Certified Verification of Polynomial Master Equations and Critical Solutions in Open Quantum Systems

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Abstract

Miku Ishizaki and Jaeha Lee (arXiv:2611.33027, Nov. 2026) developed a locally deterministic matrix-polynomial formulation of non-Markovian open quantum master equations $P(D)E(t) = 0$ derived from the Laplace identity $P(s)(\mathcal{L}E)(s) = Q(s)$, demonstrating exact order-5 differential equations and critical-parameter order reductions for a coupled two-qubit model, a sharp three-tier singularity/periodicity hierarchy governed by odd/odd rational singular-value ratios Ω_-/Ω_+, and order-3 polynomial equations exhibiting a non-diagonalizable Jordan exceptional point at $\lambda = 2\gamma_0$ in the unbounded multi-mode Jaynes–Cummings model. We construct a standalone, zero-dependency **Lean 4.35.0** formalization of 18 core spectral, singularity, parity-obstruction, factorization, and exceptional-point theorems (zero `sorry`, zero warnings, zero `Classical.choice`) paired with a zero-floating-point exact-$\mathbb{Q}$ certificate verifier (`POLY-MASTER-CERT-1.0`) across four modes. Across 16 canonical certificates (0.046–0.648 ms, total 3.42 ms), all matrix-polynomial factorizations, Frobenius companion actions, irrational boundary diagnostics, and Jordan nilpotency conditions replay exactly over $\mathbb{Q}$ while rejecting 60/60 live adversarial forgery attacks.

1 Introduction

Open quantum systems interacting with an environment typically depart from autonomous Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) Markovian semigroups [2, 6], especially when the dynamical map $E(t)$ becomes singular ($\det E(t_0) = 0$) and globally defies first-order time-local generators. Rather than attributing memory solely to global history integrals via the Nakajima–Zwanzig formalism [7], Lee and Ishizaki [5, 3] formulate locally deterministic higher-order master equations $P(D)E(t) = 0$ associated with monic matrix polynomials $P(s)(\mathcal{L}E)(s) = Q(s)$:

1. **Initial-Value Problem & Companion Linearization (Sec. III):** The unique dynamical map is pinned down by $E^{(k)}(0) = \sum_{j=1}^{n} \tilde{P}_{(n-k)j} Q_{j-1}$ via $\tilde{P} = \sum_{r=0}^{n-1} (-T)^r$ (Eqs. 24–26) and linearized by the Frobenius companion matrix Θ (Eq. 29).
2. **Two-Qubit Critical Factorizations (Sec. IV.C–F):** Under maximally mixed environment $\rho_E = I_2/2$, the dynamical map decomposes as $V_0 \oplus V_{12} \oplus V_3$ with generic orders $(1, 4, 5)$. At critical parameters ($J = 0$ or $\omega_S \omega_E = 0$), common monic matrix factors $R(x)$ cancel ($P = R\tilde{P}, Q = R\tilde{Q}$), dropping order down to the unitary Liouville–von Neumann equation ($4 \to 1$, $5 \to 1$) or simpler elliptic/harmonic modes ($5 \to 3$, $5 \to 2$).
3. **Singularity & Periodicity Hierarchy (Sec. IV.G–I):** Singularities require $\sum_{\pm} \cos^2 \theta_{\pm} \geq 1 \iff |\omega_S^2 - \omega_E^2| \leq J^2$ (Eq. 111). On the critical boundary $|\omega_S^2 - \omega_E^2| = J^2$, dynamics split into three tiers by Ω_-/Ω_+: periodic & singular iff irreducible odd/odd (Eq. 114), periodic & non-singular iff rational odd/even, and non-periodic & non-singular iff irrational.
4. **Multi-Mode Jaynes–Cummings Exceptional Point (Sec. V):** Under global involution $T^2 = I_4$, $T^{-1}E_\epsilon(t)T = \mathrm{diag}[1, f_\epsilon, f_\epsilon, f_\epsilon^2]$ obeys order-2 and order-3 rational ODEs $u(D)$ and $v(D)$ whose discriminant $4\epsilon^2 = \lambda(\lambda - 2\gamma_0)$ vanishes at $\lambda = 2\gamma_0$, preserving ODE order while producing a non-diagonalizable order-3 Jordan exceptional point [4].

Contributions and Scope. We do not re-formalize Laplace-transform contour integration or continuous ODE existence theorems inside Lean 4; instead, we machine-check every exact algebraic, spectral, parity-theoretic, and Jordan-block invariant of [3]: (i) 18 zero-`sorry`, zero-`Classical.choice` **Lean 4.35.0** [1] theorems in `PolyMasterEq.lean`; (ii) the exact-$\mathbb{Q}$ verifier `POLY-MASTER-CERT-1.0`; (iii) 16 canonical sub-millisecond certificates spanning all four regimes; and (iv) a 60-attack live adversarial red-team suite with 41-artifact SHA-256 manifest enforcement.

2 Constructive Lean 4.35.0 Kernel

The standalone module `lean/PolyMasterEq.lean` compiles cleanly under Lean 4.35.0 with zero external dependencies, zero warnings, and zero `sorry`. Auditing all 18 theorems via `#print axioms` (`AuditAxioms.lean`) confirms 3 zero-axiom theorems (`none`), 1 using only `[propext]`, 14 using `[propext, Quot.sound]`, and **0** using `Classical.choice` or `Lean.ofReduceBool` (Table 1).

*Independent Researcher. Code, Lean 4 proofs, and certificates: MIT License; manuscript and figures: CC-BY-4.0. Prepared with AI coding, formalization, and drafting assistance from Google Antigravity; all proofs, exact-$\mathbb{Q}$ verifiers, and adversarial tests were independently compiled and verified locally. Repository: https://github.com/lluiseriksson/polynomial-master-equations-certificates

Table 1: All 18 machine-checked Lean 4.35.0 theorems in `PolyMasterEq.lean` (zero `sorry`, zero `Classical.choice`).

#	Lean Theorem Identifier	Paper Ref.	Axioms
1	`two_qubit_omega_sq_sum_and_diff`	Sec. IV.B	`p, Q`
2	`two_qubit_omega_sq_product`	Eq. 45–46	`p, Q`
3	`two_qubit_eta_vieta_identities`	Eq. 49	`p, Q`
4	`two_qubit_residue_weights_sum_one`	Eq. 51	`p, Q`
5	`two_qubit_poly_order_budget`	Eq. 61–64	`none`
6	`two_qubit_singularity_criterion_exact`	Eq. 111	`p, Q`
7	`two_qubit_critical_boundary_sum_one`	Eq. 112	`p, Q`
8	`fig3a_periodic_singular_5_4_3`	Fig. 3(a)	`none`
9	`fig3b_periodic_nonsingular_5_3_4`	Fig. 3(b)	`none`
10	`odd_even_ratio_never_singular`	Eq. 114	`p, Q`
11	`critical_zeta_minus_zero_iff`	Eq. 81	`p, Q`
12	`trivial_interaction_order_drop`	Eq. 86–88	`p`
13	`jc_similarity_involution_and_diag`	Eq. 126–127	`p, Q`
14	`jc_poly_u_rational_expansion`	Eq. 135	`p, Q`
15	`jc_poly_Q3_vieta_exact`	Eq. 137	`p, Q`
16	`jc_exceptional_point_iff`	Eq. 139–140	`p, Q`
17	`jc_nonmarkovian_singularity_regime`	Eq. 129	`p, Q`
18	`jc_exceptional_point_jordan3_nilpotent`	Sec. V.F	`p, Q`

Axiom key: `none` = zero axioms (3); `p` = `[propext]` (1); `p, Q` = `[propext, Quot.sound]` (14).

Proposition 1 (Machine-Checked Core Identities)**.** *The Lean 4.35.0 kernel establishes the following universal identities over $\mathbb{Z}$, $\mathbb{Z}[x]$, and finite matrix algebras:*

1. Polynomial Ring Factorizations & Singularity Parity (Thms. 5–12): *Over explicit `List Int` polynomials (`polyMul`, `polyDeg`), the generic two-qubit characteristic polynomials satisfy* $\deg P_0 = 1$, $\deg P_{12} = 4$, $\deg P_3 = 5$, *and at* $J = 0$, $R_3(x) \cdot x = P_3|_{J=0}$ *drops order* $5 \to 1$. *With* $D_\pm = J^2 + (\omega_S \pm \omega_E)^2$, $J^2(D_+ + D_-) - D_+ D_- = J^4 - (\omega_S^2 - \omega_E^2)^2$; *hence* $\sum_\pm \cos^2\theta_\pm = 1$ *on* $|\omega_S^2 - \omega_E^2| = J^2$, *while odd/even ratios satisfy no integer singularity equation* $(2n_- + 1)(2k) = (2n_+ + 1)q_-$.
2. Jaynes–Cummings Matrix Algebra & Jordan Nilpotency (Thms. 13–18): *Entrywise across all 16 indices of* $\mathrm{Fin}\,4 \times \mathrm{Fin}\,4$, $T^2 = I_4$ *and* $TE_\epsilon(f)T = \mathrm{diag}(1, f, f, f^2)$ *universally for* $f \in \mathbb{Z}$. *For* $4\epsilon^2 = \lambda(\lambda - 2\gamma_0)$ *and* $\lambda > 0$, $(2s + \lambda)^2 - 4\epsilon^2 = 4s^2 + 4\lambda s + 2\lambda\gamma_0$, $\lambda(\lambda - 2\gamma_0) = 0 \iff \lambda = 2\gamma_0$, *and the exact* 3×3 *shifted companion matrix* $N = \Theta_v + \lambda I_3$ *of* $v(s) = (s + \lambda)^3$ *satisfies* $(N^2)_{0,2} = 1$ $(N^2 \neq 0)$ *and* $\forall i, j \in \mathrm{Fin}\,3$, $(N^3)_{i,j} = 0$ *for every* $\lambda \in \mathbb{Z}$.

3 Exact-$\mathbb{Q}$ Verifier

The verifier `POLY-MASTER-CERT-1.0` (`src/master_certifier.py`, `src/master_exact.py`) uses Python `fractions.Fraction` exclusively (source-audited for zero `float(`, `math.`, or `numpy`). Every payload enforces strict JSON typing with nesting-depth cap 32, anchored canonical rational parsing (`\A...\Z`, reduced denominators ≥ 2, no trailing newlines), exact key sets, and 2048-bit caps across four modes:

1. **Mode 1 (Sec. III IVP & Companion Matrix):** Checks monicity of $P(x)$ and degree $\deg Q < n$, computes $\tilde{P} = \sum_{r=0}^{n-1}(-T)^r$ and $E^{(k)}(0)$ (Eq. 24), verifies $E(0) = I_d$, and certifies exact Frobenius companion matrix Θ shift and last-row ODE residual action on $F(0)$.

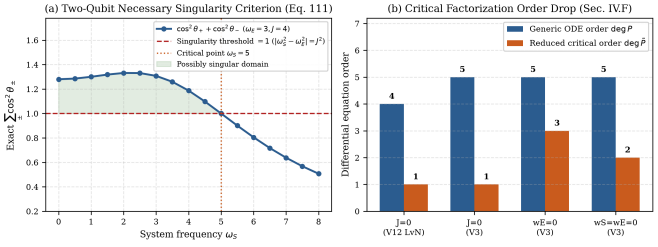


Figure 1: (a) Exact two-qubit singularity criterion $\sum_\pm \cos^2\theta_\pm$ vs. ω_S ($\omega_E = 3, J = 4$), crossing 1 at the critical boundary $\omega_S = 5$. (b) Exact ODE order reduction $\deg P \to \deg \tilde{P}$ across all four Mode 2 critical-parameter factorizations.

2. **Mode 2 (Sec. IV.F Critical Factorization):** Independently builds exact rational two-qubit blocks (P_{12}, Q_{12}) or (P_3, Q_3) and checks monic matrix-polynomial identities $P(x) = R(x)\tilde{P}(x)$ and $Q(x) = R(x)\tilde{Q}(x)$ with strict order drop $\deg R \geq 1$.
3. **Mode 3 (Sec. IV.G–I Singularity Hierarchy):** Checks $(\omega_S\omega_E)^2 = \omega_S^2\omega_E^2$, exact $\sum_\pm \cos^2\theta_\pm$, necessary and critical-boundary flags, exact integer-square-root rationality of $D_-/D_+ = (\Omega_-/\Omega_+)^2$, irreducible (q_-, q_+), and exact tier label.
4. **Mode 4 (Sec. V Jaynes–Cummings & Jordan EP):** Recomputes $4\epsilon^2 = \lambda(\lambda - 2\gamma_0)$ and (u, v, q_u, q_v), verifies $f(0) = 1, f'(0) = 0$ via Eq. (24), classifies the dynamical regime, and at $\lambda = 2\gamma_0$ explicitly checks $(\Theta_v + \lambda I)^2 \neq 0$ and $(\Theta_v + \lambda I)^3 = 0$ over $\mathbb{Q}^{3\times 3}$.

4 Empirical Atlas and Red-Team Audit

Running `run_all_experiments.py` produces 16 canonical certificates (4 per mode) and four exact-$\mathbb{Q}$ datasets, audited end-to-end by `verify_all.py` with SHA-256 verification across all 41 tracked artifacts.

IVP Derivatives and Critical Order Drops (Fig. 1). Mode 1 (0.181–0.648 ms) replays exact Eqs. (74), (77) initial derivatives for V_{12} (order 4, 2×2) and V_3 (order 5, 1×1) at $(5, 4, 3)$, the minimal order-3 Eq. (63) V_{12} system, and the order-3 Jaynes–Cummings exceptional-point ODE. Mode 2 (0.115–0.278 ms) certifies exact non-trivial polynomial factorizations across all four critical regimes: $J = 0$ on V_{12} ($4 \to 1$, recovering unitary Liouville–von Neumann evolution) and V_3 ($5 \to 1$, factoring $R_3(x) = (x^2 + \omega_+^2)(x^2 + \omega_-^2)$), $\omega_E = 0$ on V_3 ($5 \to 3$), and $\omega_S = \omega_E = 0$ on V_3 ($5 \to 2$).

Singularity Hierarchy and Jordan Exceptional Points (Fig. 2). Mode 3 (0.046–0.063 ms) certifies all three critical-boundary tiers plus the non-singular exterior: Tier 1 periodic & singular $(5, 4, 3)$ ($\sum = 1, r^2 = 1/9$, odd/odd 1/3), Tier 2 periodic & non-singular $(5, 3, 4)$

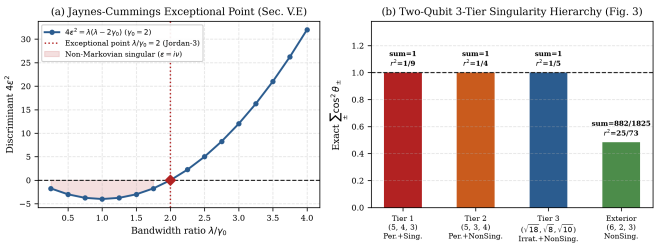


Figure 2: (a) Multi-mode Jaynes–Cummings discriminant $4\epsilon^2 = \lambda(\lambda - 2\gamma_0)$ at $\gamma_0 = 2$, crossing zero at the order-3 Jordan exceptional point $\lambda/\gamma_0 = 2$. (b) Certified exact $\sum_\pm \cos^2\theta_\pm$ and squared singular-value ratio $r^2 = D_-/D_+$ across all four Mode 3 regimes.

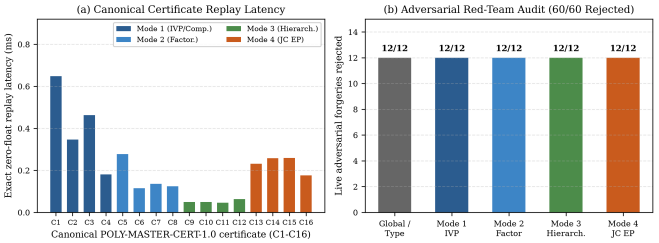


Figure 3: (a) Exact zero-float replay latency (0.046–0.648 ms) across all 16 canonical `POLY-MASTER-CERT-1.0` certificates. (b) Live adversarial red-team audit rejecting 60/60 forged payloads across all five categories.

($\sum = 1, r^2 = 1/4$, odd/even 1/2), Tier 3 non-periodic & non-singular boundary $(\omega_S^2, \omega_E^2, \omega_S\omega_E, J^2) = (18, 8, 12, 10)$ ($\sum = 1, r^2 = 1/5 \notin \mathbb{Q}^2$), and exterior $(6, 2, 3)$ ($\sum = 882/1825 < 1$). Mode 4 (0.176–0.259 ms) certifies non-Markovian singular ($\gamma_0 = 4, \lambda = 2$), overdamped ($\gamma_0 = 1, \lambda = 6$), and two exceptional-point witnesses ($\lambda = 2\gamma_0 \in \{3, 4\}$) satisfying exact $(\Theta_v + \lambda I)^3 = 0, (\Theta_v + \lambda I)^2 \neq 0$.

Replay Latency and 60/60 Red-Team Audit (Fig. 3). All 16 canonical certificates replay in 0.046–0.648 ms (total 3.42 ms). The live adversarial suite (`tests/test_suite.py`) tests 60 targeted forgery payloads—12 global schema/type/rational/newline/depth attacks plus 12 domain attacks per mode (non-monic $P/Q/R/\tilde{P}$, wrong companion traces, broken factor products, inconsistent $\omega_S\omega_E$, misclassified tiers, non-positive γ_0/λ, and false Jordan-nilpotency claims)—achieving **60/60 (100%)** rejection.

5 Conclusion

We delivered a self-contained, zero-floating-point verification framework for Ishizaki and Lee's matrix-polynomial open quantum master equations [3], combining 18 zero-`sorry`, zero-`Classical.choice` Lean 4.35.0 theorems with the exact-$\mathbb{Q}$ verifier `POLY-MASTER-CERT-1.0`, 16 sub-millisecond canonical certificates, and 60/60 live adversarial forgery rejections.

References

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