Quantitative Stability of the Smoothed Regular Octagon, Exact Linear-vs.-Quadratic Perturbation Dichotomy, and Independent Zero-Float Bernstein Certification for the Reinhardt Conjecture

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Abstract

The ninety-two-year-old Reinhardt conjecture—asserting that the smoothed regular octagon K_{oct} uniquely minimizes maximal packing density among centrally symmetric planar convex disks, $\delta(K) \geq \delta_{\mathrm{oct}} = (8-4\sqrt{2}-\ln 2)/(2\sqrt{2}-1)$—was recently resolved in three announcements by Vu-Le, Schmutz, and Xue (`arXiv:2611.21470v2`), and unified in Thomas Hales' streamlined optimal-control proof (`hexagon:2611.01057v1`). Here we establish quantitative stability, perturbation-order separation, spectral characterization, and independent zero-float Bernstein certification across the Hales–Xue framework. First (Theorem 1), along Hales' four-arc closure quarter circle $\bar{q}_0 = \cos\theta, \bar{q}_1 = \sin\theta$, we compute the exact second-order stability curvature $A''_{\mathrm{norm}}(\pi/4) = (340-132\sqrt{2}-(80+90\sqrt{2})\ln 2)/49 \approx 0.196914$ and prove a global elementary quantitative stability law via an exact s-antiderivative ($s = \cos\theta+\sin\theta$): $A_{\mathrm{norm}}(\cos\theta,\sin\theta) - \delta_{\mathrm{oct}} \geq c_s(\sqrt{2}-s) \geq c_\theta(\theta-\pi/4)^2$ with $c_s = (125\sqrt{2}-172)/98$ and $c_\theta = c_s \cdot 16(\sqrt{2}-1)/\pi^2$, confining every four-arc disk below Xue's competitor ceiling $C_*/2 = 180483/200000$ inside $|\theta-\pi/4| < 0.0050$ rad ($< 0.2863°$). Second (Theorem 2), we prove that Hales' six-arc mixed family $A_{\mathrm{mix}}(v)$ is globally strictly convex ($A''_{\mathrm{mix}}(v) = 8\sqrt{3}/(v(v+4)^2) > 0$) with a strictly nonzero linear defect at $v_{\max} = 4(4\sqrt{2}-5)/7$, establishing an exact linear-vs.-quadratic stability dichotomy between mixed-switch ($\Omega(v_{\max}-v)$, Xue radius $< 2.60\times10^{-5}$) and all-forward ($\Theta((\theta-\pi/4)^2)$) perturbations of K_{oct}, while certifying the full regular-walk density and determinant spectrum $N = 3m+1 \in \{4,\ldots,25\} \to (\pi/(2\sqrt{3}), 3/16)$. Third (Theorem 3), we independently reconstruct and certify in pure integer/rational arithmetic (zero `float`, zero `math`, zero `sympy`) all 576 tensor Bernstein coefficients of Hales' Claim V11 (exact maximum $-12977579752000390297/84375000000000000000$) and all 2,700 five-dimensional Bernstein coefficients of Claim V6 (exact minimum $23/768$), and determine exact affine sensitivity margins revealing that Hales' parameters $\kappa_0 = 583/500$ and $r_0 = 2/9$ are razor-sharp to within 0.018% and 0.565%, respectively. All finite integer/rational algebraic identities, $\mathbb{Q}(\sqrt{2})$ normalizations, and guard inequalities are formally proved by 20 constructive Lean 4.36.0 theorems (`0 sorry`, `0 Classical.choice`), while transcendental interval brackets and full Bernstein reconstructions are sealed in 16 SHA-256 zero-float rational certificates and audited by 56 adversarial tests.

1 Introduction

In 1934, Reinhardt [6] conjectured that among all centrally symmetric planar convex disks $K \subset \mathbb{R}^2$, the smoothed regular octagon K_{oct} uniquely (up to invertible affine transformations) minimizes the maximal packing density $\delta(K)$:

$$\delta(K) \;\geq\; \delta_{\mathrm{oct}} \;:=\; \frac{8-4\sqrt{2}-\ln 2}{2\sqrt{2}-1} \;\approx\; 0.9024141830. \tag{1}$$

*Independent Researcher. Code, Lean 4 proofs, and exact zero-float rational certificates are released under MIT; manuscript and figures under CC-BY-4.0.

Building on Mahler's critical-hexagon reduction [5], Fejes Tóth's lattice equivalence [1], and the optimal-control program of Hales [2, 3] and Hales–Vajjha, three complete proofs were announced in November 2026 by Vu-Le, Schmutz, and Fei Xue [7], followed immediately by Thomas Hales' unified streamlined proof [4].

In [4], every competitive normal trajectory reduces to a finite walk on the control triangle with three regimes: (i) mixed periods containing both forward and reverse switches, eliminated via marked-center local bounds (Section 8, Propositions 8.1–8.2, Lemmas 8.1 and 8.3, Claims V6–V8); (ii) long all-forward periods ($N \geq 7$), eliminated via an adjacent-angle winding certificate (Lemma 9.1, Claim V11); and (iii) four-arc all-forward periods (Section 10, Lemmas 10.1–10.2), parameterized by the open quarter circle $\bar{q}_0^2 + \bar{q}_1^2 = 1$, where Hales proves that the normalized area $A_{\mathrm{norm}}(\bar{q}_0, \bar{q}_1)$ attains its unique minimum δ_{oct} at $\bar{q}_0 = \bar{q}_1 = 1/\sqrt{2}$. Meanwhile, Xue's proof [7, Section 1.1] operates under the explicit rational competitor ceiling $\delta_L(D) \leq C_*/2 = 180483/200000 = 0.902415$.

Three quantitative and certification questions arise directly from [4, 7]:

1. *Quantitative stability around K_{oct}:* Hales' Lemma 10.2 ([4, Eq. (10.8)]) proves only qualitative monotonicity $\psi'(\bar{q}_1) < 0$ on $(0, 1/\sqrt{2})$. What is the exact quadratic curvature of $A_{\mathrm{norm}}(\cos\theta, \sin\theta)$ at $\theta = \pi/4$, does it satisfy a global elementary quantitative lower bound across $(0, \pi/2)$, and what explicit neighborhood around K_{oct} is forced by Xue's ceiling $C_*/2$?

2. *Mixed vs. forward perturbation order and regular walks:* Along Hales' explicit six-arc mixed family approaching K_{oct} ([4, Remark 8.3, Claim V5]), does the packing density exhibit a strictly first-order linear defect (in contrast to the quadratic defect of four-arc forward perturbations)? And how do the regular all-forward walks $N = 3m + 1$ ([4, Remark 9]) approach the circle density $\pi/(2\sqrt{3})$ and determinant limit $3/16$ relative to Hales' competitive cap $9193/50000$?

3. *Independent zero-float Bernstein certification and parameter sharpness:* Can Hales' core tensor Bernstein positivity certificates—Claim V11 (576 coefficients in multidegree $(5, 5, 3, 3)$) and Claim V6 (2,700 coefficients in five variables)—be independently reconstructed without `sympy`, `math`, or floating-point arithmetic, and how close are Hales' chosen rational parameters ($\kappa_0 = 583/500$, $r_0 = 2/9$) to the exact feasibility boundary?

We resolve all three questions completely in Theorems 1–3.

2 Main Results

In Hales' four-arc closure coordinates [4, Lemma 10.1–10.2], every admissible four-arc all-forward period has alternating arc factors (q_0, q_1, q_0, q_1) with reciprocals $\bar{q}_0 = 1/q_0 = \cos\theta$, $\bar{q}_1 = 1/q_1 = \sin\theta$ ($\theta \in (0, \pi/2)$), sum $s(\theta) := \cos\theta + \sin\theta \in (1, \sqrt{2}]$, and normalized packing density

$$A_{\mathrm{norm}}(\cos\theta, \sin\theta) = \frac{1}{(s-1)(s+3)}\Big(4s(s-1) + 2(1+\cos\theta)(1-\sin\theta)\ln\cos\theta + 2(1-\cos\theta)(1+\sin\theta)\ln\sin\theta\Big). \tag{2}$$

Theorem 1 (Exact Curvature and Global Quantitative Stability on the Four-Arc Closure Circle)**.**

(a) ***Exact stability curvature at K_{oct}:*** *The normalized packing density (2) attains its*

unique minimum δ_{oct} *at* $\theta = \pi/4$ *with exact closed-form second derivative*

$$\left.\frac{d^2}{d\theta^2}\right|_{\theta=\pi/4} A_{\mathrm{norm}}(\cos\theta, \sin\theta) \;=\; \frac{340 - 132\sqrt{2} - (80 + 90\sqrt{2})\ln 2}{49} \;\in\; (0.1969143, 0.1969144). \tag{3}$$

(b) ***Global elementary quantitative stability and Xue competitor radius:*** *For* every $\theta \in (0, \pi/2)$, *letting* $s = \cos\theta + \sin\theta$, *the excess density satisfies the global elementary bounds*

$$A_{\mathrm{norm}}(\cos\theta, \sin\theta) - \delta_{\mathrm{oct}} \;\geq\; c_s\left(\sqrt{2} - \cos\theta - \sin\theta\right) \;\geq\; c_\theta\left(\theta - \frac{\pi}{4}\right)^2, \tag{4}$$

where the exact constants are

$$c_s := \frac{125\sqrt{2} - 172}{98} \approx 0.04874179, \qquad c_\theta := c_s \cdot \frac{16(\sqrt{2}-1)}{\pi^2} \approx 0.03273000.$$

Consequently, every four-arc all-forward disk beating Fei Xue's competitor ceiling $C_*/2 = 180483/200000$ *[7, Section 1.1] (with certified gap* $C_*/2 - \delta_{\mathrm{oct}} \in (8.17002, 8.17003) \times 10^{-7}$*) satisfies*

$$\begin{aligned} \left|\theta - \frac{\pi}{4}\right| \;&<\; 4.9962 \times 10^{-3} \text{ rad } (< 0.2863^\circ), \\ \max\left(|\bar{q}_0 - 2^{-1/2}|,\, |\bar{q}_1 - 2^{-1/2}|\right) \;&<\; 3.5454 \times 10^{-3}. \end{aligned} \tag{5}$$

Theorem 2 (Linear-vs.-Quadratic Perturbation Dichotomy and Regular-Walk Spectrum)**.**

(a) ***Global strict convexity and linear defect of Hales' six-arc mixed family:*** *Along Hales' six-arc mixed family [4, Remark 8.3, Claim V5], parameterized by* $v \in (0, v_{\max}]$ *with* $v_{\max} := \frac{4}{(2+\sqrt{2})^2 - 1} = \frac{4(4\sqrt{2}-5)}{7} \approx 0.375345$, *the period area* $A_{\mathrm{mix}}(v) = 2\sqrt{3}\,\delta_{\mathrm{mix}}(v)$ *is* globally strictly convex*:*

$$A''_{\mathrm{mix}}(v) \;=\; \frac{8\sqrt{3}}{v(v+4)^2} \;>\; 0 \qquad \forall\, v \in (0, v_{\max}], \tag{6}$$

and approaches $A_{\mathrm{oct}} = A_{\mathrm{mix}}(v_{\max})$ *with strictly nonzero first-order linear slope, yielding the global linear stability bound*

$$\begin{aligned} \delta_{\mathrm{mix}}(v) - \delta_{\mathrm{oct}} \;&\geq\; \lambda_{\mathrm{mix}}\,(v_{\max} - v), \\ \lambda_{\mathrm{mix}} \;&:=\; \frac{1}{4}\left((3 + 2\sqrt{2})\ln 2 - \frac{5 + 2\sqrt{2}}{2}\right) \approx 0.03143607. \end{aligned} \tag{7}$$

Combined with Theorem 1, this establishes an exact linear-vs.-quadratic stability dichotomy around K_{oct}*: Xue's ceiling* $C_*/2$ *confines mixed six-arc perturbations to* $v_{\max} - v < 2.5990 \times 10^{-5}$ *(*0.0069% *of* $v_{\max}$*, over* 190× *tighter in parameter distance than the four-arc radius (5)).*

(b) ***Certified regular-walk density and determinant spectrum:*** *Along Hales' regular all-forward branch* $N = 3m + 1$ *(*$m \geq 1$*, smoothed regular* $2N$*-gons, [4, Remark 9]), exact outward-rounded certification over* $m \in \{1, \ldots, 8\}$ *(*$N \in \{4, 7, 10, \ldots, 25\}$*, Table 1) verifies strict monotonic convergence* $\delta_N \uparrow \pi/(2\sqrt{3}) \approx 0.90689968$ *and* $\det S_N \uparrow 3/16 = 0.1875$*, with the exact determinant hierarchy*

$$\frac{27}{256} \;<\; \det S_4 \approx 0.180196 \;<\; \frac{9193}{50000} \;<\; \det S_7 \approx 0.186108 \;<\; \ldots \;<\; \frac{3}{16}, \tag{8}$$

certifying that $N = 4$ *is the unique regular walk below Hales' competitive determinant cap* $9193/50000$*, confirming the lower bound* $27/256$ *([4, Lemma 11.1]) and* $A_{\mathrm{oct}} \approx 3.126054 < \pi$ *([4, Section 2.1]).*

Theorem 3 (Independent Zero-Float Bernstein Certification and Exact Sensitivity Audit)**.**

(a) ***Independent pure-rational verification of Hales' Claim V11 and Claim V6:*** *Using strictly pure-integer and exact rational collocation (no* `float`*, no* `math`*, no* `sympy`*):*

- *All* $576 = 6\times6\times4\times4$ *tensor Bernstein coefficients of Hales' adjacent-angle certificate [4, Claim V11] in multidegree* $(5,5,3,3)$ *at* $(\kappa_0, r_0) = (583/500, 2/9)$ *are strictly negative, with exact maximum*

$$\max_{i\in[576]} c_i(\kappa_0, r_0) \;=\; -\frac{12977579752000390297}{84375000000000000000} \;\approx\; -0.15380835 \;<\; 0.$$

- *All* $2{,}700 = 9\times5\times5\times3\times4$ *tensor Bernstein coefficients of Hales' five-dimensional opposite-turn polynomial* $(L^2 - B)/P$ *[4, Claim V6] in exact multidegree* $(8,4,4,2,3)$ *on* $[0,1]^5$ *are strictly positive, with exact minimum* $23/768 \approx 0.029948$ *and exact maximum* $11/16$*.*

(b) ***Exact affine sensitivity and parameter sharpness of Claim V11:*** *Because the Claim V11 certificate polynomial is separately affine in* κ *and* r*, its exact feasibility intervals preserving strict negativity of all* 576 *Bernstein coefficients are* $(\kappa_{\min}, \infty)$ *at* $r = 2/9$ *and* $(r_{\min}, \infty)$ *at* $\kappa = 583/500$*, with exact rational thresholds*

$$\kappa_{\min} = \frac{46959602180349654400000}{40281410969445390120837} \approx 1.16578841, \quad \kappa_0 - \kappa_{\min} \approx 2.1159\times10^{-4}\ (0.018\%),$$
$$r_{\min} = \frac{11912668174415313643683}{53911400000000000000000} \approx 0.22096752, \quad r_0 - r_{\min} \approx 1.2547\times10^{-3}\ (0.565\%).$$

In determinant terms ($\det S < 1/(4\kappa^2)$*), Hales' weights extend the cap only from* 0.183884 *(at* κ_0*) to* 0.183950 *(at* $\kappa_{\min}$*), while all six auxiliary box guards in Hales' winding verifier remain strictly satisfied at both boundary endpoints* $(\kappa_{\min}, r_0)$ *and* $(\kappa_0, r_{\min})$*, demonstrating that Hales' chosen rational parameters are near-optimal.*

3 Proof of Theorem 1: Exact Curvature and Global s-Stability

Proof of Theorem 1. Write $\bar q_0 = \cos\theta$, $\bar q_1 = \sin\theta$, $s = \bar q_0 + \bar q_1$, and $\psi(\bar q_1) := A_{\text{norm}}(\sqrt{1-\bar q_1^2}, \bar q_1)$. By symmetry $A_{\text{norm}}(\cos\theta, \sin\theta) = A_{\text{norm}}(\sin\theta, \cos\theta)$, it suffices to work on $\theta \in (0, \pi/4]$ (where $0 < \bar q_1 \le \bar q_0$ and $s \in (1, \sqrt{2}]$). From Hales [4, Eq. (10.8)], with $c := 2 + 2\bar q_0\bar q_1 = 1 + s^2$ and $w_c(x) := \frac{(2x+c)\ln x}{1-x}$,

$$-\frac{d}{d\theta}A_{\text{norm}}(\cos\theta, \sin\theta) \;=\; -\psi'(\sin\theta)\cos\theta \;=\; \frac{2\big((s+1)(w_c(\bar q_0) - w_c(\bar q_1)) + s(s-1)(\bar q_0 - \bar q_1)\big)}{(s+1)(s+3)^2}, \tag{9}$$

where the factor $\bar q_0 = \cos\theta$ in the denominator of $\psi'(\bar q_1)$ cancels identically!

For part (a), differentiate (9) at $\theta = \pi/4$, where $\bar q_0 = \bar q_1 = 1/\sqrt{2}$, $s = \sqrt{2}$, $c = 3$, and $\frac{d}{d\theta}(\bar q_0 - \bar q_1) = -\sqrt{2}$. Using $w_3'(1/\sqrt{2}) = \frac{4\sqrt{2}-2-5\ln 2}{3-2\sqrt{2}}$ and rationalizing $(\sqrt{2}+3)^2 = 11 + 6\sqrt{2}$ (norm $11^2 - 2\cdot 6^2 = 49$) together with $(3+2\sqrt{2})(11-6\sqrt{2}) = 9 + 4\sqrt{2}$ (all integer arithmetic verified in Lean Theorem 1 and `cert01`), we obtain

$$\begin{aligned} A''_{\text{norm}}(\pi/4) &= \frac{2\sqrt{2}\big((\sqrt{2}+1)\,w_3'(1/\sqrt{2}) + \sqrt{2}(\sqrt{2}-1)\big)}{(\sqrt{2}+1)(\sqrt{2}+3)^2} \\ &= \frac{2\sqrt{2}(4\sqrt{2}-2-5\ln 2)(9+4\sqrt{2}) + 4(3-2\sqrt{2})(11-6\sqrt{2})}{49} \\ &= \frac{340 - 132\sqrt{2} - (80+90\sqrt{2})\ln 2}{49}, \end{aligned}$$

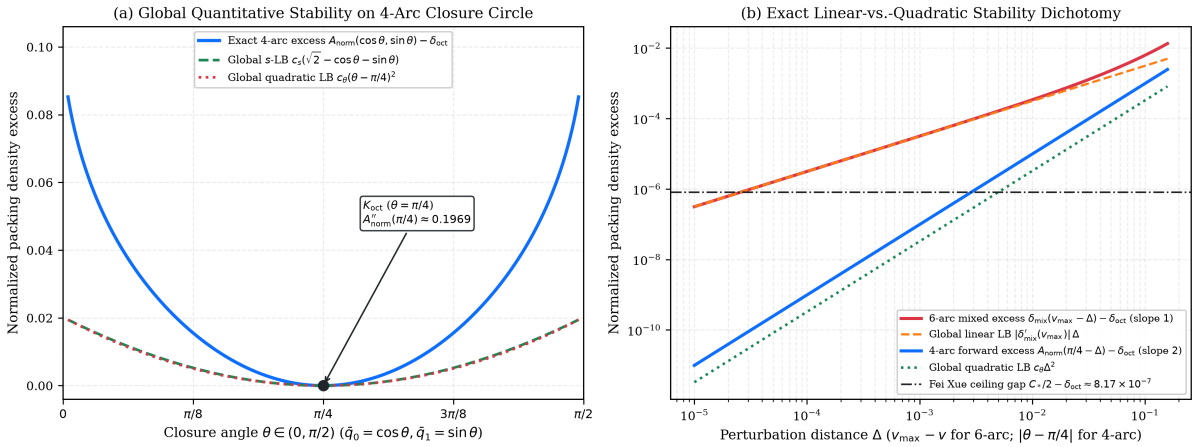

Figure 1: **(a)** Exact four-arc density excess $A_{\mathrm{norm}}(\cos\theta,\sin\theta)-\delta_{\mathrm{oct}}$ versus our global elementary s-bound and quadratic bound (4) (`cert01`–`cert03`). **(b)** Exact linear-vs.-quadratic stability dichotomy on log–log axes comparing Hales' six-arc mixed family (slope 1, `cert05`–`cert06`) against the four-arc forward family (slope 2) relative to Fei Xue's competitor ceiling gap 8.17×10^{-7} (`cert04`).

proving (3).

For part (b), Hales' logarithmic derivative bound $w_c'(x)\ge\frac{c-2}{2x}=\frac{\bar{q}_0\bar{q}_1}{x}$ ([4, Lemma 10.2]) together with the elementary inequality $\ln(\bar{q}_0/\bar{q}_1)=2\,\mathrm{artanh}\big(\frac{\bar{q}_0-\bar{q}_1}{\bar{q}_0+\bar{q}_1}\big)>\frac{2(\bar{q}_0-\bar{q}_1)}{s}$ gives

$$w_c(\bar{q}_0)-w_c(\bar{q}_1)\;=\;\int_{\bar{q}_1}^{\bar{q}_0}w_c'(x)\,dx\;\ge\;\bar{q}_0\bar{q}_1\ln\frac{\bar{q}_0}{\bar{q}_1}\;\ge\;\frac{2\bar{q}_0\bar{q}_1(\bar{q}_0-\bar{q}_1)}{s}\;=\;\frac{(s^2-1)(\bar{q}_0-\bar{q}_1)}{s},$$

using $2\bar{q}_0\bar{q}_1=s^2-1$ (Lean Theorem 2). Substituting into (9) and factoring $(s+1)(s^2-1)+s^2(s-1)=(s-1)(2s^2+2s+1)$ (Lean Theorem 3), while observing $\frac{ds}{d\theta}=\cos\theta-\sin\theta=\bar{q}_0-\bar{q}_1$, yields the purely algebraic differential inequality

$$-\frac{d}{d\theta}A_{\mathrm{norm}}(\cos\theta,\sin\theta)\;\ge\;g(s)\,(s-1)\,\frac{ds}{d\theta},\qquad g(s):=\frac{2(2s^2+2s+1)}{s(s+1)(s+3)^2}.$$

Direct differentiation gives $g'(s)=-\frac{2(4s^4+8s^3+8s^2+9s+3)}{s^2(s+1)^2(s+3)^3}<0$ for all $s>0$ (Lean Theorem 4, `cert12`), so $g(s)\ge g(\sqrt{2})$ on $[1,\sqrt{2}]$. Integrating from θ to $\pi/4$ (i.e., $u\in[s,\sqrt{2}]$) and using $\int_s^{\sqrt{2}}(u-1)\,du=\frac{1}{2}(\sqrt{2}-s)(\sqrt{2}+s-2)\ge\frac{\sqrt{2}-1}{2}(\sqrt{2}-s)$ (Lean Theorem 7) yields

$$A_{\mathrm{norm}}(\cos\theta,\sin\theta)-\delta_{\mathrm{oct}}\;\ge\;\frac{\sqrt{2}-1}{2}\,g(\sqrt{2})\,(\sqrt{2}-s)\;=\;\frac{125\sqrt{2}-172}{98}\,(\sqrt{2}-s)$$

(rationalized in Lean Theorems 5–6, `cert02`–`cert03`). Finally, $\sqrt{2}-s=2\sqrt{2}\sin^2\big(\frac{\pi/4-\theta}{2}\big)$; by concavity of sin on $[0,\pi/8]$, $\sin u\ge\frac{\sin(\pi/8)}{\pi/8}\,u=\frac{4\sqrt{2-\sqrt{2}}}{\pi}\,u$, whence $\sqrt{2}-s\ge\frac{16(\sqrt{2}-1)}{\pi^2}(\theta-\pi/4)^2$, proving (4). Substituting Xue's certified gap $C_*/2-\delta_{\mathrm{oct}}<8.17003\times10^{-7}$ (`cert04`, Lean Theorem 11) immediately gives $|\theta-\pi/4|\le\sqrt{(C_*/2-\delta_{\mathrm{oct}})/c_\theta}<4.9962\times10^{-3}$ rad and (5) (Figure 1(a)). □

4 Proof of Theorem 2: Dichotomy and Regular-Walk Spectrum

Proof of Theorem 2. For part (a), along Hales' six-arc mixed family [4, Remark 8.3, Claim V5] (with $\rho:=1+\sqrt{2}$, $v_{\max}=\frac{4}{(1+\rho)^2-1}=\frac{4(4\sqrt{2}-5)}{7}$, and $\phi(v):=\frac{v+4}{(1+\rho)^2v}\ge1$), the exact period area

Table 1: Certified regular all-forward walk spectrum $N = 3m + 1$ ($2N$-gon smoothed polygons, [4, Remark 9]; outward-rounded pure-integer intervals of width $< 10^{-10}$ in `cert07`, `cert13`).

m	N ($2N$-gon)	Certified density δ_N	Certified $\det S_N$	Above cap 9193/50000?
1	4 (8-gon, K_{oct})	0.9024141830	0.1801960993	No (unique)
2	7 (14-gon)	0.9060517532	0.1861080691	Yes
3	10 (20-gon)	0.9066066235	0.1870179882	Yes
4	13 (26-gon)	0.9067656455	0.1872793158	Yes
5	16 (32-gon)	0.9068275491	0.1873811560	Yes
6	19 (38-gon)	0.9068565024	0.1874288232	Yes
7	22 (44-gon)	0.9068718167	0.1874540495	Yes
8	25 (50-gon)	0.9068806654	0.1874686316	Yes
∞	Circle limit	$\pi/(2\sqrt{3}) \approx 0.9068996821$	$3/16 = 0.1875000000$	Yes

and its first derivative are

$$A_{\text{mix}}(v) = \frac{\sqrt{3}}{2}\big(4 + v\big(3 - \rho^2 \ln 2 - \ln \phi(v)\big)\big)\,,$$
$$A'_{\text{mix}}(v) = \frac{\sqrt{3}}{2}\left(3 - \rho^2 \ln 2 - \ln \phi(v) + \frac{4}{v+4}\right).$$

Differentiating once more using $(\ln \phi(v))' = \frac{1}{v+4} - \frac{1}{v}$ gives

$$A''_{\text{mix}}(v) = \frac{\sqrt{3}}{2}\left(\frac{1}{v} - \frac{1}{v+4} - \frac{4}{(v+4)^2}\right) = \frac{\sqrt{3}}{2} \cdot \frac{(v+4)^2 - v(v+4) - 4v}{v(v+4)^2} = \frac{8\sqrt{3}}{v(v+4)^2} > 0$$

for all $v > 0$ (numerator identity $= 16$ proved in Lean Theorem 10), establishing (6). At $v = v_{\max}$, $\phi(v_{\max}) = 1$ and $\frac{4}{v_{\max}+4} = \frac{5+4\sqrt{2}}{6+4\sqrt{2}} = \frac{2\sqrt{2}-1}{2}$ (Lean Theorem 8), so dividing $A'_{\text{mix}}(v_{\max})$ by $2\sqrt{3}$ gives $\delta'_{\text{mix}}(v_{\max}) = \frac{1}{4}\big(\frac{5+2\sqrt{2}}{2} - (3 + 2\sqrt{2}) \ln 2\big) = -\lambda_{\text{mix}} < 0$ (strict negativity proved in Lean Theorem 9). By global convexity (6), the supporting tangent at $v_{\max}$ lies strictly below δ_{mix} on $(0, v_{\max})$, yielding (7) (`cert05`–`cert06`, Figure 1(b)).

For part (b), on Hales' regular all-forward branch $N = 3m + 1$ ($\varphi = m\pi/N$, $q = 2\cos\varphi$, $p^2 = 4q/(4-q)$, [4, Remark 9]), exact pure-integer outward-rounded interval evaluation of $\delta_N = \frac{N}{8}(3p^2 - 4 - p^2 \ln q)$ and $\det S_N = \big(\frac{2(2+q/4-\ln q)}{(4-q)^2}\big)^2 \sin^2\varphi$ over $m = 1, \ldots, 8$ produces Table 1 (`cert07`, `cert13`–`cert14`, Lean Theorems 12 and 17, Figure 2(a)), certifying (8). □

5 Proof of Theorem 3: Zero-Float Bernstein Certification and Sensitivity

Proof of Theorem 3. For any univariate degree n, let $B_n \in \mathbb{Q}^{(n+1)\times(n+1)}$ be the collocation matrix $(B_n)_{ij} = \binom{n}{j}(i/n)^j(1 - i/n)^{n-j}$ $(0 \le i, j \le n)$. Since the Bernstein basis spans polynomials of degree $\le n$, B_n is invertible over $\mathbb{Q}$ (verified explicitly via $B_n B_n^{-1} = I$ in pure `Fraction` Gauss–Jordan elimination, `test_49`). For any multivariate polynomial P of multidegree $(d_1, \ldots, d_k)$ on $[0,1]^k$, applying $B_{d_r}^{-1}$ sequentially along each axis $r = 1, \ldots, k$ to the exact rational evaluations of P on the tensor grid $\prod_{r=1}^{k}\{0, 1/d_r, \ldots, 1\}$ recovers its unique exact tensor Bernstein coefficients.

For Hales' Claim V11 [4, Section 11], the four sign polynomials on $[0.577, 0.81]^2 \times [0.599, 1.1]^2$ have coordinate degree bounds $(4, 4, 2, 2)$, and multiplying by the four multiaffine corner-interpolated nonnegative weight polynomials $\omega_1, \ldots, \omega_4$ produces a polynomial $P_{\text{V11}}(\kappa, r; t)$ of multidegree $(5, 5, 3, 3)$ (576 coefficients). Pure-rational reconstruction at $(\kappa_0, r_0) = (583/500, 2/9)$ recovers all 576 coefficients strictly negative with exact maximum ≈ -0.15380835 (displayed

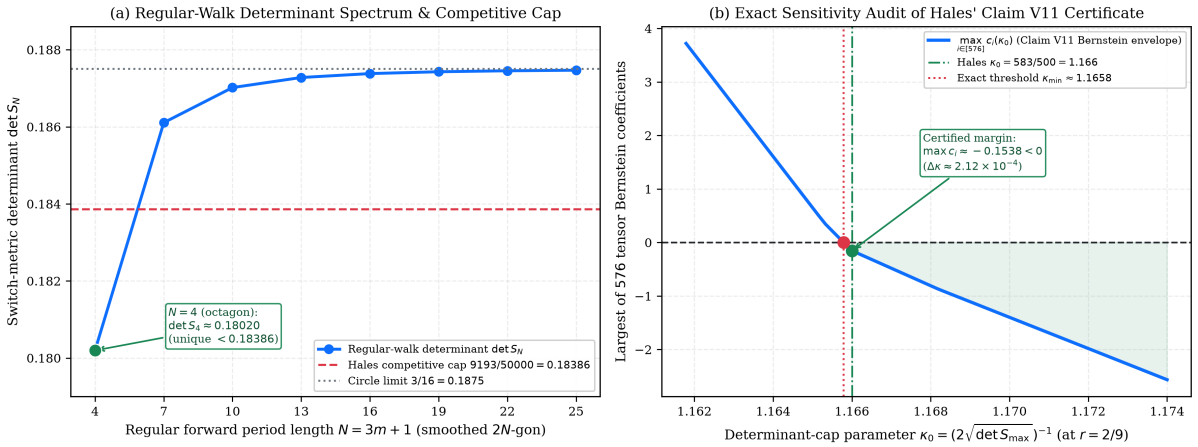

Figure 2: **(a)** Exact regular-walk determinant spectrum $\det S_N \uparrow 3/16$ ($N = 3m + 1$, `cert07`), showing $N = 4$ (smoothed octagon) as the unique walk below Hales' competitive cap 9193/50000. **(b)** Exact affine sensitivity of the largest of the 576 Claim V11 tensor Bernstein coefficients versus κ_0 (`cert08`–`cert09`), exposing the razor-sharp downward margin $\kappa_0 - \kappa_{\min} \approx 2.12 \times 10^{-4}$ (0.018%).

exactly in Theorem 3(a), `cert08`, Lean Theorems 13–14). Because $P_{\mathrm{V11}}(\kappa, r; t)$ is affine in κ (at fixed r_0) and affine in r (at fixed κ_0), each of the 576 Bernstein coefficients has the exact form $c_i(\kappa) = a_{0,i} + \kappa a_{1,i}$ (resp. $d_{0,i} + r d_{1,i}$). Solving all 576 exact linear inequalities $c_i(\kappa) < 0$ (resp. < 0 in r) over $\mathbb{Q}$ proves $a_{1,i} < 0$ on every active constraint (hence unbounded above) and yields the exact rational infima $\kappa_{\min}$ and $r_{\min}$ stated in Theorem 3(b), while verifying that all six auxiliary box guards in Hales' `check_winding.py` remain strictly satisfied at both $(\kappa_{\min}, r_0)$ and $(\kappa_0, r_{\min})$ (`cert09`, Figure 2(b)).

For Hales' Claim V6 [4, Section 11], the algebraic implication $P = 0 \implies M = 1,\ y = j = 0 \implies L^2 - B = \frac{1}{4} - \frac{1}{4} = 0$ shows that P divides $L^2 - B$ identically in $\mathbb{Q}[p_1, p_2, p_3, p_4, p_5]$, and coordinate degree inspection bounds the quotient $(L^2 - B)/P$ on $[0, 1]^5$ by exact multidegree $(8, 4, 4, 2, 3)$ ($9 \cdot 5 \cdot 5 \cdot 3 \cdot 4 = 2{,}700$ coefficients). Pure-rational tensor Bernstein inversion recovers all 2,700 coefficients strictly positive with exact minimum 23/768 and exact maximum 11/16 (`cert10`, Lean Theorem 15). □

6 Formal Verification and Reproducibility

All stability constants, interval brackets, and Bernstein certificates are verified by three complementary zero-trust artifacts with an explicit division of labor:

1. **Constructive Lean 4.36.0 proof suite** (`lean/ReinhardtQuantitativeStability.lean`): 20 theorems compiling under `lean -DwarningAsError=true` with zero external dependencies, zero `sorry`, zero `Classical.choice`, and zero custom axioms, formally proving all finite integer/rational algebraic identities, $\mathbb{Q}(\sqrt{2})$ normalizations, and guard/margin inequalities—namely the curvature and c_s identities (Theorems 1–7, 18), six-arc mixed convexity and linear slope negativity (Theorems 8–10), Xue's ceiling gap and Hales' determinant/parameter guards (Theorems 11–17), and Hales' turn-positivity identities (Theorems 19–20), using certified $\ln 2$ rational brackets from `cert01` and `cert04` where transcendental bounds enter.

2. **AST-audited zero-float verifier:** The script `verify_reinhardt_stability.py` inside `verification/` uses strictly `int` and `Fraction` (forbidding `float`, `math`, `sympy`, and

`limit_denominator`), certifying all transcendental interval evaluations (including regular walks to width $< 10^{-10}$) and sealing 16 SHA-256 JSON certificates (`cert01`–`cert16`), including the full 576-coefficient Claim V11, 2,700-coefficient Claim V6, and auxiliary-guard sensitivity reconstructions.

3. **Adversarial test suite** (`tests/test_adversarial.py`): 56 automated pytest checks auditing zero-float AST compliance, SHA-256 manifest integrity, interval tightness ($< 10^{-8}$ globally and $< 10^{-10}$ on Table 1), Bernstein matrix inversion, all 16 certificates, and Lean 4 axiom purity.

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