# From Dobrushin Comparison to a Quasi-Local C*-State: A Lean-Checked Construction for the Two-Dimensional Ising Model

Lluis Eriksson

**Abstract**

We give a Lean 4 formalization of a thermodynamic-limit argument for the anisotropic nearest-neighbour Ising model on odd centred squares. The new analytic brick is a comparison theorem for two finite Gibbs specifications on one product space. A random-scan telescoping identity and the classical Dobrushin resolvent turn single-site total-variation defects into an expectation bound. When every defect row is at graph distance at least $R$ from the support of a local observable $g$, the result is

$$|\mu(g) - \nu(g)| \leq \frac{\alpha^R}{1-\alpha} \sum_j \delta_j(g), \qquad 0 \leq \alpha < 1.$$

The order of summation is essential: the defect sites are summed inside the resolvent, so no cardinality of the ambient volume appears.

We then construct exact restriction and reindexing maps between genuine Gibbs measures on two different squares. For a kernel $g$ supported in the square of radius $r$ and $r \leq n \leq m$, the machine-checked comparison is

$$|\mathbb{E}_m g - \mathbb{E}_n g| \leq \frac{\alpha^{n-r}}{1-\alpha}, |\Lambda_r| \operatorname{osc}(g).$$

It yields convergence of the complete free-boundary sequence, not merely of a subsequence. The limit is first packaged as a compatible family of positive, normalized, real-linear cylinder functionals. It is independent of the admissible auxiliary envelope $\alpha$, of every cofinal sampling of the centred-square sequence, and of enlargement of the declared support. Finally, a second application of the comparison theorem proves stability under arbitrary symmetric boundary-row perturbations receding from the support and, as a concrete corollary, equality of the free and periodic thermodynamic limits.

We then construct finite-site cylinder presentations directly on $\mathbb{Z}^2$ and quotient them by equality of the represented function on the full spin space. Pointwise addition, multiplication and real scalar multiplication descend to this algebraic local carrier. Translation of supports gives a literal additive $\mathbb{Z}^2$ action, including inverses. Exact cyclic reindexing of every finite periodic Gibbs sum, combined with the free–periodic limit theorem, produces a positive normalized real-linear functional $\omega$ satisfying

$$\omega(\tau_z f) = \omega(f), \qquad z \in \mathbb{Z}^2.$$

The real cylinder algebra is then given its intrinsic uniform norm, faithfully complexified as a star algebra, and represented in the bounded continuous functions on the product spin space. The closure is a commutative C*- algebra, and the cylinder functional extends continuously to a positive complex linear functional of norm one. We also construct, for every finite ambient spin system and every finite conditioning set, the exact Gibbs conditional kernel. Lean verifies

positivity, normalization, exterior locality, idempotence, and the finite-volume DLR tower identity by an involutive reindexing of the finite sum. The passage of this finite-set identity to the infinite-volume C*-state is not claimed here. All concrete results use only

$$2\tanh|\beta| + 2\tanh|\gamma| \leq \alpha < 1.$$

No Kotecky–Preiss hypothesis enters. The parameter region remains the classical Dobrushin region; no beyond-Dobrushin phase-diagram improvement and no Yang–Mills consequence is claimed.

# 1 Introduction

Dobrushin's uniqueness method replaces a global comparison of Gibbs measures by a matrix of one-site influences. Its finite-volume form is classical [1, 2]; covariance estimates based on the same resolvent appear, for example, in Follmer's work [3]. The mathematical theorem proved here is therefore not a new high-temperature phase diagram. The contribution is an end-to-end formal chain whose interfaces are usually suppressed on paper:

1. two probability weights and their one-site kernels are compared on a common finite product space;
2. differences of conditionals are telescoped site by site;
3. the resolvent is summed in an order that avoids a spurious volume factor;
4. Gibbs measures on genuinely different rectangles are transported to a common chart without changing either Hamiltonian or normalization;
5. the resulting pairwise estimate constructs a whole-sequence limit;
6. positivity, normalization, linearity, support-presentation coherence, cofinal schedule independence, and free–periodic boundary independence are all passed to the limit;
7. intrinsic integer-site cylinders are quotiented by equality of their global realization, and exact periodic translations prove invariance of the resulting state under the full group $\mathbb{Z}^2$.
8. the local quotient is complexified and completed in its faithful uniform representation, and the state is extended to a positive norm-one functional on the resulting commutative C*-algebra;
9. arbitrary finite conditioning sets produce positive normalized exterior-local idempotent Gibbs kernels, and the finite Gibbs measure is proved to satisfy their exact tower identity.

The formalization was motivated by a precise gap in an earlier finite-volume Dobrushin development: exponential estimates held uniformly in the volume, but no infinite-volume state value had been constructed. A uniform finite-volume estimate is not itself a thermodynamic limit. The missing statement is a comparison of *two volumes* with an error controlled by the distance between a fixed support and the smaller boundary.

The main pitfall is easy to state. Bounding the effect of every changed row separately and then summing can produce a term such as $|\Lambda_m|\alpha^{m-r}$. Such a bound may still vanish along some exhaustions, but it obscures the mechanism and does not give the uniform boundary stability one expects from Dobrushin comparison. The proof below instead sums the defect field against the resolvent before estimating the geometric tail.

### What is machine-checked

Every theorem labelled "machine-checked" is a Lean declaration compiled at the source anchor `f587b35001f4ae87c7fe0383662864df6bc9a1c0`. The terminal source contains no `sorry`. A theorem-to-source table is given in Section 13. The logical audit reports only Lean's standard classical foundations: `propext`, `Classical.choice`, and `Quot.sound`.

### What is not claimed

The paper constructs both an algebraic local state with a full $\mathbb{Z}^2$ action and a positive norm-one state on a concrete commutative C* completion; it also proves free–periodic equality for the centred square exhaustion. It does not compare every possible exhaustion shape, extend the full $\mathbb{Z}^2$ action to the completed algebra, or prove that the completed state satisfies the infinite-volume DLR equations. The finite-set tower theorem below is a load-bearing finite identity, not a relabelling of the completed state as DLR.

## 2 Finite products and one-site influence

Let $I$ be a finite nonempty site set and $S$ a finite nonempty spin set. A configuration is $\eta \in S^I$. For a probability weight $\mu$ and an observable $f : S^I \to \mathbb{R}$, write

$$\mathbb{E}_\mu f = \sum_{\eta \in S^I} \mu(\eta) f(\eta).$$

The one-coordinate oscillation is

$$\delta_i(f) = \sup_{\eta \in S^I,\, s \in S} |f(\eta) - f(\eta^{i \leftarrow s})|.$$

All suprema are finite suprema in the Lean development.

A one-site kernel $p_i(\eta, \cdot)$ is normalized and local: it depends on $\eta$ only outside $i$. Its conditional-expectation operator is

$$P_i f(\eta) = \sum_{s \in S} p_i(\eta, s) f(\eta^{i \leftarrow s}).$$

The matrix $C = (C_{ik})$ is an influence majorant when, for $k \neq i$ and configurations differing only at $k$,

$$\mathrm{TV}\big(p_i(\eta, \cdot), p_i(\eta', \cdot)\big) \leq C_{ik}.$$

The formal comparison theorem permits any nonnegative majorant with zero diagonal and row sums

$$\sum_k C_{ik} \leq \alpha < 1.$$

### 2.1 Random-scan telescoping

Define the random-scan operator

$$Pf = \frac{1}{|I|} \sum_{i \in I} P_i f.$$

If $\mu$ is invariant under all $P_i$, it is invariant under $P$. For a second specification $q$ with invariant weight $\nu$, the difference $\mu(f) - \nu(f)$ is telescoped through iterates of the two scan operators. At a single step, the discrepancy separates into:

- propagation of the current coordinate oscillations through $C$; and
- a source term $b_i\delta_i(f)$, where $\mathrm{TV}(p_i(\eta), q_i(\eta)) \le b_i$.

Iteration produces partial sums of $I + C + C^2 + \cdots$. The remainder vanishes because the scan contracts the total oscillation under the row-sum condition. This is formalized in `DobrushinMeasureComparison.lean`.

**Proposition 2.1** (resolvent comparison; machine-checked)**.** *Under the hypotheses above,*

$$|\mathbb{E}_\mu f - \mathbb{E}_\nu f| \le \sum_i b_i \sum_j \left(\sum_{q\ge 0} (C^q)_{ji}\right) \delta_j(f).$$

*No Cauchy sequence, limiting state, or boundary-tail estimate occurs among the hypotheses.*

## 3 The volume-free boundary tail

Assume a graph distance $d$ on $I$ and range-one support:

$$d(j,i) > 1 \quad \Longrightarrow \quad C_{ji} = 0.$$

Matrix multiplication and the triangle inequality imply

$$q < d(j,i) \quad \Longrightarrow \quad (C^q)_{ji} = 0.$$

The row-sum hypothesis propagates to every power,

$$\sum_i (C^q)_{ji} \le \alpha^q.$$

Suppose $0 \le b_i \le 1$ and every nonzero defect lies at distance at least $R$ from coordinate $j$. Then

$$\begin{aligned}\sum_i b_i \sum_{q\ge 0} (C^q)_{ji} &= \sum_{q\ge 0}\sum_i b_i (C^q)_{ji} \\ &= \sum_{q\ge R}\sum_i b_i (C^q)_{ji} \\ &\le \sum_{q\ge R} \alpha^q = \frac{\alpha^R}{1-\alpha}.\end{aligned}$$

The finite sum over sites is moved inside the infinite sum before the row-sum bound is applied. No estimate of the number of defect sites is used.

**Theorem 3.1** (boundary comparison; machine-checked)**.** *Let $\mu, \nu$ be invariant probability weights for two one-site specifications $p, q$. Let $C$ majorize the influence of $p$, with $0 \le \alpha < 1$, row sums at most $\alpha$, and range one in $d$. Suppose*

$$\mathrm{TV}(p_i(\eta), q_i(\eta)) \le b_i \le 1$$

*and*

$$b_i \ne 0, \quad \delta_j(f) \ne 0 \quad \Longrightarrow \quad R \le d(j,i).$$

*Then*

$$|\mathbb{E}_\mu f - \mathbb{E}_\nu f| \le \frac{\alpha^R}{1-\alpha}\sum_j \delta_j(f).$$

The terminal Lean declaration is `measure_comparison_boundary_decay`. The theorem is genuinely two-specification: its only cross-model input is the pointwise total-variation defect. In particular, it does not assume the conclusion in the form of a Cauchy estimate or a pre-existing boundary tail.

## 4 Specialization to Ising Hamiltonians

For a symmetric zero-diagonal coupling $J : I \times I \to \mathbb{R}$ and two-state spins $\sigma : S \to \{-1, +1\}$, the finite Ising weight is

$$w_J(\eta) = \exp\left(\frac{1}{2}\sum_{i,k\in I} J_{ik}\sigma(\eta_i)\sigma(\eta_k)\right).$$

The associated heat-bath conditional is strictly positive and normalized. Changing spin $k$ moves the local field at $i$ by at most $2|J_{ik}|$, and the two-point conditional calculation yields the classical envelope

$$c_{ik} \le \tanh|J_{ik}|.$$

Let $J$ and $K$ live on the same finite site set. If their complete rows agree at $i$, their heat-bath kernels agree exactly at $i$. On a changed row, total variation is bounded by one. Thus Theorem 3.1 applies with the defect predicate

$$D(i) \iff \exists k,\ J_{ik} \neq K_{ik}.$$

**Theorem 4.1** (Ising row-perturbation comparison; machine-checked)**.** *Assume the envelope row sums of $J$ are at most $\alpha < 1$, and $J$ has range one. If every row changed between $J$ and a second symmetric zero-diagonal coupling $K$ is at distance at least $R$ from the active coordinates of $f$, then*

$$|\mathbb{E}_J f - \mathbb{E}_K f| \le \frac{\alpha^R}{1-\alpha}\sum_j \delta_j(f).$$

*No Dobrushin condition is required of $K$.*

The last observation is useful: comparison is oriented. The reference model $J$ supplies the resolvent; the perturbed Hamiltonian needs positivity and normalization, but its own influence matrix need not be controlled. The Lean endpoint is `ising_boundary_comparison`.

## 5 Two genuine rectangles

Let

$$\Lambda_n = \{0, \ldots, 2n\}^2$$

with anisotropic free nearest-neighbour coupling $\beta$ horizontally and $\gamma$ vertically. The square $\Lambda_n$ is embedded centrally in $\Lambda_m$ by adding $m - n$ to each coordinate.

Comparing two different finite products requires more than identifying their cardinalities. The formal development performs three exact operations.

**Restriction.** Delete every bond of the $m$-coupling with at least one endpoint outside the central copy of $\Lambda_n$. The outside spins become uniform spectators. The partition sum factors, and normalized expectations of observables that ignore the spectators are unchanged.

**Reindexing.** The active subtype of the large chart is explicitly equivalent to $\Lambda_n$. Under that equivalence the restricted coupling is exactly the free $n$-coupling, not merely isomorphic in an informal sense. Gibbs sums and normalizations are transported through the equivalence.

**Observable lifting.** For $r \le n$, a kernel $g$ on $\Lambda_r$ is lifted to $\Lambda_n$ by restriction to the centred core and reindexing. These lifts compose exactly. Outside the core, every coordinate oscillation vanishes. Inside,

$$\sum_{j\in\Lambda_n} \delta_j(g^{\uparrow n}) \le |\Lambda_r|\,\mathrm{osc}(g).$$

This is the second place where the ambient volume disappears.

The files `DobrushinVolumeRestriction.lean`, `DobrushinVolumeEquiv.lean`, and `DobrushinRectangleVolume.lean` implement these steps separately. Their separation is deliberate: it makes the normalization and spectator cancellation auditable independently of the comparison bound.

**Theorem 5.1** (two-volume estimate; machine-checked)**.** *Let $g : (\{0,1\}^{\Lambda_r}) \to \mathbb{R}$, and let $g^{\uparrow n}$ denote its canonical lift to $\Lambda_n$. If $r \le n \le m$ and*

$$0 \le \alpha < 1, \qquad 2\tanh|\beta| + 2\tanh|\gamma| \le \alpha,$$

*then*

$$|\mathbb{E}_m(g^{\uparrow m}) - \mathbb{E}_n(g^{\uparrow n})| \le \frac{\alpha^{n-r}}{1-\alpha},|\Lambda_r|\,\mathrm{osc}(g).$$

This is `centered_local_observable_comparison`. Every quantity on the right is fixed once the observable and the smaller radius are fixed, except for the decaying distance factor.

### 5.1 The visible coupling window

Every site of a free rectangle has at most two horizontal and two vertical neighbours. Hence the envelope row sum is bounded by

$$2\tanh|\beta| + 2\tanh|\gamma|.$$

In the isotropic case $\beta = \gamma$, the condition becomes

$$4\tanh|\beta| < 1, \qquad |\beta| < \operatorname{atanh}(1/4) \approx 0.2554.$$

This is a sufficient high-temperature window, not the exact critical point and not a sharp uniqueness criterion. Its role here is transparency: the same macroscopic Dobrushin window is visible in the finite comparison, the limit theorem, and the boundary-independence results. No cluster-expansion or Kotecky–Preiss constant is inherited from the proof architecture.

## 6 Whole-sequence thermodynamic limit

Fix $r$ and $g$ on $\Lambda_r$. Define the complete sequence

$$E_n(g) = \mathbb{E}_{r+n}(g^{\uparrow(r+n)}), \qquad n \in \mathbb{N}.$$

Theorem 5.1 gives, for $n \le m$,

$$|E_m(g) - E_n(g)| \le \alpha^n \frac{|\Lambda_r|\,\mathrm{osc}(g)}{1-\alpha}.$$

Since $\alpha^n \to 0$, this is a direct Cauchy proof for the entire sequence. Completeness of $\mathbb{R}$ selects a value $\omega_r(g)$, and the same construction proves $E_n(g) \to \omega_r(g)$.

**Theorem 6.1** (complete free-boundary sequence; machine-checked)**.** *In the window of Theorem 5.1, the sequence $E_n(g)$ is Cauchy and converges to a constructed real value $\omega_r(g)$.*

The Cauchy statement is `cauchySeq_centeredLocalGibbsExpectation`; the convergence statement is `tendsto_infiniteCenteredLocalGibbsExpectation`. The declaration order is itself a circularity check: the finite comparison is imported by the Cauchy module, the Cauchy theorem constructs the limit, and the limit is consumed by convergence. No earlier declaration takes a Cauchy sequence, a limit, or a tail estimate as input.

# 7 A compatible positive normalized cylinder-state family

At each fixed radius $r$, the finite-volume expectation is linear, positive, and normalized. These properties pass to the complete-sequence limit by uniqueness of limits and order closure.

**Theorem 7.1** (state properties at fixed radius; machine-checked)**.** *For kernels $f, g : \{0,1\}^{\Lambda_r} \to \mathbb{R}$ and $c \in \mathbb{R}$,*

$$\begin{aligned} \omega_r(f+g) &= \omega_r(f) + \omega_r(g), \\ \omega_r(cf) &= c\,\omega_r(f), \\ \omega_r(\mathbf{1}) &= 1, \\ f \geq 0 &\Longrightarrow \omega_r(f) \geq 0. \end{aligned}$$

The four declarations begin at `infiniteCenteredLocalGibbsExpectation_add` and end at `infiniteCenteredLocalGibbsExpectation_nonneg`.

A local observable may be presented at many radii. If $r \leq s$, let $\iota_{r,s} g$ be its canonical lift from $\Lambda_r$ to $\Lambda_s$. The finite sequences associated with the two presentations are not termwise identical: their $n$th terms live in squares of radii $r+n$ and $s+n$. The two-volume estimate nevertheless shows that their difference tends to zero. This proves the coherence relation

$$\omega_s(\iota_{r,s} g) = \omega_r(g).$$

**Theorem 7.2** (compatible cylinder-state family; machine-checked)**.** *The family $(\omega_r)_{r\in\mathbb{N}}$ consists of positive normalized real-linear functionals and is compatible with every canonical support enlargement.*

Lean bundles exactly these fields in `infiniteCenteredLocalGibbsStateFamily`. Section 9 constructs the corresponding quotient-level carrier and proves that this compatible family descends to it.

## 7.1 Two further canonicality statements

First, the construction is independent of the auxiliary $\alpha$. If two numbers $\alpha, \alpha' < 1$ both dominate the same Ising row sum, both completeness constructions are limits of the same finite sequence and hence are equal. This is `infiniteCenteredLocalGibbsExpectation_independent_alpha`.

Second, if $v : \mathbb{N} \to \mathbb{N}$ is cofinal, then

$$E_{v(k)}(g) \longrightarrow \omega_r(g).$$

Thus skipping radii, changing the rate at which centred squares grow, or repeating finitely many volumes does not change the limit, provided the schedule tends to infinity. The theorem is `tendsto_infiniteCenteredLocalGibbsExpectation_comp`. This is schedule independence within the same shape family; it is not yet a comparison with arbitrary non-square Følner sets.

# 8 Boundary-condition stability

The finite comparison gives more than free-boundary convergence. For each $n$, let $K_n$ be any symmetric zero-diagonal coupling on $\Lambda_{r+n}$. Let $D_n$ be a set of rows such that

$$i \notin D_n \quad \Longrightarrow \quad K_n(i,k) = J_n^{\mathrm{free}}(i,k) \quad \text{for every } k.$$

Suppose every active coordinate of the lifted observable is at distance at least $n$ from every row in $D_n$. No row-sum hypothesis is imposed on $K_n$.

**Theorem 8.1** (receding row perturbations; machine-checked)**.** *Under the Dobrushin hypothesis for the free reference coupling,*

$$|E_n^{\mathrm{free}}(g) - E_n^K(g)| \le \frac{\alpha^n}{1-\alpha} |\Lambda_r| \operatorname{osc}(g),$$

*and*

$$E_n^K(g) \longrightarrow \omega_r(g).$$

*The assumptions on $(K_n)$ are local row equality, symmetry, zero diagonal, and separation from the fixed support. They do not include Cauchy or convergence of $E_n^K$.*

The generic convergence endpoint is `tendsto_centeredPerturbedGibbsExpectation`. This result covers boundary modifications expressible by changed interaction rows on the common chart. Fixed external fields would require an explicit one-body or ghost-site encoding and are not silently included.

## 8.1 Periodic rectangles

For an actual boundary condition, the formalization defines cyclic adjacency on each finite coordinate and the corresponding anisotropic periodic coupling. Periodic and free rows agree away from the four edges. If the observable is supported in $\Lambda_r \subset \Lambda_{r+n}$, every edge row is at Manhattan distance at least $n$ from every active coordinate. Theorem 8.1 therefore applies directly.

**Theorem 8.2** (free–periodic equality; machine-checked)**.** *Throughout*

$$0 \le \alpha < 1, \qquad 2\tanh|\beta| + 2\tanh|\gamma| \le \alpha,$$

*the complete periodic sequence satisfies*

$$E_n^{\mathrm{per}}(g) \longrightarrow \omega_r(g),$$

*the same limit as the complete free-boundary sequence.*

The concrete endpoint is `tendsto_centeredPeriodicGibbsExpectation`. The periodic model itself need not be re-analysed by a second Dobrushin argument: the oriented comparison uses the free resolvent and the fact that periodicity changes only boundary rows.

# 9 The integer site-cylinder carrier and full covariance

The centred family is now converted into a state on an intrinsic local carrier. Write $\mathcal{X} = \{0,1\}^{\mathbb{Z}^2}$. A finite presentation consists of a finite set $S \subset \mathbb{Z}^2$ and a kernel

$$F_S : \{0,1\}^S \longrightarrow \mathbb{R}.$$

It represents the global cylinder function

$$\widehat{F}_S(\sigma) = F_S(\sigma|_S), \qquad \sigma \in \mathcal{X}.$$

Presentations are equivalent precisely when their represented functions agree on every $\sigma \in \mathcal{X}$. Lean forms the quotient

$$\mathcal{A}_{\mathrm{loc}} = \{\text{finite site-cylinder presentations}\}/\sim .$$

This is an actual quotient carrier, not merely the assertion that a compatible family could be quotiented. Union of finite supports defines pointwise addition and multiplication; negation and real scalar multiplication preserve the support. All operations descend because their global realizations are pointwise operations. The realization map from the quotient to $\mathcal{X} \to \mathbb{R}$ is injective.

For $z \in \mathbb{Z}^2$, translation sends the support $S$ to $S + z$ and satisfies

$$\widehat{\tau_z F}(\sigma) = \widehat{F}\big(q \mapsto \sigma(q+z)\big).$$

The quotient carries a literal additive action:

$$\tau_0 F = F, \qquad \tau_{z+w} F = \tau_z(\tau_w F), \qquad \tau_{-z}(\tau_z F) = F.$$

The construction is typed for Ising sites. It does not coerce the separate gauge-link observable type used elsewhere in the repository.

## 9.1 Exact finite periodic translations

Let $\rho_N$ be cyclic successor on $\operatorname{Fin} N$. Direct arithmetic shows that cyclic nearest-neighbour adjacency is invariant under $\rho_N$. Consequently the anisotropic periodic coupling obeys

$$J_N^{\mathrm{per}}(\rho_N p, \rho_N q) = J_N^{\mathrm{per}}(p, q)$$

for a shift in either coordinate. Reindexing the finite configuration sum then gives, for every test function $H$ and without a small-coupling premise,

$$\mathbb{E}_N^{\mathrm{per}}(H \circ \tau_{e_i}) = \mathbb{E}_N^{\mathrm{per}}(H), \qquad i = 1, 2.$$

These are the machine-checked declarations `expect_periodic_torusShiftX` and `expect_periodic_torusShiftY`.

Each integer presentation has a canonical radius large enough to contain its support. In any larger centred chart, adding the chart centre to the integer coordinates realizes its kernel. Enlarging the chart agrees exactly with the old canonical lift. Thus the compatible centred state descends to a well-defined functional $\omega$ on $\mathcal{A}_{\mathrm{loc}}$; equality of global cylinder functions, including presentations with redundant sites, implies equality of state values.

**Theorem 9.1** (fully translation-invariant local state; machine-checked)**.** *Suppose*

$$0 \le \alpha < 1, \qquad 2\tanh|\beta| + 2\tanh|\gamma| \le \alpha.$$

*There is a functional* $\omega : \mathcal{A}_{\mathrm{loc}} \to \mathbb{R}$ *such that*

$$\begin{aligned} \omega(F+G) &= \omega(F) + \omega(G), & \omega(cF) &= c\,\omega(F), \\ \omega(\mathbf{1}) &= 1, & F \ge 0 &\Longrightarrow \omega(F) \ge 0, \\ \omega(\tau_z F) &= \omega(F) & &\textit{for every } z \in \mathbb{Z}^2. \end{aligned}$$

*Negative translations are included as group inverses. The hypothesis list contains neither a thermodynamic-limit premise nor translation invariance.*

The quotient descent is `infiniteValue_eq_of_equivalent`; the full group identity is `infiniteStateValue_vadd`; and the terminal bundled object is `infiniteTranslationInvariantSiteCylinderState`. The proof of the group identity first compares both translated presentations in one sufficiently large periodic chart, uses the exact finite symmetry, and then invokes Theorem 8.2 twice. Hence Dobrushin comparison participates through the free–periodic convergence theorem, while the translation step itself introduces no narrower analytic window.

# 10 Uniform completion and finite-set Gibbs specification

## 10.1 A concrete commutative C*-algebra and its state

The quotient realization is now used as a norm producer, rather than as informal notation. For $F \in \mathcal{A}_{\mathrm{loc}}$ put

$$\|F\|_\infty = \sup_{\sigma \in \mathcal{X}} |\widehat{F}(\sigma)|.$$

Faithfulness of the global realization makes this a genuine norm. Positivity and normalization of the state imply the machine-checked estimate

$$|\omega(F)| \le \|F\|_\infty.$$

The proof applies positivity to the two nonnegative cylinders $\|F\|_\infty \mathbf{1} \pm F$; contractivity is therefore derived from the state laws and is not an extra continuity hypothesis.

A complex local cylinder is the explicit pair $F + iG$ with $F, G \in \mathcal{A}_{\mathrm{loc}}$. Multiplication is $(F+iG)(H+iK) = (FH - GK) + i(FK + GH)$ and the involution is pointwise conjugation. The map

$$F + iG \longmapsto \big(\sigma \mapsto \widehat{F}(\sigma) + i\widehat{G}(\sigma)\big)$$

is proved to be an injective star-algebra homomorphism into the bounded continuous complex functions on the compact product space $\mathcal{X}$. Define

$$\mathcal{A} = \overline{\mathcal{A}_{\mathrm{loc}} + i\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|_\infty} \subset C_b(\mathcal{X}; \mathbb{C}).$$

In Lean, $\mathcal{A}$ is literally the topological closure of the range star subalgebra. Its closed-subalgebra instance supplies completeness and the C* identity; neither is postulated in a record field. The local embedding is isometric and has dense range.

**Theorem 10.1** (quasi-local C*-state; machine-checked)**.** *Under the hypotheses of Theorem 9.1, there is a complex linear functional $\widetilde{\omega} : \mathcal{A} \to \mathbb{C}$ satisfying*

$$\begin{aligned} \widetilde{\omega}(F + iG) &= \omega(F) + i\omega(G), \qquad F, G \in \mathcal{A}_{\mathrm{loc}}, \\ x \geq 0 &\Longrightarrow \widetilde{\omega}(x) \geq 0, \\ \widetilde{\omega}(\mathbf{1}) &= 1, \qquad \|\widetilde{\omega}\| = 1. \end{aligned}$$

*Thus $\widetilde{\omega}$ is a positive normalized state on the explicit commutative C*-algebra $\mathcal{A}$.*

The faithful representation is `realizeGlobalBCFStarAlgHom_injective`; density is `denseRange_localToQuasilocal`; positivity on all positive elements is `quasilocalStateCLM_nonneg`; and the terminal positive map is `quasilocalCStarState`. The norm-one identity is separately proved at `norm_quasilocalStateCLM`.

### 10.2 Arbitrary finite conditioning sets

Let $I$ and $S$ be finite nonempty types, let $w : S^I \to \mathbb{R}_{>0}$, and let $\Lambda \subset I$ be any finite set. If $\eta[x]_\Lambda$ replaces precisely the spins of $\eta$ in $\Lambda$ by $x \in S^\Lambda$, define

$$Z_\Lambda(\eta) = \sum_{x \in S^\Lambda} w(\eta[x]_\Lambda), \qquad \gamma_\Lambda(x \mid \eta) = \frac{w(\eta[x]_\Lambda)}{Z_\Lambda(\eta)},$$

and

$$(\Gamma_\Lambda F)(\eta) = \sum_{x \in S^\Lambda} \gamma_\Lambda(x \mid \eta) F(\eta[x]_\Lambda).$$

No rectangular geometry is built into these definitions.

**Theorem 10.2** (finite-set Gibbs tower; machine-checked)**.** *For every positive finite weight and every finite $\Lambda \subset I$, $\gamma_\Lambda$ is strictly positive, sums to one, and depends on $\eta$ only through $I \setminus \Lambda$. The operator $\Gamma_\Lambda$ preserves one and positivity, is exterior-local and idempotent, and the Gibbs probability $\mu_w = w / \sum_\xi w(\xi)$ satisfies*

$$\mathbb{E}_{\mu_w}[\Gamma_\Lambda F] = \mathbb{E}_{\mu_w}[F]$$

*for every $F : S^I \to \mathbb{R}$.*

The tower identity is not passed as an antecedent. Its Lean proof reindexes the double sum by the involution

$$(\eta, x) \longmapsto (\eta[x]_\Lambda, \eta|_\Lambda),$$

then cancels the positive local and global partition functions. The main endpoints are `finiteSetGibbsKernel_sum_one`, `finiteSetCondExp_idem`, and `expect_finiteSetCondExp`. This theorem supplies the finite algebraic DLR layer. Identifying the stable nearest-neighbour kernel on $\mathbb{Z}^2$ and passing the identity through the thermodynamic limit remain separate obligations.

## 11 Proof architecture and kill tests

The proof was designed around failure modes that can survive informal notation.

**No hidden limit premise.** The abstract comparison theorem is finite. The rectangle theorem imports it. Only then is Cauchy proved, and only Cauchy is passed to completeness. The boundary-perturbed sequence is shown convergent from a quantitative comparison with the already constructed free sequence; its convergence is not assumed.

**No synthetic common-volume shortcut.** The smaller Gibbs measure is not defined by assertion on the larger space. Spectator spins are eliminated by an exact partition-sum factorization, and an explicit equivalence reindexes the remaining Gibbs sum. These modules participate in the terminal comparison theorem.

**No volume factor.** The sum over defect sites is performed inside the resolvent. The vanishing of short matrix powers and the row-sum estimate are used before any cardinality bound. The only cardinality in Theorem 5.1 is the fixed support size $|\Lambda_r|$.

**No drifting observable norm.** The canonical lift has zero coordinate oscillation outside the core. Its total coordinate oscillation is bounded by the fixed core cardinality times the global oscillation of the original kernel.

**No hidden cluster-expansion window.** The terminal theorem types mention only $0 \leq \alpha < 1$ and the explicit Ising row-sum inequality. The proof reuses the architecture of a separate thermodynamic-limit development but none of its Kotecky–Preiss hypotheses.

**No false translation coercion.** The new support coordinates are Ising sites in $\mathbb{Z}^2$. The action laws, finite periodic symmetry and infinite-state invariance are proved on that carrier itself. The gauge-link action elsewhere in the repository is not imported into any statement of the construction.

**No asserted C$^*$ completion.** The norm is induced by an injective realization into bounded continuous functions. The quasi-local carrier is the topological closure of the actual local range, and density is proved. Positivity is first proved on local star-squares, propagated by continuity, and then converted to positivity on all positive elements using the C$^*$ spectral order. Normalization and the operator-norm lower and upper bounds are separate theorems.

**No DLR conclusion as premise.** The finite-set kernel is defined from the positive weight and a replacement map. The fixed-point equation for the finite Gibbs measure is the conclusion of an explicit involutive sum reindexing. Conversely, no infinite-volume DLR label is attached to the completed state until the stable lattice kernel and limit passage have been formalized.

## 12 Relation to classical work

Dobrushin's 1968 paper formulates regularity and uniqueness conditions through conditional probabilities [1]; the 1970 paper develops the prescription of random fields by conditional distributions and the associated comparison machinery [2]. The resolvent-based covariance estimate is classically associated with Follmer [3]. The finite-spin Ising specialization used here is standard: the single-bond conditional has a hyperbolic-tangent variation, and summing those envelopes gives the Dobrushin coefficient.

The new content is not a stronger uniqueness region. It is the explicit, kernel-checked assembly of the following interfaces in Lean 4:

$$\begin{gathered}\text{sitewise TV defects} \Longrightarrow \text{two-specification comparison} \Longrightarrow \text{two genuine rectangles}\\ \Longrightarrow \text{whole-sequence Cauchy} \Longrightarrow \text{compatible positive normalized family}\\ \Longrightarrow \text{receding-boundary stability} \Longrightarrow \text{free–periodic equality}\\ \Longrightarrow \text{integer-site quotient} \Longrightarrow \text{exact periodic translations} \Longrightarrow \text{full } \mathbb{Z}^2\text{-invariant local state}\\ \Longrightarrow \text{uniform complex star closure} \Longrightarrow \text{positive norm-one C}^*\text{-state,}\end{gathered}$$

Alongside this thermodynamic chain, the finite product layer now constructs arbitrary finite-set conditional kernels and proves their Gibbs tower identity. This auxiliary theorem is parameter-free and does not enlarge the classical Dobrushin uniqueness region. We make no categorical priority claim about formalizations in other proof assistants. The reproducible contribution is the particular Lean theorem chain and its audited source anchor.

## 13 Formal audit and reproducibility

The development uses Lean 4 and Mathlib [4, 5]. Table 1 lists the principal terminal declarations. The source links are immutable because they include the source commit rather than a moving branch.

Table 1: Machine-checked API at the source freeze.

| Lean declaration | Mathematical role |
|---|---|
| `measure_comparison_boundary_decay` | Abstract volume-free boundary tail. |
| `ising_boundary_comparison` | Two Ising couplings on one chart. |
| `centered_local_observable_comparison` | Genuine two-rectangle estimate. |
| `cauchySeq_centeredLocalGibbsExpectation` | Cauchy property of the complete sequence. |
| `tendsto_infiniteCenteredLocalGibbsExpectation` | Construction and convergence of the limit. |
| `infiniteCenteredLocalGibbsExpectation_independent_alpha` | Independence of the auxiliary envelope. |
| `infiniteCenteredLocalGibbsExpectation_lift` | Coherence under support enlargement. |
| `infiniteCenteredLocalGibbsStateFamily` | Positive normalized compatible family. |
| `tendsto_centeredPerturbedGibbsExpectation` | Generic receding-boundary stability. |
| `tendsto_centeredPeriodicGibbsExpectation` | Free–periodic exhaustion equality. |
| `LocalCylinderAlgebra` | Quotient of finite site presentations by global equality. |
| `neg_vadd_vadd` | Literal inverse law for the $\mathbb{Z}^2$ action. |
| `expect_periodic_torusShiftX` | Exact horizontal periodic Gibbs symmetry. |
| `expect_periodic_torusShiftY` | Exact vertical periodic Gibbs symmetry. |
| `infiniteStateValue_vadd` | Invariance under every integer displacement. |

| Lean declaration | Mathematical role |
|---|---|
| infiniteTranslationInvariantSiteCylinderState | Positive normalized fully invariant local state. |
| abs_infiniteStateValue_le_uniformNorm | Intrinsic uniform contractivity. |
| realizeGlobalBCFStarAlgHom_injective | Faithful complex star representation. |
| denseRange_localToQuasilocal | Dense local inclusion in the C* closure. |
| quasilocalStateCLM_nonneg | Positivity on every positive completed element. |
| norm_quasilocalStateCLM | Operator norm exactly one. |
| quasilocalCStarState | Terminal positive normalized C* functional. |
| finiteSetCondExp_idem | Idempotent arbitrary finite-set Markov operator. |
| expect_finiteSetCondExp | Exact finite Gibbs tower/DLR identity. |

### Build protocol

The release protocol is:

1. clone the public repository and detach at the frozen source commit;
2. restore Mathlib/Lake artifacts with `lake exe cache get`;
3. build the terminal module `YangMills.OS.DobrushinFiniteSetDLR`, which imports the uniform and C* state layers;
4. build `YangMillsCore`;
5. run `lake env lean oracle_check.lean`;
6. compile this manuscript twice with `pdflatex`;
7. render every PDF page to PNG and inspect page geometry and typography.

The release manifest accompanying the PDF records exact commands, exit codes, times, byte counts, and SHA-256 digests. A source file being present is not reported as verified unless its target was materialized successfully.

## 14 Limitations and next theorem

The current result is materially stronger than a centred scalar limit: it is a positive normalized real-linear functional on an explicit quotient of finite $\mathbb{Z}^2$ site cylinders, invariant under the complete translation group and stable under the free–periodic boundary change, together with a positive norm-one extension to a concrete commutative C*-algebra. It also includes the exact finite-system, finite-set Gibbs tower. Three boundaries remain.

1. **Infinite-volume DLR passage.** The arbitrary finite-set kernel, its Markov laws and the finite Gibbs fixed-point identity are formalized. The stable nearest-neighbour kernel on the infinite product space and the limit theorem $\widetilde{\omega}(\Gamma_\Lambda F) = \widetilde{\omega}(F)$ are not yet proved.

2. **Arbitrary exhaustion geometry.** Cofinal schedules of centred squares and the free–periodic pair are covered. Arbitrary aspect ratios, general Følner exhaustions, and all one-body boundary fields are not.

3. **Completed covariance and parameter frontier.** The local real state has the full $\mathbb{Z}^2$ invariance theorem, while the continuous automorphism action and its invariance on the C$^*$ closure are not bundled. The analytic window remains the classical, conservative entrywise Dobrushin envelope. No "non-classical" region is inferred from the new algebraic completion.

The next theorem is therefore the infinite-volume interface: define the finite-range Ising specification on $\mathbb{Z}^2$, prove its cylinder locality, and pass Theorem 10.2 through the already established complete thermodynamic-limit sequence. Completed covariance can then be obtained by extending the isometric local translations. Any genuine enlargement beyond the classical Dobrushin region requires a separate analytic mechanism and is not supplied by terminology.

# 15 Conclusion

The thermodynamic limit is obtained from the estimate that was actually missing: a two-volume comparison whose cost is the distance from a fixed support to a changed boundary. Random-scan telescoping produces the Dobrushin resolvent; finite propagation kills powers shorter than that distance; row sums bound the remaining geometric tail. Exact restriction and reindexing then identify the comparison with two genuine Gibbs rectangles.

The final Lean development proves the complete free sequence convergent, packages its values into a compatible positive normalized cylinder-state family, removes dependence on the auxiliary envelope and on cofinal sampling, and proves that periodic and free rectangles select the same limit. It then constructs the integer-site quotient, proves exact periodic symmetries and obtains a positive normalized state invariant under every translation in $\mathbb{Z}^2$. The quotient is then faithfully complexified and uniformly closed; the state extends to a positive functional of norm one on the resulting commutative C$^*$-algebra. Arbitrary finite conditioning sets satisfy an exact finite Gibbs tower identity. The visible region remains the classical Dobrushin window, rather than a constant imported from a cluster expansion; infinite-volume DLR passage, completed covariance, and a beyond-Dobrushin region remain explicit future work.

## A Dependency map

The terminal import chain is

$$
\begin{array}{c}
\texttt{DobrushinConditional},\ \texttt{DobrushinComparison} \\
\texttt{DobrushinGibbs},\ \texttt{DobrushinIsing},\ \texttt{DobrushinLattice} \\
\Downarrow \\
\texttt{DobrushinMeasureComparison} \longrightarrow \texttt{DobrushinIsingComparison} \\
\Downarrow \\
\texttt{DobrushinVolumeRestriction} \quad + \quad \texttt{DobrushinVolumeEquiv} \\
\Downarrow \\
\texttt{DobrushinRectangleVolume} \longrightarrow \texttt{DobrushinThermodynamicLimit} \\
\Downarrow \\
\texttt{DobrushinInfiniteState} \\
\Downarrow \\
\texttt{DobrushinSiteCylinder} \longrightarrow \texttt{DobrushinPeriodicTranslation} \\
\Downarrow \\
\texttt{DobrushinCovariantState} \longrightarrow \texttt{DobrushinUniformState} \\
\Downarrow \\
\texttt{DobrushinCStarState} \longrightarrow \texttt{DobrushinFiniteSetDLR}.
\end{array}
$$

Each arrow corresponds to an actual Lean import and theorem consumption. The Kotecky–Preiss thermodynamic-limit development is an architectural precedent, not an analytic dependency of the Dobrushin terminal theorem.

## B Glossary of scopes

**Complete sequence.** Every $n \in \mathbb{N}$ is used; no compactness subsequence is selected.

**Cofinal schedule.** A map $v : \mathbb{N} \to \mathbb{N}$ tending to infinity; monotonicity is unnecessary.

**Centred cylinder presentation.** A real kernel on configurations of one finite centred square, with canonical lifts to larger centred squares.

**Compatible state family.** One positive normalized linear functional at each radius, with equality under canonical lift.

**Integer site-cylinder quotient.** Finite supports in $\mathbb{Z}^2$, identified exactly when their represented functions on $\{0,1\}^{\mathbb{Z}^2}$ agree.

**Full covariance.** The identity $\omega(\tau_z F) = \omega(F)$ for every $z \in \mathbb{Z}^2$, including negative displacements as group inverses, on the real local cylinder algebra.

**Quasi-local C$^*$-algebra.** The uniform topological closure of the faithfully represented finite-support complex star algebra inside $C_b(\{0,1\}^{\mathbb{Z}^2}; \mathbb{C})$.

**C$^*$-state.** The positive complex-linear norm-one extension $\widetilde{\omega}$ to that closure. Completed translation covariance is not part of this term here.

**Finite-set Gibbs tower.** The fixed-point identity $\mathbb{E}_{\mu_w}\Gamma_\Lambda F = \mathbb{E}_{\mu_w} F$ for every positive finite weight and every finite conditioning set, derived by a finite involution.

**Receding boundary perturbation.** A coupling whose changed rows are at graph distance at least $n$ from the fixed support at stage $n$.

**Periodic exhaustion.** The explicit cyclic nearest-neighbour coupling on every odd square; not an informal boundary-condition label.

**Not included.** Arbitrary Følner shapes, infinite-volume DLR for the completed state, completed $\mathbb{Z}^2$ covariance, a beyond-Dobrushin parameter region, or a Yang–Mills conclusion.

# References


[1] R. L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, *Theory of Probability and Its Applications* 13 (1968), 197–224. doi:10.1137/1113026.

[2] R. L. Dobrushin, Prescribing a system of random variables by conditional distributions, *Theory of Probability and Its Applications* 15 (1970), 458–486. doi:10.1137/1115049.

[3] H. Follmer, A covariance estimate for Gibbs measures, *Journal of Functional Analysis* 46 (1982), 387–395. doi:10.1016/0022-1236(82)90053-2.

[4] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction – CADE 28*, LNCS 12699, 2021, 625–635. doi:10.1007/978-3-030-79876-5_37.

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