

Interior-zero complexity of polynomial differentiation residuals

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Abstract

We classify the numbers of interior zeros that permit vanishing and superalgebraic differentiation residuals for polynomials whose zeros lie in the closed unit disk. If a degree- n polynomial has exactly M strictly interior zeros, then, for every complex slope, its normalized H^2 residual is at least $n/[2\sqrt{2}16^M(n+2M+4)^{2M+2}]$. Consequently a prescribed budget b_n permits a superalgebraically small residual sequence if and only if $b_n \rightarrow \infty$. Sufficiency holds with real coefficients, normalization at the origin, the slope fixed by that normalization, and exactly b_n interior zeros. In contrast, one interior zero already permits residuals $O(n^{-1/6})$, with family-specific asymptotic constant $\sqrt{15}/4^{4/3}$, whereas the circle-only infimum tends to $1/\sqrt{8}$. For one interior zero we also prove a universal $cn^{-13/6}$ lower bound; the optimal rate remains open. The finite-budget obstruction combines a reciprocal identity with classical Mahler, truncated-shift and Poisson-kernel estimates, and extends to a stable factor times a self-inversive cofactor. For unrestricted interior counts we prove $\exp(-C\sqrt{n})$ lower bounds for every $C > \sqrt{(8/3)\log 17}$. The upper construction corrects both the exact slope and degree rounding at every large degree. The resulting refinement of exterior power-sum bounds uses a known residual method; the negative answer to Erdős Problem 973 was already known. No matching fixed-budget rates or stretched-exponential upper bounds are claimed.

1 Statement and relation to previous work

All logarithms are natural. The measure on the unit circle is normalized to have mass one. Thus for $P(z) = \sum_{j=0}^n a_j z^j$,

$$\|P\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^2 dt = \sum_{j=0}^n |a_j|^2, \quad \|P\|_\infty = \max_{|z|=1} |P(z)|.$$

Write $\mathcal{S}_n$ for the nonzero polynomials of exact degree n whose zeros are in $\overline{\mathbb{D}}$, including multiplicity and boundary zeros. Define

$$s_n = \inf_{\substack{P \in \mathcal{S}_n \\ \lambda \in \mathbb{C}}} \frac{\|P' - \lambda P\|_2}{\|P\|_2}.$$

There is no normalization at the origin in this definition.

For an individual nonzero polynomial write $\rho_\lambda(P) = \|P' - \lambda P\|_2 / \|P\|_2$. Our principal classification concerns the number of strictly interior zeros; all root counts include multiplicity.

Theorem 1 (Finite interior-root budget). *Let $P \in \mathcal{S}_n$ have exactly M zeros in $\mathbb{D}$, where $n \geq 1$ and $0 \leq M \leq n$. For every $\lambda \in \mathbb{C}$,*

$$\rho_\lambda(P) \geq \frac{n}{2\sqrt{2}16^M(n+2M+4)^{2M+2}}. \quad (1)$$

In particular, for $n \geq 2M+4$, $\rho_\lambda(P) \geq 2^{-6M-4}n^{-2M-1}$. Neither the exponent nor the constant is asserted to be sharp.

Theorem 2 (Exact criterion for a growing budget). *Let (b_n) be any integer sequence with $0 \leq b_n \leq n$. The following are equivalent:*

1. $b_n \rightarrow \infty$.
2. *There exist $P_n \in \mathcal{S}_n$, with at most b_n strictly interior zeros, and complex slopes λ_n , such that $\rho_{\lambda_n}(P_n) = o(n^{-A})$ for every fixed $A > 0$.*
3. *There exist real-coefficient $P_n \in \mathcal{S}_n$, with $P_n(0) = 1$ and exactly b_n strictly interior zeros, such that $\rho_{P'_n(0)}(P_n) = o(n^{-A})$ for every fixed $A > 0$.*

For residuals merely tending to zero under the “at most” budget, the necessary and sufficient condition is instead $b_n \geq 1$ eventually, with the same choice of restricted slope available for sufficiency.

No monotonicity of the budget is assumed. The sufficiency argument is existential: no explicit quantitative thresholds for an arbitrary budget are supplied, and extra interior zeros may lie arbitrarily close to the circle. Section 7 proves these statements and gives a necessary count for stretched-exponential decay. Their ingredients include classical multiplication and restricted Jensen estimates [8, 9, 10], and the logarithmic derivative of a finite Blaschke product [11, Lemma 4.3]. These tools themselves are not new.

Theorem 3. *Put $C_0 = \sqrt{(8/3)\log 17}$. For every $C > C_0$ there is N_C such that*

$$s_n \geq e^{-C\sqrt{n}} \quad (n \geq N_C). \quad (2)$$

The following explicit bound also holds:

$$s_n \geq e^{-3\sqrt{n}} \quad (n \geq 10^4). \quad (3)$$

For comparison, Tan, Wang, Huang and Chen [1, Theorem 1.1] considered the restricted infimum

$$r_n = \inf_{\substack{P \in \mathcal{S}_n \\ P(0)=1}} \frac{\|P' - P'(0)P\|_2}{\|P\|_2}$$

and proved $r_n \geq \exp(-(1+o(1))\sqrt{n}\log n)$. For an individual normalized polynomial we write $\rho(P) = \|P' - P'(0)P\|_2/\|P\|_2$. Theorem 5 identifies the positive limiting residual when every root lies on the circle. Theorem 13 shows that one interior root suffices to make it vanish, establishing the minimal zero-count threshold. These statements concern the global weighted boundary norm, including the exact restricted slope.

Since $s_n \leq r_n$, Theorem 3 strengthens that quantitative estimate and allows an arbitrary scalar slope. Their maximum-point normalization, segment integration and projected Newton identity are retained here, with attribution. The two changes are a local analytic bound for the logarithmic derivative and a bounded test polynomial. These remove the logarithmic factor in the exponent.

Luo, Yang and Zhu [2, Theorem 1.1] had already proved that exterior power sums cannot be exponentially small in the degree, answering Erdős Problem 973 negatively. The application in Section 5 is a quantitative refinement of that established conclusion. No new solution of the original yes-or-no problem is asserted.

Theorem 4 (Superalgebraic decay). *For every $A > 0$, there are constants C_A, N_A such that*

$$r_n \leq C_A n^{-A} \quad \text{for every integer } n \geq N_A.$$

The polynomials can be chosen with real coefficients, exactly M_A zeros strictly inside the unit disk, where M_A is independent of n , and all other zeros on its boundary. The residual of these polynomials themselves, with the restricted slope $P'_n(0)$, is at most $C_A n^{-A}$. The interior zeros approach 1 with distance and radial gap $\Theta(n^{-1/3})$. Together with Theorem 3, this implies superalgebraic decay of both s_n, r_n , while $s_n^{1/n}, r_n^{1/n} \rightarrow 1$.

The proof in Section 6 fixes the cancellation order and finite integer multiplicities first, then lets the reciprocal root scale grow as $n^{1/3}$. No explicit quantitative bounds are supplied here on M_A, C_A, N_A or uniform control as A grows. Section 8 explains why a fixed strict root radius precludes superalgebraic decay of a single sequence.

Approximation by logarithmic derivatives and finite moment realization have substantial prior history. Komarov [3] obtains geometric approximation on each fixed subdisk by logarithmic derivatives of circle-root polynomials; [4] also realizes prescribed initial moments with unit weights. Those interior estimates do not supply the global weighted boundary norm used here, with a target slope growing with degree. For example, $1 + z^n$ approximates zero logarithmically on subdisks but has restricted residual $n/\sqrt{2}$. The present contribution is the combination of exact slope, global concentration and correction at every large degree, rather than moment realization alone.

2 The sharp barrier when every zero is on the circle

Let s_n° and r_n° denote the same infima as s_n, r_n , with every zero required to lie on $|z| = 1$.

Theorem 5. *The unrestricted and restricted circle-root residuals satisfy*

$$\lim_{n \rightarrow \infty} s_n^\circ = \lim_{n \rightarrow \infty} r_n^\circ = \frac{1}{\sqrt{8}}.$$

The lower asymptotic bound also holds in the larger class of self-inversive polynomials. No finite-degree extremizer is asserted.

Proof. We first prove the lower bound, allowing an arbitrary complex slope. Rotating the variable makes this slope real and nonnegative, denoted λ . Self-inversivity permits a constant phase choice such that

$$P(e^{it}) = e^{int/2} p(t), \quad p(t) \in \mathbb{R}, \quad p(t + 2\pi) = (-1)^n p(t).$$

These operations preserve the unrestricted normalized residual. The familiar phase derivative $n/2$ for circle-root polynomials appears explicitly in [5, (1.3)–(1.4)]; here we retain the additional scalar subtraction. Direct differentiation gives

$$P' - \lambda P = e^{i(n/2-1)t} [(n/2 - \lambda \cos t)p - i(p' + \lambda \sin t p)].$$

Integration of the cross term by parts is legitimate because p^2 is periodic. Consequently the squared residual is exactly

$$\frac{\int_0^{2\pi} [p'^2 + (n^2/4 + \lambda^2 - (n+1)\lambda \cos t)p^2] dt}{\int_0^{2\pi} p^2 dt}. \quad (4)$$

Write $E(L)$ for the lowest periodic eigenvalue of $-d^2/dt^2 - L \cos t$, for $L \geq 0$. In odd degree, $|p|$ is periodic in H^1 and has the same quadratic-form values as p . Thus in both parities (4) is at least

$$g_n(\lambda) = n^2/4 + \lambda^2 + E((n+1)\lambda).$$

This is a variational lower bound, not an identification of the periodic and antiperiodic spectra.

The change of variables $t = 2x + \pi$ gives $E(L) = a_0(2L)/4$, in the standard Mathieu convention $-d^2/dx^2 + 2q \cos(2x)$; see [7, §28.2]. The use of this ground state in circular variational problems is classical; compare [6, §VI.A.2]. The fixed-index expansion [7, (28.8.1), $m = 0, s = 1$] is

$$a_0(q) = -2q + 2\sqrt{q} - \frac{1}{4} + O(q^{-1/2}), \quad E(L) = -L + \sqrt{L/2} - \frac{1}{16} + O(L^{-1/2}).$$

We must minimize uniformly over λ , rather than applying this expansion only at a fixed shift. The variational principle gives $E(L) \geq -L$ and Lipschitz continuity, so g_n is continuous and coercive, with

$$g_n(\lambda) \geq (\lambda - (n+1)/2)^2 - (2n+1)/4.$$

The trial value $\lambda = n/2 + 1/4$ gives $g_n = 1/8 + O(n^{-1})$. Every minimizer therefore has the form $n/2 + c_n$, where $c_n = O(\sqrt{n})$. On this window $(n+1)\lambda \asymp n^2$, and the displayed single-variable Mathieu remainder is uniformly $O(n^{-1})$. Expanding its square root uniformly on the same window yields

$$g_n(n/2 + c) = (c - 1/4)^2 + \frac{1}{8} + O((c^2 + 1)/n).$$

The upper bound on the minimum and absorption of the c_n^2/n term imply $c_n = O(1)$. Applying the expansion again proves $\inf_{\lambda \geq 0} g_n(\lambda) = 1/8 + O(n^{-1})$. This supplies the required lower limit, also for the restricted slope.

For the upper bound, write $n = 6m + k$, $0 \leq k \leq 5$, and for $m \geq 3$ choose

$$\begin{aligned} c_{m,k} &= \cos t_{m,k} = \frac{1}{2} + \frac{2k-1}{8m}, \\ B_t(z) &= (1+z)^4(1-2\cos t z + z^2), \\ P_{m,k}(z) &= B_{t_{m,k}}(z)^m(1+z)^k. \end{aligned}$$

All zeros lie on the circle, $P_{m,k}(0) = 1$, and

$$P'_{m,k}(0) = n/2 + 1/4, \quad P'_{m,k}(1)/P_{m,k}(1) = n/2.$$

The limiting factor is $B_* = (1+z)^4(1-z+z^2)$. With $s = \sin^2(\theta/2)$, its squared normalized modulus is $\cos^6(\theta/2)\cos^2(3\theta/2)$, with a unique maximum at $\theta = 0$, and its logarithm is $-3\theta^2 + O(\theta^4)$. More explicitly, for the perturbed factor this modulus is

$$H_c(s) = (1-s)^4 \left(1 - \frac{2s}{1-c}\right)^2.$$

Its secondary maximum has value $(1+c)^6/[729(1-c)^2] < 1$ for $0 \leq c \leq 3/4$. All six parameter sequences eventually lie in this interval. Local negative curvature and the strict outer gap are uniform as $c \rightarrow 1/2$. They give Gaussian domination near zero and an exponentially small outer tail.

For $F_{m,k} = P'_{m,k}/P_{m,k} - P'_{m,k}(0)$, the identity

$$B'_*/B_* - 3 = -\frac{3z(z-1)^2}{z^3+1}$$

shows, on putting $\theta = u/\sqrt{m}$, that

$$F_{m,k}(e^{i\theta}) \longrightarrow -\frac{1}{4} + \frac{3}{2}u^2, \quad |P_{m,k}(e^{i\theta})/P_{m,k}(1)|^2 \longrightarrow e^{-3u^2}.$$

Indeed the constant term of $F_{m,k}$ at 1 is exactly $-1/4$, its linear term is $O(1)$, and its quadratic term is $-3m/2 + O(1)$. A fixed zero-free neighborhood gives the local bound $C(1 + |\theta| + m\theta^2)$; after rescaling, a polynomial times a Gaussian dominates. On the outer interval, differentiate the polynomial product, removing one factor. Uniformly bounded normalized factor derivatives give an outer squared numerator $O(m^2 q^{2m-2})$, for some $q < 1$. Thus no estimate at a pole of the logarithmic derivative is required. The ratio of the complete integrals tends to

$$\frac{\int_{\mathbb{R}} (-1/4 + 3u^2/2)^2 e^{-3u^2} du}{\int_{\mathbb{R}} e^{-3u^2} du} = \frac{1}{16} - \frac{3}{4} \cdot \frac{1}{6} + \frac{9}{4} \cdot \frac{1}{12} = \frac{1}{8}.$$

Uniformity over the six values of k covers every sufficiently large degree. This proves the upper bound for r_n° , and $s_n^\circ \leq r_n^\circ$ completes both limits. $\square$

3 A finite residual certificate

We first give conditions that can be checked at a fixed degree. The asymptotic and explicit choices will then be elementary.

Lemma 6 (Normalization and separation). *Suppose $P \in \mathcal{S}_n$, $\lambda \in \mathbb{C}$, $y > 0$, and*

$$\frac{\|P' - \lambda P\|_2}{\|P\|_2} \leq e^{-y}, \quad \sqrt{n+1} e^{-y} \leq 1.$$

Set

$$\gamma_n = 2 + \log(2\sqrt{n+1}), \quad \eta = \frac{y - \gamma_n}{n}. \quad (5)$$

If $\eta > 0$, there are numbers $v_1, \dots, v_n$ and moments $m_k = n^{-1} \sum_j v_j^k$ such that

$$m_1 \in \mathbb{R}, \quad m_1 \geq \frac{1}{2}, \quad \operatorname{Re} v_j \geq \frac{1}{2}, \quad |v_j|^2 \leq \frac{\operatorname{Re} v_j}{\eta}. \quad (6)$$

They are the reciprocals $v_j = (1 - \beta_j)^{-1}$ of the complex displacements from the zeros to a normalized maximum point.

Proof. This is the normalization and separation argument of [1, Sections 2.1–2.2], with the scalar slope left arbitrary. Choose $|\tau| = 1$ at a maximum of $|P|$, and put

$$G(w) = \frac{P(\tau w)}{P(\tau)}, \quad \mu = \tau\lambda, \quad E = G' - \mu G.$$

The denominator is nonzero because P is nonzero. We have $G(1) = \|G\|_\infty = 1$, $\|G\|_2 \leq 1$, and $\|E\|_2 \leq e^{-y}$. Differentiating $|G(e^{it})|^2$ at its maximum shows that $d = G'(1)$ is real. Bernstein's polynomial inequality gives $|d| \leq n$. The coefficient bound $\|E\|_\infty \leq \sqrt{n+1}\|E\|_2$ therefore yields

$$|\mu - d| = |E(1)| \leq 1, \quad |\mu| \leq n + 1. \quad (7)$$

Let β be any zero of G , and set $\delta = 1 - \operatorname{Re} \beta$. Integrating the exact ODE along the segment from β to 1 gives

$$1 = \int_\beta^1 e^{\mu(1-w)} E(w) dw.$$

That segment lies in $\overline{\mathbb{D}}$. On it, $\operatorname{Re}(\mu(1-w)) \leq n\delta + 2$, using $d \in \mathbb{R}$, $|d| \leq n$, $|\mu - d| \leq 1$, and $|1 - \beta| \leq 2$. Consequently

$$1 \leq 2\sqrt{n+1} e^{n\delta+2-y}, \quad \delta \geq \eta.$$

No zero equals 1, since $G(1) = 1$. For $v = (1 - \beta)^{-1}$,

$$\operatorname{Re} v = \frac{1 - \operatorname{Re} \beta}{|1 - \beta|^2} \geq \frac{1}{2}, \quad \frac{|v|^2}{\operatorname{Re} v} = \frac{1}{1 - \operatorname{Re} \beta} \leq \frac{1}{\eta}.$$

Finally $nm_1 = G'(1)/G(1) = d$, so m_1 is real and at least $1/2$. □

Lemma 7 (Local analytic estimate). *With the normalization above, let $r > 0$ and suppose*

$$q = r\sqrt{n+1} \exp(-y + (2n+1)r) \leq \frac{1}{2}. \quad (8)$$

Then G has no zero in $|w - 1| \leq r$, and, for every integer $k \geq 2$,

$$|m_k| \leq \frac{2\sqrt{n+1}}{n} \exp(-y + (2n+1)r) r^{1-k}. \quad (9)$$

Proof. The disk centered at 1 can extend beyond $\mathbb{D}$; the needed control there comes from the coefficients. For $|u| \leq r$,

$$|E(1+u)| \leq \sqrt{n+1}(1+r)^n \|E\|_2 \leq \sqrt{n+1} e^{-y+nr}.$$

Solving the ODE from $G(1) = 1$ gives

$$G(1+u) = e^{\mu u}(1+h(u)), \quad h(u) = \int_0^u e^{-\mu s} E(1+s) ds.$$

Using the straight segment and (7), we obtain $|h(u)| \leq q \leq 1/2$. Thus G is nonzero on that closed disk and on a neighborhood of it. The analytic function $F = G'/G - \mu = E/G$ satisfies

$$|F(1+u)| \leq 2\sqrt{n+1} e^{-y+(2n+1)r} \quad (|u| \leq r).$$

Cauchy's derivative estimate and logarithmic differentiation give

$$\frac{|F^{(k-1)}(1)|}{(k-1)!} \leq 2\sqrt{n+1} e^{-y+(2n+1)r} r^{1-k}, \quad F^{(k-1)}(1) = (-1)^{k-1} (k-1)! nm_k.$$

This proves (9). There is no restriction $k \leq n+1$: F is an analytic rational function on the specified disk. $\square$

Proposition 8 (Finite certificate). *Fix $n \geq 1$, $y, r > 0$, and an integer $K \geq 2$. Define η by (5) and set $a = 2\eta$. Suppose $0 < \eta \leq 2$, $\sqrt{n+1}e^{-y} \leq 1$, and (8) holds. Put*

$$B = \frac{2\sqrt{n+1}r}{Kan} \exp\left(-y + (2n+1)r + K \log\left(1 + \frac{a}{r}\right)\right). \quad (10)$$

If

$$\frac{1}{K\eta} + B < \frac{1}{2}, \quad (11)$$

then $s_n \geq e^{-y}$.

Proof. Suppose an admissible P, λ had residual at most e^{-y} . For the numbers in Lemma 6,

$$|1 - av_j|^2 = 1 - 2a \operatorname{Re} v_j + a^2 |v_j|^2 \leq 1.$$

Use the bounded Cesàro test

$$\Psi(v) = \frac{1 - (1 - av)^K}{Ka} = v + \sum_{k=2}^K \frac{(-1)^{k+1}}{K} \binom{K}{k} a^{k-1} v^k. \quad (12)$$

It has zero constant coefficient, linear coefficient one, and $|\Psi(v_j)| \leq 2/(Ka) = 1/(K\eta)$. By Lemma 7, its remaining averaged terms have modulus at most

$$\frac{2\sqrt{n+1}r}{Kan} e^{-y+(2n+1)r} \left[\left(1 + \frac{a}{r}\right)^K - 1 - \frac{Ka}{r} \right] \leq B.$$

The triangle inequality and $m_1 \geq 1/2$ now imply

$$\frac{1}{2} \leq m_1 \leq \left| \frac{1}{n} \sum_j \Psi(v_j) \right| + B \leq \frac{1}{K\eta} + B,$$

contrary to (11). Each individual residual is therefore larger than e^{-y} ; taking its infimum gives the asserted non-strict bound. $\square$

4 Choice of parameters

Proof of (2). Fix $C > C_0$, and choose $t > 0$ so small that

$$C^2 > \frac{4}{3}(2+t) \log 17.$$

Set

$$y = C\sqrt{n}, \quad r = \frac{y}{8n}, \quad \eta = \frac{y - \gamma_n}{n}, \quad K = \left\lceil \frac{2+t}{\eta} \right\rceil.$$

For all sufficiently large n , the preliminary conditions of Proposition 8 hold: $\eta \sim C/\sqrt{n}$, $q \rightarrow 0$, and $\sqrt{n+1}e^{-y} \rightarrow 0$. Also $K\eta \geq 2+t$, leaving a fixed positive margin in (11). We have

$$K \sim \frac{2+t}{C}\sqrt{n}, \quad \frac{a}{r} \rightarrow 16,$$

and hence

$$\frac{-y + (2n+1)r + K \log(1 + a/r)}{\sqrt{n}} \rightarrow -\frac{3C}{4} + \frac{2+t}{C} \log 17 < 0.$$

The prefactor in (10) is $O(n^{-1})$. Thus $B \rightarrow 0$, and the finite certificate applies. $\square$

Proof of (3). Take $n \geq 10^4$ and choose

$$y = 3\sqrt{n}, \quad r = \frac{3}{8\sqrt{n}}, \quad K = \left\lceil \frac{21}{10\eta} \right\rceil.$$

First,

$$\gamma_n \leq \frac{3}{20}\sqrt{n}, \quad \frac{19}{20} \frac{y}{n} \leq \eta \leq \frac{y}{n}. \quad (13)$$

Indeed $\gamma_n/\sqrt{n}$ decreases for $n \geq 10^4$, since the sign of its derivative is that of $n/(n+1) - \gamma_n < 0$. At $n = 10^4$, $\gamma_n < 10 < 15$: $e^8 > 2^8 = 256 > 2\sqrt{10001}$ suffices for the first inequality. In particular $0 < \eta < 2$. The condition $\sqrt{n+1}e^{-y} < 1$ is immediate, and

$$q = \frac{3}{8} \sqrt{1 + \frac{1}{n}} \exp\left(-\frac{9}{4}\sqrt{n} + \frac{3}{8\sqrt{n}}\right) < \frac{1}{2}.$$

Moreover $K\eta \geq 21/10$, so

$$\frac{1}{2} - \frac{1}{K\eta} \geq \frac{1}{42}.$$

From (13) we obtain $a/r \leq 16$ and $K \leq (14/19)\sqrt{n} + 1$. Using $\log 17 < 3$,

$$\begin{aligned} -y + (2n+1)r + K \log(1 + a/r) &\leq \left(-\frac{9}{4} + \frac{14}{19} \log 17\right) \sqrt{n} + \log 17 + \frac{3}{8\sqrt{n}} \\ &< -\frac{3}{76}\sqrt{n} + 3 + \frac{3}{8\sqrt{n}} < 0. \end{aligned}$$

The last expression decreases with n and is negative at 10^4 . For an elementary check of $\log 17 < 3$, the exponential series through degree five gives $e^3 > 18 > 17$. Finally $Ka \geq 21/5$, and therefore

$$\frac{2\sqrt{n+1}r}{Kan} \leq \frac{5}{28n} \sqrt{1 + \frac{1}{n}} < \frac{1}{5n}.$$

It follows that $B < 1/(5n) \leq 1/50000 < 1/42$. All conditions of Proposition 8 are satisfied. $\square$

Remark 9. The constants 3 and 10^4 are convenient explicit choices. The same proof with $r = sy/n$, $0 < s < 1/2$, gives the sufficient asymptotic condition

$$C^2 > \frac{2 \log(1 + 2/s)}{1 - 2s}.$$

We use $s = 1/8$ to state a simple constant. Neither this expression nor C_0 is asserted to be the optimal residual constant.

5 Exterior power sums and Hardy norms

Corollary 10. *Let $|z_j| \geq 1$, $1 \leq j \leq n$, and write $M_n = \max_{2 \leq k \leq n+1} |\sum_j z_j^k|$. Then*

$$M_n \geq \frac{1}{n} e^{-3\sqrt{n}} \quad (n \geq 10^4).$$

For every $C > C_0$, uniformly over all such configurations, $M_n \geq n^{-1} e^{-C\sqrt{n}}$ for all sufficiently large n . No distinguished point $z_1 = 1$ is needed.

Proof. We use the projected Newton identity of [1, Proposition 3.1]. Put $P(t) = \prod_{j=1}^n (1 - z_j t)$ and $p_k = \sum_j z_j^k$. Then $P \in \mathcal{S}_n$, $P(0) = 1$, and $P'(0) = -p_1$. Logarithmic differentiation near zero, followed by coefficient projection onto degrees at most n , gives exactly

$$P' - P'(0)P = -\Pi_{\leq n} \left[\left(\sum_{k=2}^{n+1} p_k t^{k-1} \right) P \right].$$

Projection is contractive in H^2 , and multiplication by a polynomial S has H^2 norm at most $\|S\|_\infty$. Thus

$$s_n \leq \frac{\|P' - P'(0)P\|_2}{\|P\|_2} \leq \sum_{k=2}^{n+1} |p_k| \leq nM_n.$$

Apply Theorem 3. □

The factor $1/n$ is retained in these finite statements. It can be absorbed in a leading square-root exponent with any strictly larger constant. This corollary strengthens the scale in [1]; the earlier qualitative conclusion of [2] remains fully credited.

Corollary 11. *Uniformly for $1 \leq p \leq \infty$, $P \in \mathcal{S}_n$, and $\lambda \in \mathbb{C}$,*

$$\frac{\|P' - \lambda P\|_{H^p}}{\|P\|_{H^p}} \geq \frac{e^{-3\sqrt{n}}}{\sqrt{n+1}} \quad (n \geq 10^4).$$

For every $C > C_0$, the lower bound $e^{-C\sqrt{n}}$ holds for all sufficiently large n , uniformly over the same parameters.

Proof. Write $R = P' - \lambda P$, of degree at most n . For $1 \leq p \leq 2$, bounding each Fourier coefficient by $\|R\|_1$ gives

$$\|R\|_2 \leq \sqrt{n+1} \|R\|_1 \leq \sqrt{n+1} \|R\|_p, \quad \|P\|_2 \geq \|P\|_p.$$

For $p \geq 2$, use

$$\|R\|_2 \leq \|R\|_p, \quad \|P\|_p \leq \|P\|_\infty \leq \sqrt{n+1} \|P\|_2.$$

In both cases $\|R\|_p / \|P\|_p \geq (\|R\|_2 / \|P\|_2) / \sqrt{n+1}$. For the asymptotic statement, first apply Theorem 3 with a constant strictly between C_0 and C , and absorb $\frac{1}{2} \log(n+1) = o(\sqrt{n})$. □

6 Superalgebraic upper bounds

We prove the upper statement by cancelling a prescribed finite number of logarithmic-derivative terms at a strict global maximum. All constants in this section may depend on the prescribed cancellation order K . They are not asserted to be uniform as $K \rightarrow \infty$.

6.1 A finite positive moment cone

Lemma 12 (A finite positive moment cone). *Fix $K \geq 3$. For a rational point $\zeta = x + iy$, $x > 0$, $y \geq 0$, define the real column*

$$V(\zeta) = \begin{cases} (\zeta^2, \dots, \zeta^K)^T, & y = 0, \\ (2 \operatorname{Re} \zeta^2, \dots, 2 \operatorname{Re} \zeta^K)^T, & y > 0. \end{cases}$$

There is a finite collection of these columns whose cone is $\mathbb{R}^{K-1}$. The collection can contain any prescribed K distinct positive rational real points. Its column matrix A_∞ admits a strictly positive rational vector α^0 satisfying

$$A_\infty \alpha^0 = (0, 3/2, 0, \dots, 0)^T.$$

Proof. (a) *The dual cone is zero.* Let $a = (a_2, \dots, a_K)$ be a real vector with $a \cdot V(\zeta) \geq 0$ for all admissible rational ζ . Then $h(z) = \operatorname{Re} \sum_{j=2}^K a_j z^j$ is nonnegative at all such points (the factor 2 is irrelevant for the sign). By continuity $h \geq 0$ on $\{x > 0, y \geq 0\}$, and since $h(\bar{z}) = h(z)$, on all of $\operatorname{Re} z > 0$. If $a \neq 0$ and $l \geq 2$ is the largest index with $a_l \neq 0$, choose $\theta \in (-\pi/2, \pi/2)$ with $a_l \cos(l\theta) < 0$; this is possible because $l \geq 2$ and $\cos(l\theta)$ takes both signs on that interval. Then $h(te^{i\theta}) < 0$ for large t , a contradiction. Hence $a = 0$.

(b) *Finitely many columns suffice.* Let $\mathcal{C}$ be the convex cone generated by all admissible columns. By (a) the dual cone of $\mathcal{C}$ is $\{0\}$, so, by the bipolar theorem, the closure of $\mathcal{C}$ is $\mathbb{R}^{K-1}$. A convex set in $\mathbb{R}^{K-1}$ has the same interior as its closure; hence $\mathcal{C} = \mathbb{R}^{K-1}$. In particular each vector $\pm e_i$ of the standard basis is a finite nonnegative combination of admissible columns. Let $s_1, \dots, s_N$ be the finitely many columns used in these $2(K-1)$ combinations, together with the columns of the K prescribed real points. They are rational vectors, and their cone is $\mathbb{R}^{K-1}$.

(c) *A strictly positive rational kernel vector and solution.* Write $A = (s_1 | \dots | s_N)$. For each i , the polyhedron $\{\gamma \in \mathbb{R}^N : \gamma \geq 0, A\gamma = -s_i\}$ is nonempty by (b) and is defined by rational data. It contains a vertex, and vertices of such polyhedra are rational. Let $\gamma^{(i)}$ be a rational point of it and put $\kappa = \sum_i (e_i + \gamma^{(i)})$. Then $A\kappa = 0$ and every coordinate of κ is at least one. Moreover $A\alpha = w$, $w = (0, 3/2, 0, \dots, 0)^T$, has a real solution because the columns span, hence a rational solution α^* (Gaussian elimination over $\mathbb{Q}$). For a sufficiently large rational Λ the vector $\alpha^0 = \alpha^* + \Lambda\kappa$ is strictly positive, rational, and satisfies $A\alpha^0 = w$. $\square$

6.2 Estimates shared by the upper constructions

Throughout this subsection $P \in \mathcal{S}_n$ is real with $P(0) = 1$, with zeros $\beta_1, \dots, \beta_n \in \overline{\mathbb{D}}$ counted with multiplicity. Put

$$v_k = (1 - \beta_k)^{-1}, \quad S_j = \sum_{k=1}^n v_k^j, \quad G(z) = \frac{P(z)}{P(1)}, \quad w = z - 1.$$

Since $|1 - \beta|^2 \leq 2(1 - \operatorname{Re} \beta)$ for $|\beta| \leq 1$, every v_k has $\operatorname{Re} v_k \geq \frac{1}{2}$, hence $|v_k| \geq \frac{1}{2}$. Directly, $G(1+w) = \prod_k (1 + v_k w)$, $P'/P(1+w) = \sum_k v_k/(1 + v_k w)$, and $S_1 = P'(1)/P(1)$. From $v/(1 + vw) = \sum_{j=0}^{K-1} v^{j+1}(-w)^j + v^{K+1}(-w)^K/(1 + vw)$ we get the exact remainder formula

$$S_2 = \dots = S_K = 0 \implies F(z) := \frac{P'(z)}{P(z)} - S_1 = (-w)^K \sum_k \frac{v_k^{K+1}}{1 + v_k w} \quad (|v_k w| < 1 \ \forall k). \quad (14)$$

Lemma 6.1 (Normalized moduli of the factors). *Let $z = e^{i\theta}$, $s = \sin^2(\theta/2) \in [0, 1]$, and let $B = (1+z)^4(1-z+z^2)$ as in the next subsection.*

$$(a) \quad |(1+z)/2|^2 = 1 - s \leq e^{-s}; \quad |(1+z)'|/|2| = \frac{1}{2}.$$

$$(b) |B(z)/B(1)|^2 = (1-s)^4(1-4s)^2 \leq e^{-3s} \text{ and } |B'(z)|/|B(1)| \leq 3.$$

$$(c) \text{ For real } v > 1 \text{ and } f_v(z) = 1 - z/\beta, \beta = 1 - 1/v: |f_v(z)/f_v(1)|^2 = 1 + 4v(v-1)s \leq e^{Ts}, \\ T = 4v(v-1), \text{ and } |f'_v(z)|/|f_v(1)| = v.$$

$$(d) \text{ For } v = x + iy, x > \frac{1}{2}, y > 0, |v| \geq 2, \text{ and } f_v = (1 - z/\beta)(1 - z/\bar{\beta}): |f_v(z)/f_v(1)|^2 \leq e^{Ts} \\ \text{with } T = 4(|v|^2 + |v-1|^2), \text{ and } |f'_v(z)|/|f_v(1)| \leq 12|v|^2.$$

Proof. (a) is immediate. (b) Since $(1+z)(1-z+z^2) = 1+z^3$, we have $B = (1+z)^3(1+z^3)$, $|B'| \leq 3 \cdot 4 \cdot 2 + 3 \cdot 8 = 48$ on the circle and $B(1) = 16$. Moreover $|1+z|^2 = 4(1-s)$ and $|1-z+z^2| = |2\cos\theta - 1| = |1-4s|$, which gives the identity. For $0 \leq s \leq \frac{1}{2}$ we have $|1-4s| \leq 1$ and $(1-s)^4 \leq e^{-4s}$. For $\frac{1}{2} \leq s \leq 1$ put $\tau = 1-s \in [0, \frac{1}{2}]$; the quantity is $\tau^4(3-4\tau)^2$ and $e^{-3s} = e^{-3}e^{3\tau}$. If $\tau \leq \frac{1}{4}$, then $\tau^4(3-4\tau)^2 \leq 9/256 < 0.036 < e^{-3}$. If $\frac{1}{4} \leq \tau \leq \frac{1}{2}$, then $\tau^2(3-4\tau)$ is increasing (its derivative is $6\tau(1-2\tau) \geq 0$) and at most $\frac{1}{4}$, so the quantity is at most $\frac{1}{16} < 0.063$, while $e^{-3}e^{3\tau} \geq e^{-9/4} > 0.105$.

(c) For $\beta = 1 - 1/v$: $|1 - z/\beta|^2 = 1 + \beta^{-2} - 2\beta^{-1}\cos\theta$, so the ratio to $(1 - 1/\beta)^2$ is $1 + 4s\beta/(1-\beta)^2 = 1 + 4v(v-1)s$, because $\beta/(1-\beta)^2 = v(v-1)$. Also $|f'_v| = 1/\beta$ and $|f_v(1)| = (1-\beta)/\beta = 1/(v\beta)$.

(d) Let $r^2 = |\beta|^2$, $b = \operatorname{Re}\beta$, and $D = |1-\beta|^2 = 1 + r^2 - 2b$. Since f_v has real coefficients, on the circle $|f_v(z)|^2 = |z + r^2/z - 2b|^2/r^4$, and with $\cos\theta = 1 - 2s$, $\sin^2\theta = 4s(1-s)$,

$$|z + r^2/z - 2b|^2 = ((1+r^2)(1-2s) - 2b)^2 + 4(1-r^2)^2s(1-s).$$

Dividing by its value D^2 at $s = 0$ and expanding,

$$|f_v(z)/f_v(1)|^2 = 1 + \frac{4s[(1-r^2)^2 - (1+r^2)D]}{D^2} + \frac{16r^2s^2}{D^2}.$$

Now $D = 1/U$, $r^2 = V/U$, $b = 1-x/U$ with $Y = y^2$, $U = |v|^2 = x^2+Y$, $V = |v-1|^2 = (x-1)^2+Y$, and $U - V = 2x - 1$. Thus

$$|f_v(z)/f_v(1)|^2 = 1 + 8(x^2 - x - Y)s + 16UVs^2. \quad (15)$$

(We also confirmed (15) numerically to 40 digits at random points.) Since $UV - (x^2 - x - Y)^2 = (2x-1)^2Y \geq 0$, the right side is at most $(1 + 4\sqrt{UV}s)^2 \leq e^{8\sqrt{UV}s} \leq e^{4(U+V)s}$. Finally $|\beta| = |v-1|/|v| \geq 1 - 1/|v| \geq \frac{1}{2}$ and $|f'_v| \leq 2|\beta|^{-1}(1 + |\beta|^{-1}) \leq 12$ on the circle, while $|f_v(1)| = |1 - 1/\beta|^2 = 1/(|v|^2|\beta|^2)$; hence $|f'_v|/|f_v(1)| \leq 12|v|^2|\beta|^2 \leq 12|v|^2$. $\square$

Lemma 6.2 (From local cancellation to a residual bound). *Let $P \in \mathcal{S}_n$ be real with $P(0) = 1$ and let v_k, S_j, G, F be as above. Let $K \geq 2$ be an integer and assume $S_1 = P'(0)$ and $S_2 = \dots = S_K = 0$. Let $R \geq 1$, $\delta \in (0, \frac{1}{2}]$ and $\kappa, C_1, C_2, \Lambda > 0$, and suppose*

- (i) $|v_k| \leq R/(2\delta)$ for all k , and $R \leq \delta\sqrt{n}$;
- (ii) $\sum_k |v_k|^{K+1} \leq \Lambda$ and $\sum_k |v_k|^2 \leq C_1 n$;
- (iii) $|G(e^{i\theta})|^2 \leq \exp(-\kappa n \sin^2(\theta/2))$ for all θ ;
- (iv) $|P'(e^{i\theta})|^2 \leq C_2 n^3 |P(1)|^2 \exp(-\kappa n \sin^2(\theta/2))$ for $\delta/R \leq |\theta| \leq \pi$.

Then, with C depending only on $K, \kappa, C_1, C_2, \delta$,

$$\rho(P)^2 = \frac{\|P' - P'(0)P\|_2^2}{\|P\|_2^2} \leq C \left(\Lambda^2 n^{-K} + n^{7/2} \exp(-\kappa \delta^2 n / (\pi^2 R^2)) \right).$$

Proof. Put $\lambda = P'(0) = S_1$. Then $\rho(P)^2 = \int |GF|^2 d\theta / \int |G|^2 d\theta$, the factor $|P(1)|^2$ cancelling, and $|\lambda| \leq \sum |v_k| \leq 2 \sum |v_k|^2 \leq 2C_1 n$.

Denominator. The function $g(\theta) = \log |G(e^{i\theta})|^2 = 2 \operatorname{Re} \sum_k \log(1 + v_k w)$, $w = e^{i\theta} - 1$, is even because P is real, so $g(0) = g'(0) = 0$. For $|\theta| \leq \delta/R$ we have $|w| \leq |\theta| \leq \delta/R$, hence $|v_k w| \leq \frac{1}{2}$ by (i). Differentiating twice ($|w'| = |w''| = 1$) gives $|g''| \leq 2 \sum_k (2|v_k| + 4|v_k|^2) \leq 16 \sum_k |v_k|^2 \leq 16C_1 n$, using $|v_k| \geq \frac{1}{2}$. Since $n^{-1/2} \leq \delta/R$ by (i), Taylor's formula gives $g(\theta) \geq -8C_1 n \theta^2 \geq -8C_1$ for $|\theta| \leq n^{-1/2}$, so $\|G\|_2^2 \geq \pi^{-1} e^{-8C_1} n^{-1/2}$.

Inner arc $|\theta| \leq \delta/R$. By (14) and (i)–(ii), $|F| \leq 2|w|^K \sum |v_k|^{K+1} \leq 2\Lambda |\theta|^K$. By (iii) and $\sin^2(\theta/2) \geq \theta^2/\pi^2$,

$$\frac{1}{2\pi} \int_{|\theta| \leq \delta/R} |GF|^2 d\theta \leq \frac{2\Lambda^2}{\pi} \int_{\mathbb{R}} \theta^{2K} e^{-\kappa n \theta^2 / \pi^2} d\theta = C \Lambda^2 n^{-K-1/2}.$$

Outer arc. Here $|GF|^2 |P(1)|^2 = |P' - \lambda P|^2 \leq 2|P'|^2 + 2\lambda^2 |P|^2$, so by (iii)–(iv) the integrand is at most $C n^3 \exp(-\kappa n \sin^2(\theta/2))$, and $\sin^2(\theta/2) \geq \delta^2/(\pi^2 R^2)$ there. This arc contributes at most $C n^3 \exp(-\kappa \delta^2 n/(\pi^2 R^2))$. Dividing both contributions by the denominator bound gives the claim. $\square$

6.3 A fixed number of interior zeros

For a real reciprocal variable $v > 1$, put $\beta = 1 - 1/v$ and $f_v(z) = 1 - z/\beta$. For $v = x + iy$, $y > 0$, use the conjugate-pair factor $f_v(z) = (1 - z/\beta)(1 - z/\bar{\beta})$. Write $\nu_v = 1, 2$, respectively, and define

$$M_j(v) = \begin{cases} v^j, & v \text{ real,} \\ 2 \operatorname{Re} v^j, & \Im v > 0, \end{cases} \quad d(v) = \begin{cases} -v^2/(v-1), & v \text{ real,} \\ 2 \operatorname{Re}[-v^2/(v-1)], & \Im v > 0. \end{cases}$$

Thus $d(v) = f'_v(0) - f'_v(1)/f_v(1)$ (we use $f_v(0) = 1$), and $\operatorname{Re} v > 1/2$ is equivalent to $|\beta| < 1$; also $M_j(v) = \sum (1 - \beta')^{-j}$ over the ν_v zeros β' of f_v . Use the circle-root baseline

$$B(z) = (1+z)^4(1-z+z^2), \quad \mathbf{b}_j = \sum_{\beta: B(\beta)=0} (1-\beta)^{-j} = 4 \cdot 2^{-j} + 2 \cos(j\pi/3).$$

It has degree six, constant coefficient one, $B'(0) = B'(1)/B(1) = 3$, $\mathbf{b}_1 = 3$, $\mathbf{b}_2 = 0$, and $\mathbf{b}_3 = -3/2$.

Proof of Theorem 4. Data. Fix $K \geq 3$. Apply Lemma 12, with K designated distinct positive real nodes among the ζ_i . Clear the denominators of its strictly positive weights by an integer Q , and put $m_i = Q\alpha_i^0$. These are positive integer group multiplicities, fixed independently of n . Set

$$M = \sum_i \nu_i m_i, \quad W = \sum_i m_i M_1(\zeta_i) > 0.$$

The moments of orders $2, \dots, K$ are now $\sum_i m_i M_j(\zeta_i) = (0, 3Q/2, 0, \dots, 0)$.

Integer parameters. For every large integer n , define

$$R = (n/(6Q))^{1/3}, \quad \epsilon = R^{-1}, \quad c_n \equiv n - M \pmod{6}, \quad |c_n - 2WR| \leq 3, \\ \ell_n = (n - M - c_n)/6.$$

Choose a nearest integer in the specified congruence class; both ℓ_n, c_n are positive eventually. Then $t_n = c_n/R = 2W + O(\epsilon)$ and $u_n = \ell_n/R^3 = Q(n - M - c_n)/n = Q + O(\epsilon^2)$.

The nonlinear system and the implicit-function step. Let $\xi = (\xi_i)$ vary near $\zeta = (\zeta_i)$, varying only the K designated real nodes; all other nodes stay at their ζ_i . Define $d_\epsilon(\xi) =$

$-\xi^2/(\xi - \epsilon)$ for a real node, and twice its real part for a pair. For (ϵ, t, u, ξ) put

$$\Phi_1 = t/2 + \sum_i m_i d_\epsilon(\xi_i), \quad (16)$$

$$\Phi_2 = t\epsilon/4 + \sum_i m_i M_2(\xi_i), \quad (17)$$

$$\Phi_j = \mathfrak{b}_j u \epsilon^{j-3} + t 2^{-j} \epsilon^{j-1} + \sum_i m_i M_j(\xi_i), \quad 3 \leq j \leq K. \quad (18)$$

The map $\Phi = (\Phi_1, \dots, \Phi_K)$ is real-analytic on a neighborhood of $(0, 2W, Q, \zeta)$, in particular across $\epsilon = 0$: ϵ enters only through polynomial terms and through $\xi - \epsilon$, which is nonzero because $\xi \approx \zeta_i > 0$. The scaling $R \rightarrow \infty$ has been absorbed in ϵ, t, u , so no singular limit is involved. At $(0, 2W, Q, \zeta)$ every component vanishes: $\Phi_1 = W - W$, $\Phi_2 = \sum m_i M_2(\zeta_i) = 0$, $\Phi_3 = -\frac{3}{2}Q + \frac{3}{2}Q$, and $\Phi_j = \sum m_i M_j(\zeta_i) = 0$ for $4 \leq j \leq K$. The Jacobian with respect to the K designated real positions has columns

$$m_i(-1, 2\zeta_i, 3\zeta_i^2, \dots, K\zeta_i^{K-1})^T,$$

whose determinant is $\pm K! \prod_i m_i \prod_{i < i'} (\zeta_{i'} - \zeta_i) \neq 0$ (a scaled Vandermonde determinant). By the implicit-function theorem there are $\eta_0 > 0$ and a real-analytic map Ξ on $\{ |(\epsilon, t - 2W, u - Q)| < \eta_0 \}$ with $\Xi(0, 2W, Q) = \zeta$ and $\Phi(\epsilon, t, u, \Xi(\epsilon, t, u)) = 0$; Ξ is Lipschitz on a smaller ball. Since $(\epsilon, t_n, u_n) \rightarrow (0, 2W, Q)$ and $|t_n - 2W| + |u_n - Q| = O(\epsilon)$, for large n the point $\xi_{i,n} = \Xi(\epsilon, t_n, u_n)$ solves the system and $\xi_{i,n} = \zeta_i + O(\epsilon)$.

The polynomials and the exact identities. Put $v_{i,n} = R\xi_{i,n}$ and

$$P_n(z) = B(z)^{\ell_n} (1+z)^{c_n} \prod_i f_{v_{i,n}}(z)^{m_i}.$$

For large n , its degree is exactly $6\ell_n + c_n + M = n$, its constant coefficient is one, and its coefficients are real. Exactly M zeros, counted with multiplicity, lie strictly inside the disk ($\operatorname{Re} v_{i,n} = R \operatorname{Re} \xi_{i,n} > \frac{1}{2}$); the other $n - M$ lie on the circle. The interior zeros satisfy $|1 - \beta_{i,n}| \asymp 1 - |\beta_{i,n}| \asymp R^{-1}$. Now $S_j = \sum_k (1 - \beta_k)^{-j}$ is additive over factors, so $S_j = \mathfrak{b}_j \ell_n + c_n 2^{-j} + \sum_i m_i M_j(v_{i,n})$, and similarly the slope difference $P'_n(0) - P'_n(1)/P_n(1)$ equals $c_n/2 + \sum_i m_i d(v_{i,n})$, because B and $(1+z)$ contribute 0 and $1/2$. Multiplying $\Phi_1, \dots, \Phi_K$ by $R, R^2, \dots, R^K$, respectively, and using $R^3 u_n = \ell_n$, $R t_n = c_n$, gives the exact identities

$$\lambda_n := P'_n(0) = P'_n(1)/P_n(1) = S_1, \quad S_j = 0 \quad (2 \leq j \leq K).$$

In particular integer rounding has been corrected, and $\lambda_n = 3\ell_n + c_n/2 + R \sum_i m_i M_1(\xi_{i,n}) = O(n)$.

Verification of the hypotheses of Lemma 6.2. Take $\delta = \min\{\frac{1}{2}, (4 \max_i |\zeta_i|)^{-1}\}$; for large n , $|v_{i,n}| \leq 2R \max_i |\zeta_i| \leq R/(2\delta)$, while the baseline and the factors $1+z$ have $|v| \leq 1$. Thus (i) holds ($R \asymp n^{1/3}$). Next, the $6\ell_n + c_n \leq n$ circle reciprocals have $|v| \leq 1$, and the M interior ones $|v| \leq CR$; as $n = 6QR^3 \leq CR^{K+1}$,

$$\sum_k |v_k|^{K+1} \leq n + CR^{K+1} \leq C_K R^{K+1} =: \Lambda, \quad \sum_k |v_k|^2 \leq n + CR^2 \leq C_1 n,$$

which is (ii). For (iii), Lemma 6.1 gives, with $s = \sin^2(\theta/2)$ and since for large n all $|v_{i,n}| \geq 2$,

$$|G(e^{i\theta})|^2 \leq \exp[-s \mathcal{E}_n], \quad \mathcal{E}_n = 3\ell_n + c_n - \sum_i m_i T(v_{i,n}).$$

Here $3\ell_n + c_n = (n - M + c_n)/2 \geq (n - M)/2$, and the fixed number of interior groups costs $\sum_i m_i T(v_{i,n}) = O(R^2)$. Hence $\mathcal{E}_n \geq \kappa_0 n$ for some fixed $\kappa_0 > 0$ and large n , and

$$|P_n(e^{i\theta})/P_n(1)|^2 \leq \exp[-\kappa_0 n \sin^2(\theta/2)]. \quad (19)$$

For (iv) differentiate the product. By the product rule, P'_n is the sum of three kinds of terms: ℓ_n times the term with one factor B replaced by B' , c_n times the same for $1+z$, and m_i times the same for one factor f_{v_i} . Divide by $|P_n(1)|$ and use Lemma 6.1: the replaced factor contributes at most $3, \frac{1}{2}, v_i$ or $12|v_i|^2$, respectively, while the squared modulus of the product of all unreplaced factors (including the other copies of the differentiated factor) is at most $\exp[-s(\mathcal{E}_n - \tau)]$ with $\tau = 3, 1$ or $-T(v_i) < 0$. Hence, for large n (so that $\mathcal{E}_n - 3 \geq \kappa_0 n/2$),

$$\frac{|P'_n(e^{i\theta})|}{|P_n(1)|} \leq \left(3\ell_n + \frac{c_n}{2} + CR^2\right) \exp\left[-\frac{s}{2}(\mathcal{E}_n - 3)\right] \leq Cn \exp\left[-\frac{\kappa_0}{4}ns\right].$$

This is (iv) with $\kappa = \kappa_0/2$ (and (iii) holds with the same κ). No quotient at a boundary zero is used.

Conclusion. Lemma 6.2 gives

$$\rho(P_n)^2 \leq C_K(R^{2K+2}n^{-K} + n^{7/2}e^{-cn/R^2}) \leq C_K n^{(2-K)/3}, \quad \rho(P_n) \leq C_K n^{(2-K)/6},$$

since $n/R^2 \asymp n^{1/3}$ and $R^{2K+2} \asymp n^{(2K+2)/3}$. Choose $K \geq 6A + 2$. Then $M_A = M$ is independent of n , as required, and the polynomials P_n themselves have $\rho(P_n) \leq C_A n^{-A}$. No explicit quantitative bounds are supplied here on M_A, C_A , or the starting degree as functions of A ; the weights produced by Lemma 12 are existential and can be large. $\square$

Theorem 13 (One interior zero suffices). *For every sufficiently large integer n , there is a real degree- n polynomial P_n , with $P_n(0) = 1$, exactly one zero strictly inside the unit disk and all remaining zeros on the circle, such that $\rho(P_n) = O(n^{-1/6})$. The interior zero is real and positive, with distance and radial gap from 1 of order $n^{-1/3}$. In conjunction with Theorem 5, the minimum number of interior zeros permitting a sequence with vanishing residual is exactly one. No optimal rate for one interior zero is asserted.*

Proof. We use two adjustable circle angles in the degree-34 baseline

$$B_{a,c}(z) = (1 - 2az + z^2)^{15}(1 - 2cz + z^2)^2, \quad -1 < a, c < 1.$$

Its slope difference and second reciprocal moment are, respectively,

$$D(a, c) = -30a - 4c - 17, \quad T(a, c) = -\frac{15a}{1-a} - \frac{2c}{1-c}.$$

Both vanish at $(a_*, c_*) = (-2/3, 3/4)$. Write $n - 1 = 34m + k$, $0 \leq k \leq 33$, and let $v = v_n > 3/2$ be the unique solution of

$$v^3 - \frac{3}{2}v^2 = (n - 1)/4. \quad (20)$$

The left side is strictly increasing on this interval, and $v \sim (n/4)^{1/3}$. Set $\beta = 1 - 1/v \in (0, 1)$. For large n , $m > 0$. Define the real targets

$$d_n = \frac{v^2/(v-1) - k/2}{m} = O(n^{-2/3}), \quad t_n = \frac{-v^2 - k/4}{m} = O(n^{-1/3}).$$

The following explicit formulas select circle parameters solving $D(a_n, c_n) = d_n$, $T(a_n, c_n) = t_n$:

$$\begin{aligned} L &= (51 + d_n)/2, & U &= 17 - t_n, \\ X &= \frac{UL + 221 + \sqrt{(UL - 289)(UL - 169)}}{30U}, \\ Y &= (L - 15X)/2, & a_n &= 1 - X, & c_n &= 1 - Y. \end{aligned}$$

Indeed these equations are equivalent to $15X + 2Y = L$, $15/X + 2/Y = U$; elimination gives $15UX^2 - (UL + 221)X + 15L = 0$. The positive-square-root branch above tends to $X = 5/3, Y =$

1/4, since $UL \rightarrow 867/2$ and the square root tends to $391/2$. The denominators and discriminant stay positive, and $a_n, c_n \in (-1, 1)$ eventually, with deviations $O(n^{-1/3})$ from (a_*, c_*) . These statements are uniform over the 34 values of k .

Put

$$P_n(z) = B_{a_n, c_n}(z)^m (1+z)^k (1-z/\beta).$$

It has exactly the claimed degree, constant term and zero locations. The two target equations give the exact identities

$$P'_n(0) = P'_n(1)/P_n(1) =: \lambda_n, \quad \sum_{j=1}^n v_j^2 = 0, \quad v_j = (1 - \beta_j)^{-1}.$$

Here the extra roots at -1 contribute $k/2$ to the slope difference and $k/4$ to the second moment; the interior root contributes $-v^2/(v-1)$ and v^2 , respectively. For a conjugation-symmetric multiset of N circle roots avoiding 1, its reciprocal moments satisfy

$$M_3^\circ = \frac{3}{2}M_2^\circ - N/4. \quad (21)$$

This follows for each pair $1/2 \pm iy$, and also for the real reciprocal $1/2$. Conjugation symmetry is essential and holds here. Taking $N = n - 1$ and $M_2^\circ = -v^2$, (20) gives $\sum_j v_j^3 = 0$ exactly. Moreover $\lambda_n = (n-1)/2 + v = O(n)$.

We verify the hypotheses of Lemma 6.2 with $K = 3$ and $R = v$. Here $S_1 = P'_n(0) = \lambda_n$ and $S_2 = S_3 = 0$ were shown above.

Global bound. With $s = \sin^2(\theta/2)$ and $h_a(s) = (1 - 2s/(1-a))^2$, one has $|(1 - 2az + z^2)/(2 - 2a)|^2 = h_a(s)$ on the circle, so

$$|P_n(e^{i\theta})/P_n(1)|^2 = H_{a_n, c_n}(s)^m (1-s)^k [1 + 4v(v-1)s], \quad H_{a,c} = h_a^{15} h_c^2, \quad (22)$$

using Lemma 6.1(a),(c). The limiting function is $H(s) := H_{a_*, c_*}(s) = (1 - 6s/5)^{30} (1 - 8s)^4$; apart from its zeros $1/8, 5/6$, its only interior stationary point is $5/24$, because its logarithmic derivative is $-36/(1 - 6s/5) - 32/(1 - 8s)$. Its values there and at 1 are

$$H(5/24) = (3/4)^{30} (2/3)^4 < 10^{-4}, \quad H(1) = 7^4/5^{30} < 10^{-17}.$$

We claim there are $\eta, \kappa_1 > 0$ such that $H_{a,c}(s) \leq e^{-\kappa_1 s}$ for $0 \leq s \leq 1$ and $|a - a_*|, |c - c_*| \leq \eta$. Take $\eta \leq \frac{1}{20}$ and $s_0 = \frac{1}{16}$. On $[0, s_0]$ both bases $1 - 2s/(1-a)$, $1 - 2s/(1-c)$ are positive (as $1 - c \geq \frac{1}{5}$, $2s_0/(1-c) \leq \frac{5}{8}$), so $\log H_{a,c}$ is a sum of logarithms of positive decreasing affine functions: it is concave, vanishes at 0, and its derivative at 0 is $-60/(1-a) - 8/(1-c) \leq -60$. Hence $H_{a,c} \leq e^{-60s}$ there. On $[s_0, 1]$, the function H is decreasing on $[s_0, \frac{1}{8}]$, has maximum $H(5/24)$ on $[\frac{1}{8}, \frac{5}{6}]$, and is increasing from 0 to $H(1)$ on $[\frac{5}{6}, 1]$; therefore $\sup_{[s_0, 1]} H = H(s_0) < 0.007$. By uniform continuity in the parameters, after shrinking η we have $H_{a,c} \leq \frac{1}{2}$ on $[s_0, 1]$, so $H_{a,c} \leq e^{-(\log 2)s}$. This proves the claim with $\kappa_1 = \log 2$. Since a_n, c_n converge to a_*, c_* , for large n , (22) gives

$$|P_n(e^{i\theta})/P_n(1)|^2 \leq \exp[-s(\kappa_1 m - 4v(v-1))] \leq \exp[-\kappa n \sin^2(\theta/2)],$$

because $m \sim n/34$ and $v^2 = O(n^{2/3})$. This is (iii) of the lemma.

Reciprocals. A circle root $e^{i\varphi}$ with $\cos \varphi = a$ has $|v|^2 = 1/(2(1-a))$, hence $|v| \leq 2$ for all roots of the two quadratic factors (as $1 - a \geq \frac{2}{3}$, $1 - c \geq \frac{1}{5}$), and $|v| = \frac{1}{2}$ at -1 . With $R = v \sim (n/4)^{1/3}$, hypothesis (i) holds for a small fixed δ , and (ii) holds with $\Lambda = Cn^{4/3}$, C_1 fixed: $\sum |v_k|^4 \leq 16n + v^4$, $\sum |v_k|^2 \leq 4n + v^2$.

Outer arc. Differentiate the product $\Phi_a^{15m} \Phi_c^{2m} (1+z)^k f_v$, with $\Phi_a = 1 - 2az + z^2$, $\Phi_c = 1 - 2cz + z^2$. The normalized derivatives $|\Phi'_a|/|\Phi_a(1)| \leq 2/(1-a) \leq 2$, likewise for Φ_c (at most 10), and $1/2, v$ for $1+z$ and f_v . In the term from Φ_a the remaining product has squared modulus $h_a^{15m-1} h_c^{2m} \dots \leq H_{a,c}^{m-1} h_a^{14} h_c^2 \dots$, and in the term from Φ_c it is $H_{a,c}^{m-1} h_a^{15} h_c \dots$; the extra

functions $h_a \leq 1$, $h_c \leq (2/(1-c) - 1)^2 \leq C$ are bounded. In the terms from $1+z$ and f_v one factor of H^m is kept whole. Using the claim above and $(1-s)^k \leq 1$, $1+4v(v-1)s \leq e^{4v(v-1)s}$,

$$\frac{|P'_n(e^{i\theta})|^2}{|P_n(1)|^2} \leq Cn^2v^2 \exp[-s(\kappa_1(m-1) - 4v(v-1))] \leq Cn^3 \exp[-\kappa ns],$$

after decreasing κ if necessary. This is (iv). No logarithmic quotient at a boundary zero occurs.

Conclusion. Lemma 6.2 with $K = 3$, $\Lambda = Cn^{4/3}$, $R = v \asymp n^{1/3}$ yields

$$\rho(P_n)^2 \leq C(n^{8/3}n^{-3} + n^{7/2}e^{-cn/v^2}) = O(n^{-1/3}),$$

as required. (The local remainder is $F_n(z) = -(z-1)^3 \sum_j v_j^4/(1+v_j(z-1))$ by (14).)

Finally, Theorem 5 excludes vanishing residuals with zero interior roots; this construction permits them with one. This proves the zero-count threshold, not a sharp rate or the superalgebraic conclusion with one root. No explicit starting degree is claimed. The next proposition identifies the asymptotic norm constant of this particular family. $\square$

Proposition 14 (Asymptotic constant for the explicit family). *For the polynomials constructed in Theorem 13, through all sufficiently large integer degrees,*

$$\rho(P_n) \sim \frac{\sqrt{15}}{4^{4/3}} n^{-1/6}.$$

This constant describes that family, not a universal one-root optimum.

Proof. Retain the construction's notation and put $G_n = P_n/P_n(1)$, $F_n = P'_n/P_n - \lambda_n$, and $w = z-1$. Its reciprocal zeros satisfy $S_1 = \lambda_n = (n-1)/2+v$, $S_2 = S_3 = 0$, and $v/n^{1/3} \rightarrow 4^{-1/3}$, by (20); all boundary reciprocals have modulus at most 2. By (14) with $K = 3$, whenever $|v_j w| < 1$,

$$F_n(z) = -w^3 \sum_j \frac{v_j^4}{1+v_j w}.$$

Put $z = \exp(iu/\sqrt{n})$ with u fixed. Then $w = iu/\sqrt{n} + O(1/n)$ and $vw = O(n^{-1/6}) \rightarrow 0$, so the boundary contribution to the sum is $O(n)$ and the single interior one is $v^4(1+o(1)) \sim 4^{-4/3}n^{4/3}$. Hence

$$n^{1/6}F_n(e^{iu/\sqrt{n}}) \rightarrow i4^{-4/3}u^3. \quad (23)$$

Next, $\log G_n(1+w) = \sum_j \log(1+v_j w)$, and $|\log(1+x) - x + x^2/2 - x^3/3| \leq |x|^4/2$ for $|x| \leq \frac{1}{2}$. Using $S_2 = S_3 = 0$,

$$\log G_n(1+w) = \lambda_n w + r_n, \quad |r_n| \leq \frac{1}{2} \sum_j |v_j|^4 |w|^4 = O(n^{4/3}n^{-2}) \rightarrow 0.$$

Since $\operatorname{Re}(\lambda_n w) = -2\lambda_n \sin^2(\theta/2)$ with $\theta = u/\sqrt{n}$, $\sin^2(\theta/2) = u^2/(4n)(1+O(n^{-1}))$, and $\lambda_n/n \rightarrow 1/2$, we get

$$|G_n(e^{iu/\sqrt{n}})|^2 \rightarrow e^{-u^2/2}. \quad (24)$$

For the domination we use the verification of Lemma 6.2 in the proof of Theorem 13. With $\theta = u/\sqrt{n}$, (iii) gives $|G_n|^2 \leq \exp(-\kappa u^2/\pi^2)$ for all $|u| \leq \pi\sqrt{n}$; on the inner arc $|\theta| \leq \delta/v$, that is $|u| \leq \delta\sqrt{n}/v \asymp n^{1/6}$, the bound $|F_n| \leq 2\Lambda|\theta|^3$ with $\Lambda = Cn^{4/3}$ gives $n^{1/3}|F_n|^2 \leq Cu^6$. So the functions $|G_n|^2$ and $n^{1/3}|G_n F_n|^2$, extended by zero beyond the inner arc, are dominated by $e^{-\kappa u^2/\pi^2}$ and $Cu^6 e^{-\kappa u^2/\pi^2}$, and the inner arc exhausts $\mathbb{R}$ as $n \rightarrow \infty$. The remaining outer arc contributes to $n^{5/6}\|G_n F_n\|_2^2$ at most $n^{5/6} \cdot Cn^3 \exp(-cn/v^2) \rightarrow 0$, by (iv) and the outer-arc estimate of Lemma 6.2. Since $\|G_n\|_2^2 = (2\pi\sqrt{n})^{-1} \int |G_n|^2 du$ and $n^{5/6}\|G_n F_n\|_2^2 = (2\pi)^{-1} \int n^{1/3}|G_n F_n|^2 du$ (inner arc), dominated convergence with (23)–(24) yields

$$\sqrt{n}\|G_n\|_2^2 \rightarrow \frac{1}{\sqrt{2\pi}}, \quad n^{5/6}\|G_n F_n\|_2^2 \rightarrow \frac{15 \cdot 4^{-8/3}}{\sqrt{2\pi}},$$

where 15 is the sixth standard Gaussian moment ($\int u^6 e^{-u^2/2} du = 15\sqrt{2\pi}$). Dividing gives $n^{1/3} \rho(P_n)^2 \rightarrow 15 \cdot 4^{-8/3}$, i.e. the claim. All constants in the preceding construction are uniform over the finite set of 34 degree remainders, and the limit does not depend on k , so the limit holds through all sufficiently large integers. $\square$

7 Interior-root budgets and their obstruction

We first prove the all-degree bound and then the existence criterion. The proof also identifies the structural role of the circle roots.

7.1 Multiplication by a polynomial of small degree

For $T = c \prod_j (z - \gamma_j)$, write $\mathcal{M}(T) = |c| \prod_j \max(1, |\gamma_j|)$ for its Mahler measure. The classical coefficient bound gives

$$\|T\|_2 \leq \sqrt{\binom{2\ell}{\ell}} \mathcal{M}(T) \leq 2^\ell \mathcal{M}(T), \quad \ell = \deg T. \quad (25)$$

Indeed each coefficient is bounded by the corresponding binomial coefficient times $\mathcal{M}(T)$, by expanding the product; sum their squares. See also [8, Lemma 1.6 and Exercise 1.11].

Lemma 15. *If P has degree n and $H \neq 0$ has degree at most d , then*

$$\|HP\|_2 \geq \|H\|_2 (n + d + 2)^{-d} \|P\|_2.$$

Proof. For $V = \sum_{j=0}^N a_j z^j$ and $\gamma \in \mathbb{C}$, Parseval gives

$$\|(z - \gamma)V\|_2^2 = (1 + |\gamma|^2) \|V\|_2^2 - 2 \operatorname{Re} \left(\bar{\gamma} \sum_{j=0}^{N-1} a_j \overline{a_{j+1}} \right).$$

The numerical radius of the truncated shift of dimension $N + 1$ is $c_N = \cos(\pi/(N + 2))$; this is the classical extremal shift calculation in [9]. Directly, after a diagonal phase rotation its real part is the path matrix with off-diagonal entries $1/2$. Its eigenvectors have coordinates $\sin((j + 1)k\pi/(N + 2))$, and its largest eigenvalue is c_N . Thus the squared norm ratio is at least $1 + r^2 - 2rc_N$, where $r = |\gamma|$. For $r \leq 1$, completing a square bounds this below by $1 - c_N^2$; for $r \geq 1$, divide by r^2 and use the same argument at $1/r$. Consequently

$$\|(z - \gamma)V\|_2 \geq \max(1, |\gamma|) \sin \frac{\pi}{N + 2} \|V\|_2.$$

Factor H into $\ell \leq d$ linear factors and apply this estimate successively at degrees $n, n + 1, \dots, n + \ell - 1$. Since $\sin(\pi/x) \geq 2/x$ for $x \geq 2$,

$$\|HP\|_2 \geq \mathcal{M}(H) \left(\frac{2}{n + d + 2} \right)^\ell \|P\|_2 \geq \|H\|_2 (n + d + 2)^{-\ell} \|P\|_2.$$

This implies the assertion; the constant-polynomial case is immediate. $\square$

7.2 A positive semicircle in the reciprocal identity

Proof of Theorem 1. Write $P = cAQ$, where $A(z) = \prod_{j=1}^M (z - \beta_j)$, $|\beta_j| < 1$, is monic and all roots of Q lie on the circle. For a polynomial considered with degree parameter k , set $T^*(z) = z^k \overline{T(1/\bar{z})}$. Use parameters n for P , M for A , and importantly n for $R = P' - \lambda P$, even if R has smaller actual degree. Self-inversivity of Q gives $P/P^* = \omega A/A^*$ for some $|\omega| = 1$. The

identity $z^2(P^*)' = nzP^* - R^* - \bar{\lambda}P^*$, followed by differentiation of this quotient and clearing denominators, yields

$$z^2 AA^* R + \omega A^2 R^* = HP, \quad (26)$$

$$H = z^2 \{A' A^* - A(A^*)'\} + (nz - \lambda z^2 - \bar{\lambda}) AA^*. \quad (27)$$

These are polynomial identities, including at zeros of the factors. Here $\deg H \leq d = 2M + 2$, and on the circle $|A^*| = |A| \leq 2^M$, $|R^*| = |R|$. Hence

$$\|HP\|_2 \leq 2 \cdot 4^M \|R\|_2. \quad (28)$$

We obtain a lower bound for H without a separation assumption. The finite Blaschke identity, also given in [11, Lemma 4.3], is

$$z \frac{(A/A^*)'}{A/A^*} = \sum_{j=1}^M \frac{1 - |\beta_j|^2}{|z - \beta_j|^2}, \quad |z| = 1.$$

It follows either by differentiating each factor or by its Poisson kernel formula. Since $AA^* = z^M |A|^2$ on the circle,

$$z^{-M-1} H(z) = (n - 2 \operatorname{Re}(\lambda z)) |A(z)|^2 + K_A(z), \quad (29)$$

$$K_A(z) = \sum_{j=1}^M (1 - |\beta_j|^2) \prod_{\ell \neq j} |z - \beta_\ell|^2 \geq 0.$$

Choose a semicircle E on which $\operatorname{Re}(\lambda z) \leq 0$; if $\lambda = 0$, choose either semicircle. Thus $|H| \geq n|A|^2$ on E .

Jensen's formula for the monic stable polynomial A gives $\int_{\mathbb{T}} \log |A| dt / (2\pi) = 0$. On the complementary semicircle, $\log |A| \leq M \log 2$. Therefore its average over E , with measure dt/π , is at least $-M \log 2$. Applying Jensen's inequality to $\exp(4 \log |A|)$ on E gives

$$\frac{1}{\pi} \int_E |A|^4 dt \geq 2^{-4M}, \quad \|H\|_2 \geq \frac{n}{\sqrt{2} 4^M}.$$

This restricted-arc Jensen argument is classical in origin; compare [10, Lemma 4.1]. The logarithm is continuous on the circle for each strictly interior configuration, so no limiting interchange as the roots approach the boundary is needed. In particular $H \neq 0$. Lemma 15 and (28) now give (1). For $n \geq 2M + 4$, use $n + 2M + 4 \leq 2n$ and weaken $2^{-6M-7/2}$ to 2^{-6M-4} . $\square$

Corollary 16 (Necessary quantitative count). *Under the hypotheses of Theorem 1,*

$$M \geq \frac{\log(1/\rho_\lambda(P)) - \log n - \log(98\sqrt{2})}{2 \log n + \log 784}.$$

In particular, a sequence with $\rho_{\lambda_n}(P_n) \leq \exp(-cn^\alpha)$, where $c, \alpha > 0$, necessarily has $M_n \geq (c/2 + o(1))n^\alpha / \log n$.

Proof. Since $M \leq n$, we have $n + 2M + 4 \leq 7n$. Taking logarithms in (1) gives

$$\log(1/\rho_\lambda(P)) \leq (2M + 1) \log n + M \log 784 + \log(98\sqrt{2}).$$

Rearrangement proves the result. This is only a necessary condition; Theorem 3 can also exclude a proposed decay rate. $\square$

Remark 17 (Stable factor and self-inversive cofactor). The same bound (1) holds for any degree- n $P = cAQ$ with A monic of degree M , all its zeros strictly inside the disk, and $Q = \eta Q^*$, $|\eta| = 1$. Indeed only this last identity was used for Q ; the remaining estimates use the degree of P and the zeros of A . The cofactor may therefore contain conjugate-reciprocal pairs inside and outside the circle. In this extension M means the degree of the chosen stable factor, not the total number of interior zeros of P . Not every polynomial has such a factorization. For instance, $P_R = 1 - z/R$, $R > 1$, with $\lambda = -1/R$ has residual $1/[R^2\sqrt{1+R^{-2}}] \rightarrow 0$ at fixed degree one; its unpaired exterior root falls outside the hypothesis.

7.3 Sufficiency, necessity and exact saturation

Proof of Theorem 2. If b_n does not tend to infinity, there are a finite B and infinitely many degrees with $b_n \leq B$. For every such sufficiently large n , Theorem 1 bounds any admissible residual below by $2^{-6B-4}n^{-2B-1}$, since its actual count $M \leq B$. This contradicts superalgebraic decay, so (2) implies (1).

Suppose now $b_n \rightarrow \infty$. For each integer $j \geq 1$, apply Theorem 4 with exponent $2j$, obtaining M_j, C_j, N_j . Choose strictly increasing integers $T_j \geq j$ so large that for every $n \geq T_j$,

$$b_n \geq M_j, \quad n \geq N_j, \quad n^j \geq C_j.$$

All auxiliary eventual conditions in that construction can also be included in N_j . With $j(n) = \max\{j : T_j \leq n\}$, the constructed real normalized polynomial has exactly $M_{j(n)}$ interior zeros and restricted residual at most $n^{-j(n)}$. The index is finite and tends to infinity.

To achieve exactly b_n interior zeros while retaining reality, we do *not* multiply the polynomial by anything. We only move some of its boundary zeros. Fix n with $n \geq T_1$ and put $j = j(n)$. Let P_n be the real polynomial constructed in the proof of Theorem 4 at degree n with exponent $2j$; it has degree exactly n , $P_n(0) = 1$, and

$$P_n(z) = B(z)^\ell(1+z)^c \prod_i f_i(z)^{m_i}, \quad B(z) = (1+z)^4(1-z+z^2),$$

where, in the notation of that proof, $\ell = \ell_n$ and $c = c_n$ are the integer exponents, and $\prod_i f_i^{m_i}$ collects the $M = M_j$ interior zeros. The factor $B^\ell(1+z)^c$ is part of P_n ; its $6\ell + c = n - M_j$ zeros are the boundary zeros of P_n , namely $4\ell + c$ real zeros at -1 and ℓ conjugate pairs $e^{\pm i\pi/3}$. Eventually $\ell \geq 1$ (all eventual conditions are included in N_j). Given the required additional count $s = b_n - M_j$, so that $0 \leq s \leq n - M_j = 6\ell + c$, take

$$q = \min\{\ell, \lfloor s/2 \rfloor\}, \quad r = s - 2q.$$

Then $0 \leq q \leq \ell$ and $2q + r = s$. Moreover $0 \leq r \leq 4\ell + c$: if $q = \lfloor s/2 \rfloor$ then $r \in \{0, 1\}$ and $4\ell + c \geq 4$; otherwise $q = \ell$ and $r = s - 2\ell \leq 4\ell + c$. (We also checked this bookkeeping by machine for all $1 \leq \ell \leq 24$, $0 \leq c \leq 24$, $0 \leq s \leq 6\ell + c$.) For $0 < \delta < 1$ put $\varrho = 1 - \delta$ and

$$P_{n,\delta}(z) = (1 - z + z^2)^{\ell-q} (1 - z/\varrho + z^2/\varrho^2)^q (1+z)^{4\ell+c-r} (1+z/\varrho)^r \prod_i f_i(z)^{m_i}.$$

This is P_n with q of the ℓ pairs $e^{\pm i\pi/3}$ replaced by the pairs $\varrho e^{\pm i\pi/3}$ and r of the zeros -1 replaced by $-\varrho$. Its degree is still n , its coefficients are real, $P_{n,\delta}(0) = 1$, and its zeros lie in the closed disk. For $\delta > 0$ the $2q + r = s$ displaced zeros lie strictly inside the disk, the M_j interior zeros of P_n are untouched, and the remaining zeros stay on the circle; so $P_{n,\delta}$ has exactly $M_j + s = b_n$ interior zeros. As $\delta \rightarrow 0$, $P_{n,\delta} \rightarrow P_n$ coefficientwise. The restricted residual $\rho(P) = \|P' - P'(0)P\|_2/\|P\|_2$ is a continuous function of the coefficients wherever $\|P\|_2 \neq 0$, and $\|P_n\|_2 \neq 0$. Therefore $\rho(P_{n,\delta}) \rightarrow \rho(P_n)$, and, for each fixed n , we can fix $\delta = \delta_n > 0$ with $\rho(P_{n,\delta_n}) \leq \rho(P_n) + n^{-j(n)}$. Since $\rho(P_n) \leq C_j n^{-2j} \leq n^{-j}$ (because $n^j \geq C_j$), the final polynomial

has exactly b_n interior zeros, real coefficients, $P(0) = 1$, and residual at most $2n^{-j(n)} = o(n^{-A})$ for every fixed $A > 0$; the slope used is $P'(0)$. No product with an extra factor is taken, and no continuity is needed in n . The finitely many initial degrees can be filled arbitrarily with real normalized polynomials having the prescribed count, for example $(1+z)^{n-b}(1+2z)^b$ for b interior zeros $-1/2$. This proves (1) implies (3), while (3) implies (2) immediately.

For residuals merely tending to zero, infinitely many zero budgets would contradict Theorem 5. Conversely, a budget at least one eventually accommodates the family of Theorem 13. This last assertion concerns an upper budget, not saturation of every budget by that particular family. $\square$

For a fixed M , residual $O(n^{-A})$ consequently requires $A \leq 2M + 1$. This necessary relation is not a matching tradeoff. The number M_A in Theorem 4 depends on A ; no single fixed count gives a superalgebraic sequence.

7.4 A stronger obstruction for one interior zero

The reciprocal identity works for every count. At count one, the third and fourth reciprocal moments give a stronger exponent.

Proposition 18. *If $n \geq 4096$, $P \in \mathcal{S}_n$ has exactly one strictly interior zero and all other zeros on the circle, then, for all $\lambda \in \mathbb{C}$,*

$$\rho_\lambda(P) \geq \frac{n^{-13/6}}{512\sqrt{2}e^3}.$$

Proof. Set $C = 2\sqrt{2}e^3$ and suppose for contradiction that $\rho = \rho_\lambda(P) \leq n^{-13/6}/(256C)$. Rotate a maximum point to 1 and normalize $G(1) = \|G\|_\infty = 1$ as in Lemma 6. Its initial part gives $|\mu| \leq n + 1$ and $S_1 = G'(1) \in \mathbb{R}$, where $S_k = \sum_j (1 - \beta_j)^{-k}$ uses the rotated zeros. The local ODE proof of Lemma 7, at radius $1/n$, does not require its separate separation parameter η . Here its smallness parameter is at most $e^3/n < 1/2$, and it yields the unaveraged estimates

$$|S_k| \leq 2\sqrt{n+1}\rho e^{(2n+1)/n} n^{k-1} \leq C\rho n^{k-1/2}, \quad k \geq 2.$$

In particular,

$$|S_2| \leq \frac{n^{-2/3}}{256}, \quad |S_3| \leq \frac{n^{1/3}}{256}, \quad |S_4| \leq \frac{n^{4/3}}{256}. \quad (30)$$

The $n-1$ boundary reciprocals are $w_j = 1/2 + it_j$, $t_j \in \mathbb{R}$; the remaining reciprocal is $v = x + iy$, $x > 1/2$. Since S_1 is real, $\sum_j t_j = -y$, so

$$\operatorname{Im} S_2 = (2x - 1)y,$$

and, using $\operatorname{Re} w_j^3 = (3/2) \operatorname{Re} w_j^2 - 1/4$,

$$x^3 - \frac{3}{2}x^2 - 3(x - \frac{1}{2})y^2 = \frac{n-1}{4} + \operatorname{Re} S_3 - \frac{3}{2} \operatorname{Re} S_2 \geq \frac{n}{8}.$$

The last inequality follows from (30) for $n \geq 4096$. Both subtracted terms on the left are nonnegative, so $x \geq n^{1/3}/2 \geq 8$. Consequently $2x - 1 \geq x$, $|xy| \leq |S_2|$, and

$$\operatorname{Re} v^4 = x^4 - 6x^2y^2 + y^4 \geq \frac{n^{4/3}}{16} - \frac{6}{65536}n^{-4/3}.$$

For each boundary reciprocal, $\operatorname{Re}(1/2 + it)^4 = (t^2 - 3/4)^2 - 1/2 \geq -1/2$. It follows that

$$\operatorname{Re} S_4 \geq \frac{n^{4/3}}{16} - \frac{6}{65536}n^{-4/3} - \frac{n-1}{2} \geq \frac{n^{4/3}}{64},$$

using $n/2 \leq n^{4/3}/32$. This contradicts (30). $\square$

Together with Theorem 13, this leaves a gap between the universal one-root lower exponent $13/6$ and the achieved upper exponent $1/6$. No optimality of either is asserted.

8 A fixed radial gap forces a polynomial barrier

The upper construction uses the full closed disk. A fixed strict root radius gives a complementary obstruction, by the same local estimate and bounded test as the lower-bound proof.

Proposition 19. *For $0 \leq q < 1$, let $s_n(q)$ denote the infimum defining s_n with all zeros restricted to $|z| \leq q$. Set*

$$a = 1 - q^2, \quad K = \lceil 4/a \rceil, \quad \Delta = \frac{1}{1+q} - \frac{1+q^K}{Ka} > 0, \quad c_q = \min \left\{ \frac{1}{\sqrt{2}}, \frac{\Delta Ka}{2\sqrt{2}e^3(1+a)^K} \right\}.$$

Then $s_n(q) \geq c_q n^{3/2-K}$ for every $n \geq 41$.

Proof. Write ρ for an admissible residual. If $\rho > 1/\sqrt{n+1}$, the conclusion is immediate. Otherwise the maximum normalization gives $|\mu| \leq n+1$. Take $r = 1/n$. The smallness parameter in the proof of Lemma 7 is at most $e^3/n < 1/2$. That local proof does not need the separate zero-separation parameter η , and gives

$$|m_j| \leq \frac{2\sqrt{n+1}}{n} \rho e^3 r^{1-j} \quad (j \geq 2).$$

The exact image of $|\beta| \leq q$ under $v = (1-\beta)^{-1}$ is $|v - 1/a| \leq q/a$. Thus $|1 - av| \leq q$ and $\operatorname{Re} v \geq 1/(1+q)$. Use $\Psi(v) = [1 - (1 - av)^K]/(Ka)$, whose modulus is at most $(1 + q^K)/(Ka)$. Estimating its higher averaged terms gives

$$\Delta \leq \frac{2e^3\sqrt{n+1}\rho}{Kan^2} [(1+an)^K - 1 - Kan] \leq \frac{2e^3\sqrt{n+1}\rho}{Kan^2} (1+an)^K.$$

Now $\sqrt{n+1} \leq \sqrt{2n}$ and $1+an \leq (1+a)n$. The resulting lower bound is the stated one. Finally $Ka \geq 4$, $q^K < 1$, and $1/(1+q) > 1/2$ ensure $\Delta > 0$. $\square$

In particular, any single sequence of admissible degree- n polynomials with residual $o(n^{-A})$ for every fixed $A > 0$ must have its maximum root modulus tending to one. Otherwise an infinite subsequence would satisfy a fixed polynomial lower bound. No sharp radial exponent is asserted.

9 Scope and verification

The uniform slope formulation bounds approximate eigenvectors of polynomial differentiation under a zero-location constraint. It does not describe the full unconstrained pseudospectrum. For example, for $u = n/4$ and $T_n(z) = \sum_{k=0}^n (uz)^k/k!$,

$$T'_n - uT_n = -\frac{u^{n+1}}{n!} z^n, \quad \frac{\|T'_n - uT_n\|_2}{\|T_n\|_2} \leq \frac{n}{4} \left(\frac{e}{4}\right)^n,$$

using $\|T_n\|_2 \geq 1$ and $n! \geq (n/e)^n$. Thus a root constraint is essential for a bound of our form.

The lower and upper disk bounds do not match in order. Whether the restricted exterior power-sum minima stay bounded away from zero remains unresolved here, as does the optimal order of s_n and r_n . The superalgebraic upper bound is not a stretched-exponential upper bound, and its constants are not optimized. The minimal interior-root count permitting collapse is one, but the optimal rate with one such root is not established. Theorem 1 excludes superalgebraic decay for every fixed number of interior roots. The elementary radial barrier is a consequence of the same local method. We do not assert a theorem for arbitrary starting exponents, weighted power sums, or H^p with $p < 1$. The elementary norm comparison in Corollary 11 is a consequence of the main estimate.

The analytic proof of the explicit threshold uses only the displayed elementary inequalities. It does not rely on optimization or sampling. The supplementary diagnostics check exact projected Newton identities over Gaussian rationals, the bounded test on admissible reciprocal zeros, binomial identities, and numerical instances of the finite certificate. Rational checks of constants and high-precision evaluations are included for reproducibility. The circle-root family is checked with complete exact coefficient Parseval ratios. The one-interior-root family has symbolic and high-precision checks of the cubic, slope, moments and circle parameters. A separate calculation by an AI model that also took part in preparing this manuscript, hence not independent, covers all 34 degree classes at two scales. Additional exact checks cover the reciprocal identity, its complex conjugations, the positive Poisson kernel, the one-root moment obstruction and real-root saturation.

For Proposition 14, supplementary numerical norm diagnostics use two equispaced grids, of sizes $n + 1$ and $n + 2$, at eight degrees up to 10 000 017. Such grids would integrate the squared polynomial norms exactly in exact arithmetic. The implemented evaluations instead use rounded parameters and omit underflow-scale weights. The cancellation-free remainder approximates the exact family; it is not literally the exact residual of the polynomial with all parameters rounded. Agreement between grids does not certify the common rounding or masking error. These computations are not interval-certified norm evaluations and do not prove the asymptotic limit. Finite moment searches motivated the general construction but are not premises of its arbitrary-order proof. None of the checks establishes historical priority or replaces the analytic arguments.

In version 2, the proofs of Theorem 4, Theorem 13 and Proposition 14 were rewritten with the uniform estimates displayed (Lemmas 12, 6.1 and 6.2). As a consistency check, not a premise of any proof, one explicit instance of the construction of Theorem 4 with $K = 3$ and $M = 10$ interior zeros (three real groups and one conjugate pair, with small integer multiplicities satisfying the moment conditions of Lemma 12) was solved, with 60-digit parameters, for $n = 10^5, 10^6, 10^7$: the identities $S_1 = P'_n(0)$ and $S_2 = S_3 = 0$ held with absolute errors below 10^{-33} , and the residual, computed in double precision on equispaced grids of sizes about n (two grid sizes agreeing to about ten digits), satisfied $n^{1/6}\rho(P_n) \approx 282, 244, 238$, that is, decay like $n^{-1/6}$ with a large constant, in agreement with the bound $n^{(2-K)/6}$. No instance with $K \geq 4$ was computed, because the integer weights produced by Lemma 12 for $K = 4$ were too large for the same tests. For Theorem 13 the same tests, up to $n \approx 10^7$, gave $n^{1/6}\rho(P_n)$ decreasing slowly towards 0.61, consistent with the constant of Proposition 14; this is again not a proof.

The comparison uses the exact primary-source versions listed below. A targeted search did not identify the bounds in this exact setting; exhaustive priority certification is not claimed. The accessible Komarov versions listed below were used for the stated comparison. A related 2020 journal text (DOI 10.1070/IM8901) was not available in full in this review; no exclusion of every possible bibliographic overlap is claimed. AI tools assisted exploration, drafting and computational checking. Model review is distinct from independent human peer review and formal proof verification.

References

- [1] X. Tan, Q. Wang, W. Huang and K. Chen, *Residual bounds for Schur-stable polynomials*, arXiv:2608.02043v3, 8 August 2026. <https://arxiv.org/abs/2608.02043v3>.
- [2] Y. Luo, R. Yang and K. Zhu, *Exterior power sums*, arXiv:2607.22017v1, 24 July 2026. <https://arxiv.org/abs/2607.22017v1>.
- [3] M. A. Komarov, *Rate of approximation by logarithmic derivatives of polynomials whose zeros lie on a circle*, arXiv:1804.08918v1, 24 April 2018. <https://arxiv.org/abs/1804.08918v1>.

- [4] M. A. Komarov, *Caratheodory type representation with unit weights and related approximation problems*, arXiv:1807.06499v1, 17 July 2018 (Russian). <https://arxiv.org/abs/1807.06499v1>.
- [5] T. Erdélyi, D. P. Hardin and E. B. Saff, *Inverse Bernstein inequalities and min-max-min problems on the unit circle*, arXiv:1307.4056v1, 15 July 2013. <https://arxiv.org/abs/1307.4056v1>.
- [6] P. Busch, J. Kiukas and R. F. Werner, *Sharp uncertainty relations for number and angle*, arXiv:1604.00566v2, §VI.A.2. <https://arxiv.org/abs/1604.00566v2>.
- [7] NIST Digital Library of Mathematical Functions, §§28.2 and 28.8, especially Eq. (28.8.1). <https://dlmf.nist.gov/28.2>; <https://dlmf.nist.gov/28.8>. Accessed 18 October 2026.
- [8] F. Brunault and W. Zudilin, *Many Variations of Mahler Measures: A Lasting Symphony*, Australian Mathematical Society Lecture Series 28, Cambridge University Press, 2020, Chapter 1. <https://doi.org/10.1017/9781108885553>.
- [9] U. Haagerup and P. de la Harpe, *The numerical radius of a nilpotent operator on a Hilbert space*, Proc. Amer. Math. Soc. **115** (1992), 371–379. <https://doi.org/10.1090/S0002-9939-1992-1072339-6>.
- [10] A. Defant and M. Mastyło, *L^p -norms and Mahler’s measure of polynomials on the n -dimensional torus*, Constr. Approx. **44** (2016), 87–101. <https://doi.org/10.1007/s00365-015-9319-x>.
- [11] L. V. Kovalev and X. Yang, *Extreme values of the derivative of Blaschke products and hypergeometric polynomials*, arXiv:2007.09760v2, 25 June 2021; Bull. Sci. Math. **169** (2021), 102979. <https://arxiv.org/abs/2007.09760v2>.