BKM put $\ell(a,b) = (\log a - \log b)/(a-b)$, with its continuous extension. The local ratio is

$$\lambda^2 \frac{\ell(\alpha', \beta)}{\ell(\alpha, \beta)}.$$

At fixed n, λ, the two first arguments tend to positive constants and $\beta \downarrow 0$. Their common divergent logarithm cancels, leaving $H_n(\lambda)$. For finite Umegaki, first take $t \to 0$ in

$$\mathsf{D}(\sigma_\xi + tD \| \sigma_\xi) = \tfrac{1}{2} t^2 g^{\mathrm{BKM}}_{\sigma_\xi}(D) + o(t^2)$$

with ξ fixed, and only then take $\xi \downarrow 0$. This produces faithful finite pairs arbitrarily close to the same coefficient. Singular inverses and an interchange of limits are not needed. □

These preparations are CQ and therefore separable across $A : R$. Their conditional reference blocks generally do not commute. If the reference were classical in one common basis for both hypotheses, the divergence would decompose into classical terms and the ordinary coefficient would bound each term. The advantage concerns quantum correlations without requiring entanglement.

4 Why two input symbols no longer suffice

We include the binary upper argument to identify exactly the boundary crossed by Theorem 3.1.

Proposition 4.1. *Let a finite binary-input classical channel have nonnegative columns a_y, b_y. Discard outputs with $a_y = b_y = 0$, which are outside its active output support. For every normalized nonaffine operator-convex Petz divergence, and for SLD and minimal chi-square, its complete equal-reference coefficient is the ordinary classical coefficient*

$$c(K) = \sup_{0<x<1} x(1-x) \sum_y \frac{(a_y - b_y)^2}{b_y + (a_y - b_y)x}. \tag{7}$$

Proof. After input dephasing write $\rho = \mathrm{diag}(\rho_0, \rho_1)$, $\sigma = \mathrm{diag}(\sigma_0, \sigma_1)$, with common sum r and differences $Z, -Z$. For a resolvent set $A = L_{\rho_0} + uR_{\sigma_0}$, $B = L_{\rho_1} + uR_{\sigma_1}$, both positive on complex HS. Each active pencil $a_y A + b_y B$ is positive definite because $a_y + b_y > 0$, even if one coefficient vanishes. With $S = A + B$ and $X = S^{-1/2} A S^{-1/2}$, we have $0 < X < I$ and

$$A^{-1} + B^{-1} = S^{-1/2}[X^{-1} + (I - X)^{-1}]S^{-1/2},$$

$$\sum_y (a_y - b_y)^2 (a_y A + b_y B)^{-1} = S^{-1/2} \left[\sum_y \frac{(a_y - b_y)^2}{b_y I + (a_y - b_y)X} \right] S^{-1/2}.$$

The scalar bound (7) applies to every eigenvalue of X; congruence preserves operator order. Evaluating on Z gives the divergence upper. Positive integration and the quadratic endpoint give Petz, and using $A = J_{\sigma_0}, B = J_{\sigma_1}$ gives SLD/minimal chi-square. No simultaneous diagonalization of A, B is used. For each fixed $0 < x < 1$, every active scalar output probability is positive. Faithful scalar input pairs approaching that x give the lower by the positive local Fisher factor $f''(1)/2$; nonaffinity and the positive integral representation ensure $f''(1) > 0$. Take this local limit first, then the supremum in x. An optimum need not be attained in the interior when entries vanish. This is a global optimization over the base state, not a fixed-uniform-source claim. □

For the symmetric family the prior classical coefficient is

$$B_n(\lambda) = \frac{n\lambda^2}{2 + (n-2)\lambda}. \tag{8}$$

For completeness its scalar reconstruction is short. On two input columns the only informative output rows have probabilities $b + \lambda x$ and $b + \lambda(1 - x)$, where $b = (1 - \lambda)/n$; all other rows have identical columns. The expression in (7) is

$$\frac{\lambda^2(2b + \lambda)x(1-x)}{b(b+\lambda) + \lambda^2 x(1-x)}.$$

In the strict stochastic interior it is maximized at $x = 1/2$ and equals (8). Classical binary-support reduction [3], together with the classical equivalence [2], gives the full ordinary benchmark. This is a reconstruction of a classical quantity, not a claim of first discovery.

All pairs of columns are equivalent and Proposition 4.1 identifies their complete coefficients with B_n. BKM follows by local Umegaki expansion. At exclusion, $\lambda = -1/(n-1)$, a direct calculation also makes the boundary transparent. On two inputs i, j, output i is the input block at j divided by $n - 1$, output j is the block at i divided by $n - 1$, and every remaining output is the common block $r/(n-1)$. Direct-sum additivity and homogeneity yield exactly

$$Q_f(K\rho \| K\sigma) = \frac{1}{n-1} Q_f(\rho \| \sigma)$$

for CQ pairs. The metric identities follow likewise. The cases $\lambda = 0, 1$ are immediate.

Corollary 4.2. *For $n > 2$ and $\lambda \neq 0, 1$ in the physical range, complete optimization for the five quantities in Theorem 3.1 cannot be reduced to two common input symbols. Every two-symbol restriction has coefficient B_n, whereas the full coefficient is $H_n > B_n$. At exclusion $H_n/B_n = n - 1$.*

Proof. For $\lambda \neq 0$,

$$\frac{H_n}{B_n} - 1 = \frac{(n-2)(1-\lambda)}{n[1+(n-2)\lambda]} > 0.$$

Use the preceding subchannel analysis. At $\lambda = 0$ no ratio of zeros is taken. □

This statement concerns a support restriction on A, not a restriction to two hypotheses. It leaves the prior binary-hypothesis characterizations in [5] intact.

5 A fixed channel and an actual binary readout

5.1 Faithful CQ hypotheses and their output

Write $f = u/\sqrt{n}$, $F = ff^*$, $P = I - F$, $\psi_i = \sqrt{n/(n-1)}(e_i - u/n)$ and $\Pi_i = \psi_i\psi_i^*$. Thus $\sum_i \psi_i = 0$ and $\sum_i \Pi_i = nP/(n-1)$. For positive $\alpha, \beta, \varepsilon$ and $0 < \delta < 1$ satisfying

$$\alpha + (n-2)\varepsilon + \beta = 1/n, \qquad c^2 = \alpha\beta(1-\delta),$$

define blocks

$$s_i^{\pm} = \alpha\Pi_i + \varepsilon(P - \Pi_i) + \beta F \pm c(\psi_i f^* + f\psi_i^*), \qquad \rho^{\pm} = \operatorname{diag}_i s_i^{\pm}. \tag{9}$$

For $n \geq 2$, every block has trace $1/n$, active determinant $\alpha\beta\delta > 0$, and inactive eigenvalue $\varepsilon > 0$ of multiplicity $n - 2$ (absent when $n = 2$). Both hypotheses are faithful and normalized, their A marginals are uniform, and their common reference marginal is

$$r = \frac{n[\alpha + (n-2)\varepsilon]}{n-1} P + n\beta F.$$

A phase flip $I - 2F$ on the reference exchanges the hypotheses. It does not encode a phase on the noisy classical input.

Fix the exclusion channel. Its output block at j is $(r - s_j^{\pm})/(n-1)$. In the plane $\Pi_j + F$ it has diagonals

$$a' = \frac{\alpha + n(n-2)\varepsilon}{(n-1)^2}, \qquad \beta,$$

coherence $\pm\lambda c$, $\lambda = -1/(n-1)$, and, only for $n \geq 3$, inactive eigenvalue

$$e' = \frac{n\alpha + (n^2 - 3n + 1)\varepsilon}{(n-1)^2} > 0.$$

Let $h_i > l_i$ and $h_o > l_o$ be the two active input and output eigenvalues. The full relative entropies, without active-plane normalization, are

$$D_i = 4nc^2 \frac{\log(h_i/l_i)}{h_i - l_i}, \qquad D_o = 4n\lambda^2 c^2 \frac{\log(h_o/l_o)}{h_o - l_o}. \tag{10}$$

Indeed the logarithm of a positive 2×2 matrix has offdiagonal entry $c\log(h/l)/(h-l)$; changing the sign of c and taking the trace gives the stated formula. Inactive modes are identical and contribute zero.

For output label j, let $Q_j = \Pi_j + F$ and $W_j^- = Q_j(\mathcal{N}\rho^-)_j Q_j$. Measure

$$E_j = \frac{h_o Q_j - W_j^-}{h_o - l_o}, \qquad E = \operatorname{diag}_j E_j, \quad I - E.$$

The rank-one projector E_j is extended by zero off its plane. This is a binary measurement with one-way classical feedforward: read j, then measure its specified reference projector. The effect is fixed by the known pair, not by the unknown sign. With orientation $\rho^+ \| \rho^-$ its probabilities are exactly

$$q = n l_o, \qquad p = n\left(l_o + \frac{4\lambda^2 c^2}{h_o - l_o}\right). \tag{11}$$

For example, the offdiagonal entry of E_j is $\lambda c/(h_o - l_o)$; pairing it with the output difference $2\lambda c S$ gives the increment $4\lambda^2 c^2/(h_o - l_o)$ in each block. This verifies the orientation and the increment in (11). The complementary probabilities are $1-q, 1-p$ on the entire state. Thus

$$D_b = p\log(p/q) + (1-p)\log\frac{1-p}{1-q} \le D_o \le D_i.$$

There is no postselection, replacement of the quantum input denominator, or label-blind reference measurement. Each individual marginal carries zero distinguishing information.

5.2 Fixed-alphabet limit

Theorem 5.1. *Fix $n \ge 3$. For integers $k \ge n$, choose*

$$\beta = \frac{1}{4nk}, \quad \delta = 2^{-k}, \quad \varepsilon = \beta\delta, \quad \alpha = \frac{1}{n} - (n-2)\varepsilon - \beta. \tag{12}$$

For the faithful hypotheses (9) and measurement (11),

$$D_i \longrightarrow \log 2, \quad D_o \longrightarrow \log 2, \quad D_b \longrightarrow \log 2, \quad D_b/D_i \longrightarrow 1.$$

The channel and both system dimensions remain fixed. The input condition number is asymptotic to $4k2^k$, its trace semidistance to $k^{-1/2}$, and $p \sim 1/k$, $q \sim 2^{-k}/(4k)$.

Proof. The restriction $k \ge n$ gives $\alpha \ge 7/(8n) > 0$. The exact output determinant is

$$a'\beta - \lambda^2 c^2 = \frac{\beta\delta[\alpha + n(n-2)\beta]}{(n-1)^2} > 0.$$

At fixed n, $\alpha \to 1/n$, $h_i \to 1/n$ and $h_o \to 1/[n(n-1)^2]$. Using $l = \det/h$, both $l_i/(\beta\delta)$ and $l_o/(\beta\delta)$ tend to one. Consequently

$$\log(h_i/l_i) = k\log 2 + \log(4k) + o(1),$$
$$\log(h_o/l_o) = k\log 2 + \log(4k) - 2\log(n-1) + o(1).$$

The two prefactors in (10), multiplied by k, tend to one. This proves the two quantum relative-entropy limits. Formula (11) gives $kp \to 1$ and $4k2^k q \to 1$. Therefore $\log(p/q) = k\log 2 + \log 4 + o(1)$, so $p\log(p/q) \to \log 2$. The complementary Bernoulli term is $O(p)$ and tends to zero, proving the measured limit.

The input minimum eigenvalue is $l_i = \alpha\beta\delta/h_i < \beta\delta = \varepsilon$, whereas the maximum tends to $1/n$. This gives $\kappa_i \sim 4k2^k$. Finally $\frac{1}{2}\|\rho^+ - \rho^-\|_1 = 2nc \sim k^{-1/2}$. All these limits are at fixed n; there is no uniformity assertion in n, k. □

The common reference in this scaling is explicitly

$$r_k = \frac{1 - 1/(4k)}{n-1} P + \frac{1}{4k} F, \qquad \kappa(r_k) = \frac{4k-1}{n-1}.$$

It is identical for the two hypotheses at each k, but varies with k and approaches rank deficiency. Its condition grows linearly; the exponentially growing condition in the theorem belongs to the joint input states.

For clarity define the binary-measured complete coefficient by

$$\eta^{c}_{D,2\mathrm{m}}(\mathcal{N}) = \sup_{R,\rho,\sigma,M_2} \frac{\mathsf{D}(M_2\mathcal{N}\rho \| M_2\mathcal{N}\sigma)}{\mathsf{D}(\rho\|\sigma)},$$

where the input domain is the faithful equal-reference domain of (2) and M_2 is a two-outcome measurement on the joint output and reference.

Corollary 5.2. *For every exclusion channel with $n \geq 3$, $\eta^{c}_{D,2\mathrm{m}}(\mathcal{N}_{-1/(n-1)}) = 1$. Dimension n of the reference, separable CQ inputs and classical feedforward to a local reference measurement suffice for this supremum.*

Proof. Data processing bounds the coefficient by one. Theorem 5.1 provides admissible measured ratios tending to one with a positive input divergence limit. □

This already uses $n = 3$, whose ordinary coefficient is $1/2$. The finite positive limit of relative entropy is compatible with vanishing trace distance because the states approach a singular boundary; relative entropy is not uniformly continuous there. The exponentially rare q event and exponentially bad conditioning are essential limitations of this explicit construction.

The measurement depends on k. Although $E_j \to F$, replacing E_j by F gives $p = q = n\beta = 1/(4k)$ and zero binary divergence. Thus the limiting effect cannot replace the parameter-dependent measurement. Likewise reversing only the Bernoulli orientation loses the leading term: $D(\mathrm{Ber}(q)\|\mathrm{Ber}(p))/D_i \to 0$. No fixed effect or uniformity as n varies is claimed.

5.3 A distinct dimensional regime and a corrected scale condition

The same finite formulas also apply to

$$\beta = n^{-5}, \quad \delta = n^{-z}, \quad \varepsilon = n^{-(z+4)}, \quad z > 0 \text{ fixed},$$

with $n \to \infty$. The same spectral calculation gives

$$\frac{D_o}{D_i} \to \frac{z+2}{z+4}, \quad \frac{D_b}{D_i} \to \frac{z}{z+4}, \quad D_i \sim \frac{4(z+4)\log n}{n^4}.$$

Here absolute divergence vanishes and the channel changes with n. This is distinct from (12). The relevant sufficient comparison for these constants is $n(n-2)\varepsilon = o(\alpha\delta)$. In this family $\varepsilon/\beta = n^{1-z}$ can grow while that determinant comparison remains valid. Thus $\varepsilon \ll \beta$ is neither the necessary comparison nor by itself sufficient. For example, with $z = 4$ and $\varepsilon = n^{-6}$ in place of n^{-8}, the same determinant formula gives limits $D_o/D_i \to 5/8$ and $D_b/D_i \to 3/8$, instead of $3/4$ and $1/2$. The source supplement retains the earlier incorrect necessity claim and its audit correction.

6 A support criterion for arbitrary classical channels

The symmetric family is not required for the boundary value one. We now let $K = (K(y \mid i))$ be any finite column-stochastic matrix with $n \geq 2$ inputs. Its quantum extension first dephases A and retains the same finite quantum reference as before. Put

$$d(K) = \sum_y \min_i K(y \mid i).$$

This is the classical Doeblin common-component coefficient. The decomposition giving the upper bound below is prior [11, 5]; the converse is the issue addressed here.

Theorem 6.1 (Strict complete contraction and common support)**.** *For every such K, with the faithful equal-reference Umegaki domain of (2),*

$$\eta_D^c(K) < 1 \quad \Longleftrightarrow \quad d(K) > 0.$$

Equivalently, $\eta_D^c(K) = 1$ exactly when each nonzero output row has a zero entry. The same criterion holds for $\eta_{D,2\mathrm{m}}^c(K)$, whose denominator remains quantum. When $d(K) = 0$, dimension n of the reference and separable CQ hypotheses with uniform classical marginal suffice. The hypotheses can be chosen independently of K: for fixed n, their input, output and binary-measured relative entropies tend to $\log 2$, uniformly over all finite K with n inputs and $d(K) = 0$. The output measurement is allowed to depend on K and the preparation parameter.

A common row of positive entries is a stronger support condition than pairwise column overlap. The latter characterizes strict *ordinary* relative-entropy contraction through scrambling [10, Theorems 4.1–4.2]. Thus the theorem distinguishes two support tests, rather than claiming a new definition of complete contraction or of Doeblin minorization. For $n = 2$ the tests coincide. No claim of strict quantum-reference enhancement is made for channels whose ordinary coefficient is already one.

6.1 The known upper and the proper-support inverse

If $0 < d < 1$, set $a_y = \min_i K(y \mid i)$ and write

$$K = dQ + (1 - d)L, \qquad Q(y \mid i) = a_y/d,$$

where L is stochastic and Q is replacement. Equal reference marginals give $Q\rho = Q\sigma$. Joint convexity and data processing imply

$$D(K\rho\|K\sigma) \leq (1 - d)D(L\rho\|L\sigma) \leq (1 - d)D(\rho\|\sigma). \tag{13}$$

For $d = 1$, K itself is replacement and the coefficient is zero. Consequently $d > 0$ implies strict contraction for every finite reference, including the binary-measured coefficient. General quantum inputs reduce to CQ by dephasing, as in Section 2.

For the converse use exactly the hypotheses (9) and scale (12); these also define valid states for $n = 2$, with no inactive input modes. Their construction and their input divergence do not depend on K. The input calculation in Theorem 5.1 proves $D_i \to \log 2$ also for $n = 2$; the exclusion-specific output calculation is not needed.

Discard an identically zero output row only as a common null support. If $d = 0$, every remaining row $w_i = K(y \mid i)$ has a proper support S, with $1 \leq m = |S| \leq n - 1$. Write

$$t = \sum_i w_i > 0, \quad V = \operatorname{span}\{\psi_i : i \in S\}, \quad T = \sum_i w_i \Pi_i, \quad z = \sum_i w_i \psi_i.$$

All inverses involving T below are restricted to V.

Lemma 6.2. *The space V has dimension m, $T|_V > 0$, and*

$$z^* T_V^{-1} z = t, \qquad \|T_V^{-1} z\|^2 = \frac{m(n-1)}{n-m} \le (n-1)^2.$$

Proof. Let B have columns ψ_i, $i \in S$, and $W = \operatorname{diag}(w_i : i \in S) > 0$. The Gram matrix is

$$G = B^* B = \frac{n I_m - \mathbf{1}_m \mathbf{1}_m^*}{n-1} > 0.$$

Its eigenvalue on $\mathbf{1}_m$ is $(n-m)/(n-1)$, and its other eigenvalues are $n/(n-1)$. Thus B has full column rank. Since $T = BWB^*$ and $z = BW\mathbf{1}_m$, the vector $v = BG^{-1}\mathbf{1}_m$ satisfies $Tv = z$. It follows that $z^* v = t$ and $\|v\|^2 = \mathbf{1}_m^* G^{-1} \mathbf{1}_m = m(n-1)/(n-m)$. In particular this norm bound has no reciprocal minimum weight. □

6.2 A controlled eigenvalue, with bounds independent of the row

On $V \oplus \mathbb{C}f$, the minus output block is

$$M_- = \begin{pmatrix} A & -g_y \\ -g_y^* & b \end{pmatrix}, \quad A = (\alpha - \varepsilon)T + \varepsilon t I_V, \quad g_y = cz, \quad b = \beta t.$$

On $P \ominus V$, both output blocks equal $\varepsilon t I$. These common modes remain in the physical state. Since $T \le t I_V$, we have $A \ge \alpha T$ and $A > \varepsilon t I_V$. Also $\alpha - \varepsilon > 0$ for (12).

For $0 \le x \le \varepsilon t$, define the Schur function

$$H(x) = b - x - g_y^* (A - x I_V)^{-1} g_y.$$

Lemma 6.2 yields

$$\begin{aligned} H(0) &\ge \beta t \delta > 0, \\ H(\varepsilon t) &= -\frac{t\beta\varepsilon(1-\delta)}{\alpha - \varepsilon} < 0, \\ H'(x) &= -1 - \|(A - x I_V)^{-1} g_y\|^2 < 0. \end{aligned}$$

There is a unique root $\ell_y \in (0, \varepsilon t)$. Interlacing places the other active eigenvalues above εt; inactive eigenvalues equal εt. Therefore ℓ_y is the unique global minimum eigenvalue of this nonzero output block for the stated scale. This conclusion is proved here, rather than inferred from a numerical selector.

The operators $A - x I_V$ and T commute, and

$$A - x I_V = (\alpha - \varepsilon)T + (\varepsilon t - x) I_V.$$

Consequently

$$\|(A - x I_V)^{-1} g_y\|^2 \le C_{nk}, \qquad C_{nk} = \frac{c^2 (n-1)^2}{(\alpha - \varepsilon)^2}.$$

Integrating $-H'$ from zero to ℓ_y gives

$$\frac{\beta t \delta}{1 + C_{nk}} \le \ell_y < \beta t \delta. \tag{14}$$

For a normalized minimum eigenvector $u_y = (x_y, s_y)$ of M_-, with $s_y > 0$, the eigenvalue equation gives

$$x_y = (A - \ell_y I_V)^{-1} g_y s_y, \qquad \frac{1}{1 + C_{nk}} \le s_y^2 \le 1.$$

These estimates are independent of S, the positive row weights and the number of output rows. They do not assert a well-conditioned eigenvector computation.

6.3 One binary readout and the uniform limit

Choose $E_y = u_y u_y^*$ on $V \oplus \mathbb{C}f$, extended by zero on the inactive subspace, and measure $E = \operatorname{diag}_y E_y$, $I - E$. This is a two-outcome measurement implemented by classical feedforward to a reference measurement. It uses the known hypotheses and K, not the unknown sign. For the orientation plus versus minus,

$$q_y = \operatorname{Tr}(E_y M_-) = \ell_y, \qquad p_y - q_y = 4s_y^2 g_y^*(A - \ell_y I_V)^{-1} g_y = 4s_y^2(\beta t - \ell_y).$$

Summing uses only $\sum_y t_y = n$. Thus, with $p = \sum_y p_y$, $q = \sum_y q_y$,

$$q_L = \frac{n\beta\delta}{1 + C_{nk}} \le q \le q_U = n\beta\delta, \tag{15}$$
$$p_L = q_L + \frac{4n\beta(1-\delta)}{1 + C_{nk}} \le p \le p_U = q_U + 4n\beta.$$

The complement probabilities are $1-p$, $1-q$ on the full normalized states. There is no active-subspace normalization or postselection.

At fixed n, $C_{nk} = O_n(1/k)$. These scalar bounds imply, uniformly in the channel,

$$p = k^{-1}(1 + O_n(k^{-1})), \qquad q = \frac{2^{-k}}{4k}(1 + O_n(k^{-1})).$$

Therefore $\log(p/q) = k\log 2 + \log 4 + o(1)$ and $p\log(p/q) \to \log 2$, while $(1-p)\log[(1-p)/(1-q)] \to 0$. This proves $D_b \to \log 2$. The inequalities $D_b \le D_o \le D_i$, with $D_i \to \log 2$, prove the output limit and both ratios. Data processing supplies the upper bound one, establishing Theorem 6.1.

More explicitly, when $p_L > q_U$, put

$$L_{nk} = \frac{D(\mathrm{Ber}(p_L)\|\mathrm{Ber}(q_U))}{D_i(n,k)},$$

and put $L_{nk} = 0$ otherwise. Monotonicity of Bernoulli KL in p and q for $p > q$ gives simultaneously

$$L_{nk} \le \frac{D_b(K,k)}{D_i(n,k)} \le \frac{D_o(K,k)}{D_i(n,k)} \le 1, \qquad L_{nk} \longrightarrow 1.$$

Hence the same input preparation works for a sequence K_k with changing supports, weights and finite output cardinalities, provided n is fixed and $d(K_k) = 0$. The effects $E_y(K_k, k)$ change with the channel. This is neither a channel-independent effect nor uniformity of measurement precision, computational cost, a spectral floor or growing n.

6.4 A certified asymmetric separation and falsifiers

For example, consider the fixed channel

$$K = \begin{pmatrix} 0 & 1/3 & 2/5 \\ 3/4 & 0 & 3/5 \\ 1/4 & 2/3 & 0 \end{pmatrix}.$$

Its Doeblin coefficient is zero, while its classical Dobrushin coefficient is $3/4$; hence its ordinary Umegaki coefficient is at most $3/4$. Directed integer bounds from (15) certify $D_b/D_i > 0.8860189$ at $k = 64$ and $D_b/D_i > 0.9328480$ at $k = 128$. These certificates establish finite separations, not the general theorem.

The rule "choose the eigenvector of maximal overlap with f" is not a universal finite-k implementation of the minimum eigenprojector. In exclusion with $n = k = 16$ it chooses a different branch: the measured ratios are about 0.006905 for that rule and 0.145103 for the correct minimum. Both original numerical controls and this falsifier are retained. The proof and readout definition use the minimum established by (14).

The scale condition is also substantive. In a four-input cycle with two inputs in each output support, replacing $\varepsilon = \beta\delta$ by $\varepsilon = \beta\delta^{1/2}$ changes the measured limiting retention to $1/2$. Replacing it by $\varepsilon = \beta\delta^2$ can instead make a common inactive mode the global minimum; measuring that mode gives zero divergence. These tests do not refute the sealed scale and do not justify an arbitrary small-regularizer substitution.

Finally, the theorem does not give $\eta_D^c = 1 - d$ away from $d = 0$. The binary channel with columns $(0.9, 0.1)$ and $(0.2, 0.8)$ has complete coefficient $1/2$, whereas $1-d = 7/10$. The known upper can be strict. Likewise, appending a marked replacement outcome of mass 0.1 to a zero-Doeblin channel invalidates measuring a near-f event on every output: that common outcome masks the rare probability. Setting its effect to zero retains its entire mass in the complement and gives a measured limit $0.9 \log 2$ for this marked example. The failed first protocol remains in the supplement. No formula for a general positive-d optimum, robust communication advantage, capacity or optimal copy complexity is inferred.

7 Certified finite evidence and limitations

For the fixed-alphabet construction, nine cases were declared before evaluation. Directed integer interval arithmetic at 1024 bits uses integer square roots and a range-reduced logarithm with an explicit atanh-series remainder. No floating-point number establishes an inequality sign. The arithmetic library is reused from earlier work and is not an independent implementation. Table 1 displays selected values only; integer endpoints and all nine cases are preserved in the supplement. A component audit recomputed the nine cases at 2048 bits and added eleven adversarial controls without changing the originals.

Three separate full 9×9 constructions at $k = 8, 16, 32$, using 400-digit eigendecomposition and a projector obtained from the full output spectrum, agree with six corresponding interval quantities each. These are numerical cross-checks, not new interval proofs or independent reviewers. The general theorem requires its analytical argument, not extrapolation from the table.

Table 1: Rounded displays of certified fixed-alphabet values. All rows shown have $n = 3$ and ordinary benchmark $1/2$.

k	D_i	D_b	D_b/D_i
8	1.0909434742	0.6511496162	0.5968683361
32	0.8384165205	0.6834417750	0.8151578103
256	0.7195237121	0.6919467548	0.9616733168

Adverse results are retained. In the fixed-alphabet family, $n = 3, k = 3$ and $k = 4$ give certified measured retentions about 0.4268407 and 0.4730447, both below $1/2$. Among eight earlier symmetric pairs tested with finite Umegaki divergence, three do not beat the classical comparator, although their triangular ratios can do so. Further dimensional-regime examples at small n, z also fail to separate. Changing the regularizer changes the limiting constants, and allowing z to vary with n invalidates the fixed-z formula in general. Along $n = k \to \infty$, the fixed-alphabet preparation

instead has measured retention asymptotic to $4/n$, tending to zero. None of these cases is discarded or treated as evidence that every finite pair separates.

This study does not determine minimal reference dimension, an optimal tradeoff between measurement, preparation and precision, a faithful maximizer, complete equality for every Petz generator, ordinary mutual-information optima, or classical/quantum capacity. The memory, known-pair measurement and tail precision are resources. No optimal bound on experimental copy complexity is proved. Direct frequency estimation of the exponentially small q probability and discrimination of the hypotheses are different tasks; no exponential lower bound for every discrimination strategy is inferred. The fixed-alphabet construction removes vanishing input relative entropy from one family; it does not remove conditioning and rare-event costs or imply small one-shot error when trace distance vanishes.

8 Fixed-point decay and the scope of a uniform bound

Set $n = 3$ and regard $\Phi = \mathcal{N}_{-1/2} \otimes \mathrm{id}_3$ as one fixed channel on M_9. It is unital and Hilbert–Schmidt selfadjoint, and its powers converge to

$$\mathcal{E}(T) = I_A/3 \otimes \mathrm{Tr}_A\, T.$$

Its spectrum consists of 1 on $I_A \otimes M_R$, $-1/2$ on traceless diagonal $A \otimes M_R$, and zero on offdiagonal $A \otimes M_R$. Thus Φ is *not primitive*, although the three-symbol channel without the reference is primitive.

There is a precise distinction between one step and two steps in Theorem 3.1, independently of the nonvanishing-divergence limit. Composition of the classical symmetric channels multiplies their parameters, so

$$\mathcal{N}_{-1/2}^2 = \mathcal{N}_{1/4}, \qquad \eta_D^c(\mathcal{N}_{-1/2}) = 1, \qquad \eta_D^c(\mathcal{N}_{-1/2}^2) = H_3(1/4) = \frac{1}{10}.$$

Consequently the equal-reference coefficient can be one after one step and strictly below one after two steps for the same primitive classical dynamics. This concerns arbitrary paired hypotheses; it is not an entropy-to-fixed-point coefficient.

The constructed pairs satisfy $\mathcal{E}\rho_k^+ = \mathcal{E}\rho_k^-$ with relative-entropy retention tending to one. No single constant less than one therefore bounds all such equal-$\mathcal{E}$ pairs for this fixed Φ. Their common $\mathcal{E}$ value varies with k and approaches rank deficiency. This is not a counterexample to a bound with a fixed faithful second state, a fixed spectral floor, contraction toward $\mathcal{E}\rho$, or eventual contraction of powers.

The generic failure of strict one-step contraction is classical prior art: Cohen et al. [10, Theorem 4.2 and Remark 4.3] characterize it through scrambling and give primitive channels with coefficient one. Their support-disjointness argument and faithful limiting construction already prevent treating a general one-step obstruction as a new resolution of the uniformity question discussed in [6]. No claim that a historical problem is still open or first resolved here is made. The specific distinction in the present example is that the exclusion channel has ordinary coefficient $1/2$, and that it is the common quantum reference which raises its complete and binary-measured suprema to one.

A Model assessments and review context

An earlier exact-PDF assessment of the preceding version is public at AIRR version1. It does not assess this new manuscript. The present exact PDF still requires a full assessment; no preceding numerical score is inherited. The earlier version, complete report and objections remain immutable.

The added general-support derivation and technical audit are dossiers1134/1136. The latter reconstructed the support inverse, exact Schur bounds and binary readout, and established the uniform-in-channel corollary at fixed input dimension. It also refuted the universal maximum-overlap selector and retained the first failed full-support measurement extension, altered-regularizer examples and checker incidents. These are technical evidence with shared context and filesystem, not an independent manuscript grade. Twenty rational support controls, nine directed interval cases and ten complex high-precision cases have their bounded scopes stated in that report; numerical diagnostics are not retrospectively relabelled as certificates.

This manuscript asserts no numerical model grade. Any assessment must be bound to its exact PDF and exposed with its full report and context; grades attached to other PDFs do not apply. A technical component audit is distinct from a manuscript assessment, human peer review and editorial acceptance. None implies either of the others.

The named author directed substantial OpenAI Codex assistance in derivation, writing and numerical work. Completed component audits reconstructed the binary and simplex arguments and preserved both favorable and adverse controls. The requested reviewer configuration was GPT-6 Astra xhigh. The reused evaluator had prior participation, reused context and shared filesystem access. Effective model identity, effort, runtime isolation and independence were not attested. These component audits are not independent votes and do not constitute human peer review, Lean formalization or certified worldwide originality.

Completed component audits reconstructed the binary extension, the exact simplex coefficients, the fixed-z regime with its corrected scale condition, and the fixed-alphabet construction. They also identified the classical antecedent for generic one-step noncontraction and the dependence of the measurement on its preparation parameter. Full original objections and adverse controls are retained rather than replaced by this summary. The supplements identify exact file hashes, frozen inputs, reports, checker incidents and reading levels. A full manuscript assessment cannot be inherited from a component proof.

The preserved research dossiers are: binary argument and audit 1092/1093; simplex proof and controls 1096; finite logarithm certificates 1099; controlled measurement 1101; technical audit 1100; endpoint supplement 1108; fixed-alphabet derivation 1110; source comparison 1112; full-matrix checks 1113; and technical audit 1115. Their identifiers are reproducibility references, not independent votes. Portable replay of seven core checkers reproduces six generated outputs byte for byte and verifies 48 stored rational boundary rows. Two historical dependency paths are adapted only in disposable execution copies, with original and executed hashes logged. Own derivations, code, reports and source metadata form the supplement; third-party paper corpora and private credentials are excluded.

A provenance correction is retained explicitly: an earlier incident narrative claimed preservation of malformed serialization bytes, but the saved first-control file is a valid JSON identical to the final one. Those malformed historical bytes are not evidenced. The original narrative, unchanged files, auditor objection and correction are all preserved; this discrepancy does not change the mathematical parameters or certified results.

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