AIRR.SCIENCE · ARR-2026-50TWRE67AJ9H78SM v1 [Mathematical Physics] 29 Nov 2026

EXACT ALL-COUPLING 'T HOOFT CENTER-VORTEX FREE-ENERGY BOUNDS, FOUR-LINE GINIBRE VORTEX DOMINATION, AND CONSTRUCTIVE ELECTRIC-FLUX AREA LAWS IN d-DIMENSIONAL SU(N) LATTICE YANG–MILLS THEORY

LLUÍS ERIKSSON

Abstract. We prove four self-contained, cluster-expansion-free theorems governing 't Hooft center-vortex free energies $F_{\rm vortex}(z)$ and dual electric-flux free energies $F_{\rm elec}(e)$ on d-dimensional periodic hypercubic tori $\mathbb{T}^d_L$ ($d \geq 2$, $L_\mu \geq 2$) in SU(N) ($N \geq 2$) lattice Yang–Mills theory:

(1) *Exact electric-flux Fourier positivity and* SU(2) *self-duality:* Via the commuting unitary $\mathbb{Z}_N$ center-twist operator $\Gamma(z)$ on the Osterwalder–Seiler transfer matrix $\mathcal{T}$, every electric-flux partition function $Z_{\rm elec}(e)$ is strictly positive for all pure Wilson couplings $\beta > 0$ (all $L_1 \geq 2$) and for all SU(2) fundamental–adjoint (Bhanot–Creutz) couplings $\beta_F \neq 0$, $\beta_A \in \mathbb{R}$ (all even $L_1 \geq 2$). Consequently, $F_{\rm vortex}(e^{2\pi ik/N}) > 0$ strictly for $1 \leq k \leq N-1$, and for SU(2), $F_{\rm elec}(1) = -\ln\tanh(\frac{1}{2}F_{\rm vortex}(-1)) > 2e^{-F_{\rm vortex}(-1)}$ identically.

(2) *All-coupling non-Abelian plaquette-weighted Jensen ceiling:* Charge-conjugation symmetry and Jensen's inequality yield $0 < F_{\rm vortex}(\mathbb{T}^d_L; e^{2\pi ik/N}) \leq \beta(1-\cos\frac{2\pi k}{N})P^{(1,2)}_{\mathbb{T}^d_L}({\rm SU}(N),\beta)L_\perp$ for all $\beta > 0$. Proving an exact d-way axial-gauge Hölder upper bound $f_{\Lambda_{L-1}}(\beta) \leq \frac{2}{d}\ln c_0({\rm SU}(N), \frac{d}{2}\beta)$ and combining it with free-BC ladder/axial-gauge variational lower bounds and periodic–free comparison certifies a strictly sub-trivial asymptotic ceiling $P^{\rm hi}_d({\rm SU}(N),\beta) < 1$ on isotropic tori across all couplings.

(3) *Four-line Ginibre vortex domination and Weighted-Kraft electric-flux area law uniformly in* $\beta_A \in \mathbb{R}$*:* For SU(2) Bhanot–Creutz theory on any torus $\mathbb{T}^d_L$ ($d \geq 3$), $F^{\rm SU(2)}_{\rm vortex}(-1;\beta_F,\beta_A) \leq F^{\mathbb{Z}_2}_{\rm vortex}(-1;|\beta_F|)$ holds for all $\beta_F, \beta_A \in \mathbb{R}$. A prefix-free Weighted-Kraft queue-closure encoding of connected closed mod-2 plaquette surfaces gives $1 - e^{-F^{\rm SU(2)}_{\rm vortex}(-1)} \leq 2L_\perp(B_d\tanh|\beta_F|)^{L_1L_2}$ with exact algebraic constants $B_3 = \frac{11+3\sqrt{13}}{2} \approx 10.908327$ and $B_4 \approx 42.902777$, proving a strictly positive electric-flux string tension $\underline{\sigma}^{\rm SU(2)}_{\rm elec}(\beta_F,\beta_A) \geq -\ln(B_d\tanh|\beta_F|) > 0$ uniformly in $\beta_A \in \mathbb{R}$ whenever $|\beta_F| < \operatorname{artanh}(1/B_d)$.

(4) *Exact* 2*D* SU(N) *torus identities:* On $\mathbb{T}^2_{L_1,L_2}$ of area $A = L_1L_2$, $Z^{\rm 2D}_{\rm elec}(e;A) = \sum_{|\lambda|\equiv e \pmod N} c_\lambda(\beta)^A$, proving $\sigma^{\rm 2D}_{\rm elec}({\rm SU}(2),\beta) = -\ln(I_2(\beta)/I_1(\beta)) = \sigma^{\rm 2D}_{1/2}(\beta) > 0$ for all $\beta > 0$. Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`ThooftVorticesCerts.lean`) verify all algebraic and numerical inequalities.

1. Introduction and Honest Scope

In 't Hooft's formulation of confinement in SU(N) gauge theory on a periodic torus [13, 14], inserting a center-group twist $z = e^{2\pi ik/N}I_N \in Z({\rm SU}(N)) \cong \mathbb{Z}_N$ into the plaquettes of a co-closed $(d-2)$-dimensional stack Σ^* dual to the $(1,2)$-plane forces a magnetic center-vortex flux through the $(1,2)$-plane. Its free-energy cost $F_{\rm vortex}(z) := -\ln(Z_{\rm tw}(z)/Z_0)$ is related by exact discrete $\mathbb{Z}_N$ Fourier transform to the electric-flux partition functions $Z_{\rm elec}(e)$ [14, 7, 15, 6, 5]: exponential area-law decay $F_{\rm vortex}(z) \sim L_\perp e^{-\sigma_{\rm elec}L_1L_2}$ is equivalent to a strictly positive 't Hooft electric-flux string tension $\sigma_{\rm elec} > 0$ (confinement of color-electric flux), whereas non-vanishing vortex surface tension $F_{\rm vortex}(z) \sim \tilde{\sigma}L_\perp$ signals deconfinement or screening of magnetic disorder.

Date: November 29, 2026.

While classic Mack–Petkova [7, 8] and Tomboulis–Yaffe [15] comparisons relate SU(2) vortices to $\mathbb{Z}_2$ gauge theory and reflection-positive Peierls bounds, a completely self-contained, cluster-expansion-free treatment that simultaneously covers (i) exact all-coupling electric-flux positivity across arbitrary signed adjoint couplings $\beta_A \in \mathbb{R}$, (ii) sharp non-Abelian plaquette-weighted all-coupling vortex ceilings for general SU(N), (iii) sharp prefix-free closed-surface entropy constants B_3, B_4 yielding explicit finite-torus decay and all-β_A electric-flux area laws, and (iv) machine-verified high-precision certificates across 2D, 3D, and 4D does not appear in a single elementary reference.

In this paper, we establish four self-contained theorems:

- *Exact 't Hooft electric-flux positivity and* SU(2) *self-duality* (Section 2): Explicit verification of the commuting $\mathbb{Z}_N$ twist operator $\Gamma(z)$ on the Osterwalder–Seiler transfer matrix $\mathcal{T}$, covering both pure Wilson SU(N) ($\beta > 0$, all $L_1 \geq 2$) and arbitrary-β_A SU(2) Bhanot–Creutz actions ($\beta_F \neq 0, \beta_A \in \mathbb{R}$, all even $L_1 \geq 2$).
- *All-coupling non-Abelian Jensen ceilings via d-way axial-gauge Hölder bounds* (Section 3): Charge-conjugation elimination of $\langle \operatorname{Im} \operatorname{Tr} U_p \rangle$ plus a three-line d-way axial-gauge Hölder upper bound $f_{\Lambda_{L-1}}(\beta) \leq \frac{2}{d} \ln c_0(\mathrm{SU}(N), \frac{d}{2}\beta)$ and axial-gauge Gibbs–Bogoliubov lower bounds certifying strictly sub-trivial $P_d^{\mathrm{hi}}(\mathrm{SU}(N), \beta) < 1$ across 3D/4D SU(2), SU(3), SU(4).
- *Four-line Ginibre vortex domination and Weighted-Kraft queue-closure area law* (Section 4): Exact domination $F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1; \beta_F, \beta_A) \leq F^{\mathbb{Z}_2}_{\mathrm{vortex}}(-1; |\beta_F|)$ for all $\beta_F, \beta_A \in \mathbb{R}$ combined with a prefix-free queue-closure Kraft inequality giving $B_3 = \frac{11+3\sqrt{13}}{2} \approx 10.908327$ and $B_4 \approx 42.902777$ with *no* geometric-series prefactor, proving $\sigma^{\mathrm{SU}(2)}_{\mathrm{elec}}(\beta_F, \beta_A) \geq -\ln(B_d \tanh |\beta_F|) > 0$ uniformly for all $\beta_A \in \mathbb{R}$ whenever $|\beta_F| < \operatorname{artanh}(1/B_d)$.
- *Exact* 2*D* SU(N) *torus formulas and* 24*-benchmark* 140*-digit / zero-axiom Lean 4 audit* (Sections 5–6): Exact 2D electric-flux equality $\sigma^{\mathrm{2D}}_{\mathrm{elec}} = \sigma^{\mathrm{2D}}_{1/2} = -\ln(I_2(\beta)/I_1(\beta))$, 24 zero-quadrature `mpmath.iv` certificates, and a 5-theorem zero-axiom Lean 4.36.0 verifier.

Remark 1.1 (Honest Scope and Non-Claims). All results in this paper hold for finite-volume and thermodynamic-limit lattice Yang–Mills theory at fixed lattice spacing $a = 1$. Theorems 2.1, 3.2, and 5.1 hold at *all* couplings $\beta > 0$, whereas the constructive $d \geq 3$ electric-flux area law in Theorem 4.1 holds in the strong-fundamental strip $|\beta_F| < \operatorname{artanh}(1/B_d)$ (uniformly for all $\beta_A \in (-\infty, +\infty)$), well below the 4D weak-coupling scaling window. This paper does *not* construct a 4D continuum limit or a continuum mass gap and does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Exact 't Hooft Electric-Flux Fourier Positivity and SU(2) Self-Duality

Fix $d \geq 2$, $N \geq 2$, and a periodic hypercubic torus $\mathbb{T}^d_L := \prod_{\mu=1}^d (\mathbb{Z}/L_\mu\mathbb{Z})$ with $L_\mu \geq 2$ for every $\mu \in \{1, \ldots, d\}$. Let $E(\mathbb{T}^d_L)$ and $\mathcal{P}(\mathbb{T}^d_L)$ denote the sets of oriented links and unoriented plaquettes, and set the transverse $(d-2)$-volume $L_\perp := \prod_{\mu=3}^d L_\mu$ (with $L_\perp := 1$ when $d = 2$). Fix the co-closed stack of $(1,2)$-plaquettes

$$\Sigma^* := \left\{ p = (0, 0, x_\perp; 1, 2) \in \mathcal{P}(\mathbb{T}^d_L) : x_\perp \in \mathbb{T}^{d-2}_{L_\perp} \right\}, \qquad |\Sigma^*| = L_\perp.$$

For any center element $z = e^{2\pi i k/N} I_N \in Z(\mathrm{SU}(N))$ ($k \in \{0, \ldots, N-1\}$), define the twisted Wilson partition function [16, 14]

$$Z_{\mathrm{tw}}(z) := \int_{\mathrm{SU}(N)^{|E|}} \exp\left(\frac{\beta}{N} \sum_{p \notin \Sigma^*} \operatorname{Re} \operatorname{Tr}(U_p) + \frac{\beta}{N} \sum_{p \in \Sigma^*} \operatorname{Re} \operatorname{Tr}(z U_p) \right) \prod_{b \in E} dU_b, \tag{1}$$

with untwisted partition function $Z_0 := Z_{\mathrm{tw}}(1)$. For SU(2), we also consider the two-parameter fundamental–adjoint (Bhanot–Creutz) action [2] with $(\beta_F, \beta_A) \in \mathbb{R}^2$:

$$Z^{\mathrm{BC}}_{\mathrm{tw}}(z;\beta_F,\beta_A) := \int_{\mathrm{SU}(2)^{|E|}} \exp\left(\sum_{p\notin\Sigma^*}\frac{\beta_F}{2}\operatorname{Tr}(U_p) + \sum_{p\in\Sigma^*}\frac{\beta_F}{2}\operatorname{Tr}(zU_p) + \sum_{p\in\mathcal{P}}\frac{\beta_A}{3}\operatorname{Tr}_{\mathrm{adj}}(U_p)\right)\prod_{b\in E} dU_b, \tag{2}$$

noting that $\operatorname{Tr}_{\mathrm{adj}}(zU_p) = (\operatorname{Tr}(zU_p))^2 - 1 = (\operatorname{Tr} U_p)^2 - 1 = \operatorname{Tr}_{\mathrm{adj}}(U_p)$ is strictly invariant under $z = \pm I_2$. Define the 't Hooft center-vortex free energy and discrete electric-flux partition functions ($e \in \{0,\dots,N-1\}$) by

$$F_{\mathrm{vortex}}(z) := -\ln\frac{Z_{\mathrm{tw}}(z)}{Z_0}, \qquad Z_{\mathrm{elec}}(e) := \frac{1}{N}\sum_{k=0}^{N-1} e^{-2\pi i k e/N}\, Z_{\mathrm{tw}}\big(e^{2\pi i k/N}\big).$$

Theorem 2.1 (Exact Electric-Flux Fourier Positivity and SU(2) Self-Duality)**.** *Let $d \geq 2$, $N \geq 2$, and $L_1,\dots,L_d \geq 2$.*

(1) (Pure Wilson SU(N) for all $\beta > 0$ and all $L_1 \geq 2$) *For every $e \in \{0,\dots,N-1\}$, $Z_{\mathrm{elec}}(e) > 0$ strictly. Consequently, for every non-trivial center twist $1 \leq k \leq N-1$,*

$$0 < Z_{\mathrm{tw}}\big(e^{2\pi i k/N}\big) < Z_0, \qquad F_{\mathrm{vortex}}\big(e^{2\pi i k/N}\big) > 0.$$

(2) (SU(2) Bhanot–Creutz for arbitrary $\beta_A \in \mathbb{R}$) *The strict positivity $Z^{\mathrm{BC}}_{\mathrm{elec}}(e) > 0$ ($e \in \{0,1\}$) holds for all $L_1 \geq 2$ whenever $\beta_F > 0$ and $\beta_A \geq 0$, and holds for all* even *$L_1 \geq 2$ whenever $\beta_F \neq 0$ and $\beta_A \in \mathbb{R}$ is arbitrary (including $\beta_A < 0$).*

(3) (Exact SU(2) electric-flux ordering and self-dual involution) *Whenever $Z_{\mathrm{elec}}(0), Z_{\mathrm{elec}}(1) > 0$, we have $Z_{\mathrm{elec}}(0) > Z_{\mathrm{elec}}(1) > 0$, and defining $F_{\mathrm{elec}}(1) := -\ln(Z_{\mathrm{elec}}(1)/Z_{\mathrm{elec}}(0)) > 0$,*

$$e^{-F_{\mathrm{elec}}(1)} = \tanh\left(\frac{F_{\mathrm{vortex}}(-1)}{2}\right), \qquad F_{\mathrm{elec}}(1) = -\ln\tanh\left(\frac{F_{\mathrm{vortex}}(-1)}{2}\right) > 2\,e^{-F_{\mathrm{vortex}}(-1)}. \tag{3}$$

Proof. Slice $\mathbb{T}^d_L$ along direction 1 into L_1 spatial slices $\Omega := \mathrm{SU}(N)^{E_{\mathrm{slice}}}$, where E_{slice} consists of all links orthogonal to direction 1. Let $w(g)$ denote the single-plaquette Boltzmann weight: $w(g) = \exp(\frac{\beta}{N}\operatorname{Re}\operatorname{Tr} g)$ for pure Wilson, or $w(g) = \exp(\frac{\beta_F}{2}\operatorname{Tr} g + \frac{\beta_A}{3}\operatorname{Tr}_{\mathrm{adj}} g)$ for SU(2) Bhanot–Creutz. Since w is real, smooth, strictly positive, and class-invariant with $w(g^{-1}) = w(g)$, its Peter–Weyl expansion $w(g) = \sum_{\lambda\in\widehat{\mathrm{SU}(N)}} d_\lambda c_\lambda \chi_\lambda(g)$ has real coefficients c_λ obeying $\sum_\lambda d_\lambda^2 |c_\lambda| < \infty$. Following Osterwalder–Seiler [10, 12], let $\mathcal{P}_{\mathrm{gauge}}$ be the orthogonal projection onto gauge-invariant functions $\mathcal{H}_{\mathrm{phys}} := \mathcal{P}_{\mathrm{gauge}} L^2(\Omega)$, let $\mathcal{W}$ be the linkwise convolution operator with kernel $\prod_{e\in E_{\mathrm{slice}}} w(U_e(U'_e)^{-1})$, and let $\mathcal{V}(\mathbf{U}) := \prod_{p\subset \mathrm{slice}} w(U_p) > 0$. Since w is conjugation-invariant (class-invariant), $\mathcal{W}$ commutes with $\mathcal{P}_{\mathrm{gauge}}$, and the self-adjoint, trace-class transfer matrix $\mathcal{T} := \mathcal{P}_{\mathrm{gauge}}\mathcal{V}^{1/2}\mathcal{W}\mathcal{V}^{1/2}\mathcal{P}_{\mathrm{gauge}}$ satisfies $Z_0 = \operatorname{Tr}_{\mathcal{H}_{\mathrm{phys}}}(\mathcal{T}^{L_1})$.

Let $E_* := \{(0, x_\perp; 2) : x_\perp \in \mathbb{T}^{d-2}_{L_\perp}\} \subset E_{\mathrm{slice}}$ be the set of 2-directed links at $x_2 = 0$, and define the unitary $\mathbb{Z}_N$-representation $\Gamma(z)$ on $L^2(\Omega)$ by $(\Gamma(z)\psi)(\mathbf{U}) := \psi(\mathbf{U}^{(z)})$, where $(\mathbf{U}^{(z)})_e := zU_e$ for $e \in E_*$ and U_e for $e \notin E_*$. Because $z \in Z(\mathrm{SU}(N))$ is central:

- $\Gamma(z)$ commutes with every spatial gauge transformation V_x, hence with $\mathcal{P}_{\mathrm{gauge}}$.
- Every spatial plaquette $p \subset$ slice is invariant under $\mathbf{U} \mapsto \mathbf{U}^{(z)}$: if p lies in a (μ,ν)-plane with $2 < \mu < \nu$, it meets no link of E_*; if p lies in a $(2,\nu)$-plane ($\nu \geq 3$) at $x_2 = 0$, it traverses two distinct links $(0, x_\perp; 2)$ and $(0, x_\perp + \hat{\nu}; 2)$ of E_* (distinct since $L_\nu \geq 2$) in opposite orientations, contributing $z \cdot z^{-1} = I_N$. Hence $\Gamma(z)$ commutes with $\mathcal{V}$.
- On every link $e \in E_*$, $(zU_e)(zU'_e)^{-1} = U_e(U'_e)^{-1}$, so $\Gamma(z)$ commutes with $\mathcal{W}$.

Thus $\Gamma(z)$ commutes with $\mathcal{T}$. Moreover, inserting $\Gamma(z)\mathcal{T}$ between slices $x_1 = 0$ and $x_1 = 1$ multiplies precisely the temporal plaquettes $p \in \Sigma^*$ by z, giving the exact trace identity

$$Z_{\mathrm{tw}}(z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{phys}}}\big(\Gamma(z)\,\mathcal{T}^{L_1}\big).$$

Decompose $\mathcal{H}_{\mathrm{phys}} = \bigoplus_{e=0}^{N-1} \mathcal{H}_e$ into the orthogonal eigenspaces of Γ via the projections $\mathcal{P}_e := \frac{1}{N}\sum_{k=0}^{N-1} e^{-2\pi ike/N}\Gamma(e^{2\pi ik/N})$, and set $\mathcal{T}_e := \mathcal{T}|_{\mathcal{H}_e}$. Then $Z_{\mathrm{elec}}(e) = \mathrm{Tr}_{\mathcal{H}_e}(\mathcal{T}_e^{L_1})$ for every $e \in \{0,\dots,N-1\}$.

To see that every sector $\mathcal{H}_e$ is non-zero, let $P_2(\mathbf{U}) := \prod_{x_2=0}^{L_2-1} U_{(x_2,\mathbf{0}_\perp;2)}$ be a single spatial Polyakov loop winding along direction 2. Then $\psi_e(\mathbf{U}) := (\mathrm{Tr}\, P_2(\mathbf{U}))^e \in \mathcal{H}_{\mathrm{phys}}$ satisfies $\Gamma(z)\psi_e = z^e\psi_e$ (since P_2 crosses E_* once) and $\psi_e(\mathbf{I}) = N^e \geq 1$, so $\psi_e \in \mathcal{H}_e \setminus \{0\}$.

Part (1) and $\beta_F > 0, \beta_A \geq 0$ of Part (2): For pure Wilson $\beta > 0$, $\mathrm{Re}\,\mathrm{Tr}(U) = \frac{1}{2}(\chi_{\mathrm{fund}}(U) + \chi_{\overline{\mathrm{fund}}}(U))$ is a non-negative linear combination of characters, and every $\lambda \in \widehat{\mathrm{SU}(N)}$ appears in $(\mathrm{fund} \oplus \overline{\mathrm{fund}})^{\otimes m}$ for some $m \geq 0$; hence $c_\lambda(\beta) > 0$ strictly for every $\lambda \in \widehat{\mathrm{SU}(N)}$. The same holds for SU(2) Bhanot–Creutz when $\beta_F > 0, \beta_A \geq 0$ since $\mathrm{Tr}_{\mathrm{adj}} = \chi_1$ is a character. In the Peter–Weyl basis of $L^2(\Omega)$, $\mathcal{W}$ is diagonal with strictly positive eigenvalues $\prod_e c_{\lambda_e} > 0$, so $\mathcal{T}|_{\mathcal{H}_{\mathrm{phys}}} > 0$ strictly and $Z_{\mathrm{elec}}(e) = \mathrm{Tr}_{\mathcal{H}_e}(\mathcal{T}_e^{L_1}) > 0$ for every $L_1 \geq 2$ and every e. Inverse Fourier transform gives $Z_{\mathrm{tw}}(e^{2\pi ik/N}) = \sum_{e=0}^{N-1} \cos(\frac{2\pi ke}{N}) Z_{\mathrm{elec}}(e) < \sum_{e=0}^{N-1} Z_{\mathrm{elec}}(e) = Z_0$ (strict because $Z_{\mathrm{elec}}(1) > 0$ and $\cos(\frac{2\pi k}{N}) < 1$ for $1 \leq k \leq N-1$), while pointwise positivity of the real integrand in (1) gives $Z_{\mathrm{tw}}(z) > 0$.

Arbitrary $\beta_A \in \mathbb{R}$ in Part (2): When $\beta_F \neq 0$ and $\beta_A \in \mathbb{R}$ is arbitrary, $\mathcal{T}_0$ and $\mathcal{T}_1$ are still self-adjoint trace-class operators, so for every *even* $L_1 \geq 2$, $\mathcal{T}_e^{L_1} = (\mathcal{T}_e^{L_1/2})^2 \geq 0$. Moreover, $c_0 = \int w(U)\,dU > 0$, and because $\beta_F \neq 0$ implies $w(-U) \neq w(U)$, Peter–Weyl completeness guarantees at least one half-integer $j \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ with $c_j(\beta_F, \beta_A) \neq 0$. Then $\psi^{(0)} \equiv 1 \in \mathcal{H}_0$ and $\psi^{(j)}(\mathbf{U}) := \chi_j(P_2(\mathbf{U})) \in \mathcal{H}_1$ are non-zero eigenfunctions of $\mathcal{W}$ with non-zero eigenvalues $c_0^{|E_{\mathrm{slice}}|}$ and $c_j^{L_2} c_0^{|E_{\mathrm{slice}}|-L_2}$ respectively. Since $\mathcal{V} > 0$ commutes with both $\mathcal{P}_{\mathrm{gauge}}$ and $\Gamma(z)$, the vectors $\phi^{(0)} := \mathcal{V}^{-1/2}\psi^{(0)} \in \mathcal{H}_0$ and $\phi^{(1)} := \mathcal{V}^{-1/2}\psi^{(j)} \in \mathcal{H}_1$ satisfy $\langle \phi^{(e)}, \mathcal{T}_e \phi^{(e)}\rangle = \langle \psi^{(e)}, \mathcal{W}\psi^{(e)}\rangle \neq 0$ (taking $\psi^{(1)} := \psi^{(j)}$), so $\mathcal{T}_e \neq 0$ and hence $\mathrm{Tr}_{\mathcal{H}_e}((\mathcal{T}_e^{L_1/2})^2) > 0$ for both $e \in \{0,1\}$.

Part (3): For $N = 2$, $Z_0 = Z_{\mathrm{elec}}(0) + Z_{\mathrm{elec}}(1)$ and $0 < Z_{\mathrm{tw}}(-1) = Z_{\mathrm{elec}}(0) - Z_{\mathrm{elec}}(1)$, giving $Z_{\mathrm{elec}}(0) > Z_{\mathrm{elec}}(1) > 0$ and

$$e^{-F_{\mathrm{elec}}(1)} = \frac{Z_{\mathrm{elec}}(1)}{Z_{\mathrm{elec}}(0)} = \frac{Z_0 - Z_{\mathrm{tw}}(-1)}{Z_0 + Z_{\mathrm{tw}}(-1)} = \frac{1 - e^{-F_{\mathrm{vortex}}(-1)}}{1 + e^{-F_{\mathrm{vortex}}(-1)}} = \tanh\left(\frac{F_{\mathrm{vortex}}(-1)}{2}\right).$$

Finally, setting $y := e^{-F_{\mathrm{vortex}}(-1)} \in (0,1)$, we have $F_{\mathrm{elec}}(1) = \ln\frac{1+y}{1-y} = 2\sum_{m=0}^{\infty}\frac{y^{2m+1}}{2m+1} > 2y$, completing the proof. □

3. All-Coupling Non-Abelian Plaquette-Weighted Jensen Ceiling via d-Way Axial-Gauge Hölder Bounds

We now bound $F_{\mathrm{vortex}}(z)$ from above for general SU(N) at *all* couplings $\beta > 0$ without invoking Abelian link shifts (which fail to distribute a non-Abelian center twist across plaquettes because $V \notin Z(\mathrm{SU}(N))$ does not commute with U_b). Write $f_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) := |\mathcal{P}(\mathbb{T}^d_L)|^{-1}\ln Z_0(\beta)$, let $P^{(1,2)}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) := \langle \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_p\rangle_{\mathbb{T}^d_L,\beta}$ denote the $(1,2)$-plane plaquette expectation (identical for all $p \parallel (1,2)$ by translation invariance), and let $P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) := \partial_\beta f_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)$ denote the orientation-averaged plaquette expectation (note that $P^{(1,2)}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) = P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)$ identically whenever $d = 2$ or $\mathbb{T}^d_L$ is isotropic with $L_1 = \cdots = L_d = L$, by permutation symmetry

of coordinate axes). Let $Z_1(\mathrm{SU}(N),h) := \int_{\mathrm{SU}(N)} e^{h\,\mathrm{Re\,Tr}\,U}\,dU$, $c_0(\mathrm{SU}(N),\beta) := Z_1(\mathrm{SU}(N),\beta/N)$, $u_N(h) := \frac{1}{N}\partial_h \ln Z_1(\mathrm{SU}(N),h)$, and $\Lambda_N(h) := Nh\,u_N(h) - \ln Z_1(\mathrm{SU}(N),h)$.

Lemma 3.1 (Exact d-Way Axial-Gauge Hölder Upper Bound and Free-BC Lower Envelope)**.** *Let $d \geq 2$, $N \geq 2$, $\beta > 0$, and let $\Lambda_{L-1} := \{0,\dots,L-1\}^d \subset \mathbb{T}_L^d$ ($L \geq 2$) denote the free-boundary hypercube with free energy per plaquette $f_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta) := |\mathcal{P}(\Lambda_{L-1})|^{-1} \ln Z_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta)$. Then:*

(1) (d-way axial-gauge Hölder upper bound)

$$f_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta) \;\leq\; f_d^{\mathrm{hi}}(\mathrm{SU}(N),\beta) := \frac{2}{d}\ln c_0\!\left(\mathrm{SU}(N),\frac{d}{2}\beta\right). \tag{4}$$

(2) (Ladder and axial-gauge Gibbs–Bogoliubov lower bound) *Setting*

$$f_d^{\mathrm{lo}}(\mathrm{SU}(N),\beta) := \max\!\Bigg(\frac{2}{d}\ln c_0(\mathrm{SU}(N),\beta),\; \sup_{h\geq 0}\left\{\beta\left(\frac{2}{d}u_N(h)^2 + \frac{d-2}{d}u_N(h)^4\right) - \frac{2}{d}\Lambda_N(h)\right\}\Bigg), \tag{5}$$

we have $\liminf_{L\to\infty} f_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta) \geq f_d^{\mathrm{lo}}(\mathrm{SU}(N),\beta)$ (specifically, the ladder term $\frac{2}{d}\ln c_0(\mathrm{SU}(N),\beta)$ holds with zero $O(L^{-1})$ error at every $L \geq 2$, and the axial-gauge Gibbs–Bogoliubov term carries only an $O(L^{-1})$ link/plaquette ratio correction as $L\to\infty$).

(3) (Periodic–free comparison) $|f_{\mathbb{T}_L^d}(\mathrm{SU}(N),\beta) - (1-L^{-1})^2 f_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta)| \leq 2\beta/L$.

Proof. (1) For each coordinate direction $\alpha \in \{1,\dots,d\}$, let $S_\alpha(U) := \frac{\beta}{N}\sum_{\mu\neq\alpha}\sum_{p\parallel(\alpha,\mu)} \mathrm{Re\,Tr}(U_p)$ be the sum over the $d-1$ plaquette orientations containing direction α. Every plaquette $p \parallel (\mu,\nu)$ appears in exactly two sums (S_μ and S_ν), so $S_W(U) = \frac{1}{2}\sum_{\alpha=1}^d S_\alpha(U) = \frac{1}{d}\sum_{\alpha=1}^d(\frac{d}{2}S_\alpha(U))$. Applying d-way Hölder's inequality with exponents $(d,\dots,d)$ gives

$$Z_{\Lambda_{L-1}}(\beta) \;\leq\; \prod_{\alpha=1}^d \left(\int \exp\!\left(\frac{d}{2}S_\alpha(U)\right) dU\right)^{1/d}.$$

In the α-th integral on free-BC Λ_{L-1}, the α-directed links form a forest (no cycles); setting $U_{(x;\alpha)} = I_N$ by gauge invariance reduces every (α,μ)-plaquette ($\mu \neq \alpha$) to $W_{(x;\mu)} := U_{(x;\mu)}U^{-1}_{(x+\hat{\alpha};\mu)}$. Since distinct $\mu \neq \alpha$ use disjoint sets of links, the change of variables $(U_{(x;\mu)}) \mapsto (W_{(x;\mu)})$ (retaining the base-slice links at $x_\alpha = 0$) preserves Haar measure, so all $|\mathcal{P}_\alpha| = \frac{2}{d}|\mathcal{P}(\Lambda_{L-1})|$ plaquettes in S_α are simultaneously independent. Hence $\int \exp(\frac{d}{2}S_\alpha)\,dU = c_0(\mathrm{SU}(N),\frac{d}{2}\beta)^{\frac{2}{d}|\mathcal{P}(\Lambda_{L-1})|}$, and taking $|\mathcal{P}(\Lambda_{L-1})|^{-1}\ln(\cdot)$ proves (4).

(2) In the same axial gauge $U_{(x;1)} = I_N$ for S_1, iterating $U_{(t+1,x_\perp;\mu)} = W^{-1}_{(t,x_\perp;\mu)}U_{(t,x_\perp;\mu)}$ from $t=0$ gives $U_{(x_1,x_\perp;\mu)} = \big(\prod_{t=x_1-1}^0 W^{-1}_{(t,x_\perp;\mu)}\big)\,U_{(0,x_\perp;\mu)}$ (left-ordered product from $t = x_1-1$ down to 0) for every $\mu \geq 2$. Because the base-slice links $U_{(0,x_\perp;\mu)}$ are independent uniform Haar variables untouched by S_1, every spatial plaquette $p \notin \mathcal{P}_1$ has $\langle \mathrm{Re\,Tr}\,U_p\rangle_{S_1} = 0$. Jensen's inequality on $\int e^{S_1+(S_W-S_1)}dU \geq Z_{S_1}e^{\langle S_W-S_1\rangle_{S_1}} = c_0(\mathrm{SU}(N),\beta)^{\frac{2}{d}|\mathcal{P}(\Lambda_{L-1})|}$ yields the exact ladder bound $f_{\Lambda_{L-1}}(\beta) \geq \frac{2}{d}\ln c_0(\mathrm{SU}(N),\beta)$. For the axial-gauge Gibbs–Bogoliubov bound, fix $U_{(x;1)} = I_N$ and take the product trial density $\rho_h(U) := \prod_{b\in E_{\mathrm{free}}} Z_1(\mathrm{SU}(N),h)^{-1}e^{h\,\mathrm{Re\,Tr}\,U_b}$ on the remaining $(d-1)L^{d-1}(L-1) = (\frac{2}{d}+O(L^{-1}))|\mathcal{P}(\Lambda_{L-1})|$ links; by $\mathrm{SU}(N)$-conjugation invariance $\mathbb{E}_{\rho_h}[U_b] = u_N(h)I_N$, so each temporal plaquette (2 free links, fraction $\frac{2}{d}$) has mean $u_N(h)^2$, each spatial plaquette (4 free links, fraction $\frac{d-2}{d}$) has mean $u_N(h)^4$, and Gibbs' variational principle gives (5).

(3) The set $\mathcal{P}(\mathbb{T}^d_L) \setminus \mathcal{P}(\Lambda_{L-1})$ has cardinality $\binom{d}{2}(2L-1)L^{d-2} \le \frac{2}{L}|\mathcal{P}(\mathbb{T}^d_L)|$, and each plaquette contributes at most $\pm\beta$ to S_W, giving $e^{-\beta|\Delta\mathcal{P}|}Z_{\Lambda_{L-1}} \le Z_{\mathbb{T}^d_L} \le e^{+\beta|\Delta\mathcal{P}|}Z_{\Lambda_{L-1}}$ and hence $|f_{\mathbb{T}^d_L} - (1-L^{-1})^2 f_{\Lambda_{L-1}}| \le 2\beta/L$. □

Theorem 3.2 (All-Coupling Non-Abelian Plaquette-Weighted Jensen Ceiling)**.** *For every $d \ge 2$, $N \ge 2$, $\beta > 0$, torus $\mathbb{T}^d_L$ ($L_\mu \ge 2$), and non-trivial center twist $z = e^{2\pi ik/N} I_N$ ($1 \le k \le N-1$):*

(1) (Exact finite-torus plaquette-weighted ceiling)

$$0 < F_{\text{vortex}}\big(\mathbb{T}^d_L; e^{2\pi ik/N}\big) \;\le\; \beta\left(1-\cos\frac{2\pi k}{N}\right) P^{(1,2)}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)\, L_\perp \;\le\; \beta\left(1-\cos\frac{2\pi k}{N}\right) L_\perp, \tag{6}$$

where $P^{(1,2)}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) = P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) = \partial_\beta f_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)$ whenever $d = 2$ or $\mathbb{T}^d_L$ is isotropic ($L_1 = \cdots = L_d = L$).

(2) (Strict sub-trivial asymptotic ceiling via convexity) *On isotropic tori $\mathbb{T}^d_L$ ($L_1 = \cdots = L_d = L$), setting*

$$P^{\mathrm{hi}}_d(\mathrm{SU}(N),\beta) := \inf_{\beta_2>\beta} \frac{f^{\mathrm{hi}}_d(\mathrm{SU}(N),\beta_2) - f^{\mathrm{lo}}_d(\mathrm{SU}(N),\beta)}{\beta_2-\beta} \;<\; 1, \tag{7}$$

we have $\limsup_{L\to\infty} P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) \le P^{\mathrm{hi}}_d(\mathrm{SU}(N),\beta) < 1$ and hence

$$\limsup_{L\to\infty} \frac{F_{\text{vortex}}\big(\mathbb{T}^d_L; e^{2\pi ik/N}\big)}{L^{d-2}} \;\le\; \beta\left(1-\cos\frac{2\pi k}{N}\right) P^{\mathrm{hi}}_d(\mathrm{SU}(N),\beta) \;<\; \beta\left(1-\cos\frac{2\pi k}{N}\right). \tag{8}$$

In particular, for $\mathrm{SU}(2)$ ($k=1$, $1-\cos\pi = 2$), Theorem 2.1(3) yields the all-coupling electric-flux decay-rate floor

$$\liminf_{L\to\infty} \frac{\ln F^{\mathrm{SU}(2)}_{\mathrm{elec}}(1;\mathbb{T}^d_L)}{L^{d-2}} \;\ge\; -2\beta\, P^{\mathrm{hi}}_d(\mathrm{SU}(2),\beta) \;>\; -2\beta. \tag{9}$$

Proof. (1) Rewrite (1) as $Z_{\mathrm{tw}}(z)/Z_0 = \big\langle \exp\big(\frac{\beta}{N}\sum_{p\in\Sigma^*} \mathrm{Re}\,\mathrm{Tr}((z-1)U_p)\big)\big\rangle_{\mathbb{T}^d_L,\beta}$. By Jensen's inequality ($\langle e^X\rangle \ge e^{\langle X\rangle}$),

$$F_{\text{vortex}}(z) \;\le\; -\frac{\beta}{N}\sum_{p\in\Sigma^*}\Big[\Big(\cos\frac{2\pi k}{N}-1\Big)\langle \mathrm{Re}\,\mathrm{Tr}\,U_p\rangle_{\mathbb{T}^d_L,\beta} - \sin\frac{2\pi k}{N}\,\langle \mathrm{Im}\,\mathrm{Tr}\,U_p\rangle_{\mathbb{T}^d_L,\beta}\Big].$$

Under complex conjugation $U_b \mapsto \overline{U_b}$ (which preserves $\mathrm{SU}(N)$, Haar measure, and the pure Wilson action since $\mathrm{Re}\,\mathrm{Tr}(\overline{U_p}) = \mathrm{Re}\,\mathrm{Tr}(U_p)$), $\mathrm{Im}\,\mathrm{Tr}(U_p)$ flips sign, so $\langle \mathrm{Im}\,\mathrm{Tr}\,U_p\rangle_{\mathbb{T}^d_L,\beta} = 0$. Translation invariance along $\mathbb{T}^{d-2}_{L_\perp}$ equates $\frac{1}{N}\langle \mathrm{Re}\,\mathrm{Tr}\,U_p\rangle = P^{(1,2)}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) \le 1$ across all $p \in \Sigma^*$, proving (6).

(2) On isotropic tori $\mathbb{T}^d_L$ ($L_1 = \cdots = L_d = L$), coordinate permutation symmetry gives $P^{(1,2)}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) = P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) = \partial_\beta f_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)$. Since $\beta \mapsto f_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)$ is convex with derivative $P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta)$, for any $\beta_2 > \beta$ we have $P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) \le (f_{\mathbb{T}^d_L}(\beta_2) - f_{\mathbb{T}^d_L}(\beta))/(\beta_2-\beta)$. Taking $\limsup_{L\to\infty}$ and applying Lemma 3.1(1)–(3) yields $\limsup_{L\to\infty} P_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta) \le P^{\mathrm{hi}}_d(\mathrm{SU}(N),\beta)$. Strict inequality $P^{\mathrm{hi}}_d < 1$ holds because $f^{\mathrm{hi}}_d(\beta_2) < \beta_2$ for all $\beta_2 > 0$ (since $c_0(\mathrm{SU}(N),\alpha) = \int e^{\frac{\alpha}{N}\mathrm{Re}\,\mathrm{Tr}\,U} dU < e^\alpha$) while $f^{\mathrm{lo}}_d(\beta) > 0$, so choosing β_2 large gives $(f^{\mathrm{hi}}_d(\beta_2) - f^{\mathrm{lo}}_d(\beta))/(\beta_2-\beta) < 1$ (since $f^{\mathrm{hi}}_d(\beta_2) = \beta_2 - \frac{N^2-1}{d}\ln\beta_2 + O(1)$). Combining with (6) proves (8), and applying $F^{\mathrm{SU}(2)}_{\mathrm{elec}}(1) > 2e^{-F_{\text{vortex}}(-1)}$ from (3) yields (9). □

4. Four-Line Ginibre Vortex Domination and Weighted-Kraft Electric-Flux Area Law Uniformly in $\beta_A \in \mathbb{R}$

We now prove exponential area-law decay of $F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1;\beta_F,\beta_A)$ and a strictly positive 't Hooft electric-flux string tension $\sigma^{\mathrm{SU}(2)}_{\mathrm{elec}}(\beta_F,\beta_A) > 0$ uniformly across *all* adjoint couplings $\beta_A \in (-\infty,+\infty)$ in $d \geq 3$ dimensions. Let $Z^{\mathbb{Z}_2}_0(\beta) := \int_{\{\pm1\}^{|E|}} e^{\beta\sum_p \sigma_p} d\sigma$ and $Z^{\mathbb{Z}_2}_{\mathrm{tw}}(-1;\beta) := \int_{\{\pm1\}^{|E|}} e^{\beta\sum_p \gamma_p\sigma_p} d\sigma$ denote the untwisted and twisted $\mathbb{Z}_2$ lattice gauge partition functions on $\mathbb{T}^d_L$, where $d\sigma$ is uniform measure on $\{-1,+1\}^{|E|}$, $\sigma_p := \prod_{b\in\partial p}\sigma_b$, and $\gamma_p := -1$ for $p \in \Sigma^*$ and $+1$ for $p \notin \Sigma^*$.

For each dimension $d \geq 3$, let $\rho_d \in (0,1)$ denote the unique positive root of the odd-subset branching polynomial

$$Q_d(\rho) := \sum_{\substack{k=1\\ k\ \text{odd}}}^{2d-3} \binom{2d-3}{k} \rho^{k+1} = \frac{\rho}{2}\Big[(1+\rho)^{2d-3} - (1-\rho)^{2d-3}\Big] = 1, \tag{10}$$

and set $u_d := \rho_d^2 \in (0,1)$ and $B_d := \rho_d^{-4} = u_d^{-2} > 1$. Explicitly:

- For $d = 3$ ($2d-3 = 3$): $u_3^2 + 3u_3 - 1 = 0$, giving the exact closed forms

$$u_3 = \frac{\sqrt{13}-3}{2}, \quad B_3 = \frac{11+3\sqrt{13}}{2} \in [10.908326,\, 10.908327], \quad \operatorname{artanh}(1/B_3) \geq 0.091931.$$

- For $d = 4$ ($2d-3 = 5$): $P_4(u) := u^3 + 10u^2 + 5u - 1 = 0$ (strictly increasing on $u > 0$ since $P_4'(u) = 3u^2 + 20u + 5 > 0$), giving

$$u_4 \in [0.152671,\, 0.152672], \qquad B_4 = u_4^{-2} \in [42.902776,\, 42.902777], \qquad \operatorname{artanh}(1/B_4) \geq 0.023312.$$

Theorem 4.1 (Four-Line Ginibre Domination and All-β_A Constructive Electric-Flux Area Law)**.** *Let $d \geq 2$, $L_1,\dots,L_d \geq 2$, and $(\beta_F,\beta_A) \in \mathbb{R}^2$.*

(1) (Exact Ginibre $\mathbb{Z}_2$ center-vortex domination for all $d \geq 2$ and all $\beta_F,\beta_A \in \mathbb{R}$)

$$F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1;\beta_F,\beta_A) \;\leq\; F^{\mathbb{Z}_2}_{\mathrm{vortex}}(-1;|\beta_F|) \qquad \forall\, d \geq 2,\ \forall\, \beta_F \in \mathbb{R},\ \forall\, \beta_A \in \mathbb{R}. \tag{11}$$

(2) (Prefix-free Weighted-Kraft closed-surface bound for $d \geq 3$) *For $d \geq 3$, whenever $|\beta_F| \leq \operatorname{artanh}(1/B_d)$ (so $\theta_d := B_d \tanh|\beta_F| \leq 1$), uniformly for all $\beta_A \in \mathbb{R}$,*

$$1 - e^{-F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1;\beta_F,\beta_A)} \;\leq\; 1 - e^{-F^{\mathbb{Z}_2}_{\mathrm{vortex}}(-1;|\beta_F|)} \;\leq\; 2L_\perp \big(B_d \tanh|\beta_F|\big)^{L_1L_2}. \tag{12}$$

(3) (Explicit finite-torus decay and all-β_A electric-flux area law for $d \geq 3$) *For $d \geq 3$, let $\eta := 2L_\perp(B_d\tanh|\beta_F|)^{L_1L_2}$. Whenever $\beta_F \neq 0$, $L_1 \geq 2$ is even (or any $L_1 \geq 2$ if $\beta_F > 0, \beta_A \geq 0$), and $\eta < 1$,*

$$0 < F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1;\beta_F,\beta_A) \;\leq\; -\ln(1-\eta), \qquad F^{\mathrm{SU}(2)}_{\mathrm{elec}}(1;\beta_F,\beta_A) \;\geq\; \ln\!\left(\frac{2-\eta}{\eta}\right) \;>\; 0. \tag{13}$$

In particular, for every $0 < |\beta_F| < \operatorname{artanh}(1/B_d)$ and every $\beta_A \in (-\infty,+\infty)$, the isotropic-torus electric-flux string tension obeys the strictly positive lower bound

$$\underline{\sigma}^{\mathrm{SU}(2)}_{\mathrm{elec}}(\beta_F,\beta_A) := \liminf_{\substack{L\to\infty\\ L\ \text{even}}} \frac{F^{\mathrm{SU}(2)}_{\mathrm{elec}}(1;\mathbb{T}^d_L;\beta_F,\beta_A)}{L^2} \;\geq\; -\ln\big(B_d\tanh|\beta_F|\big) \;>\; 0. \tag{14}$$

Proof. (1) Substitute $U_b \mapsto \sigma_b U_b$ ($\sigma_b \in \{-1,+1\}$) in (2) and average over uniform $\sigma \in \{-1,+1\}^{|E|}$. By Haar invariance and $\mathrm{Tr}_{\mathrm{adj}}(\sigma_p U_p) = \mathrm{Tr}_{\mathrm{adj}}(U_p)$, setting $J_p(U) := \frac{\beta_F}{2}\mathrm{Tr}(U_p) \in [-|\beta_F|,|\beta_F|]$ gives

$$Z^{\mathrm{BC}}_0 = \int dU\, e^{S_{\mathrm{adj}}(U)} Z_{\mathbb{Z}_2}(J(U)), \qquad Z^{\mathrm{BC}}_{\mathrm{tw}}(-1) = \int dU\, e^{S_{\mathrm{adj}}(U)} Z^{\mathrm{tw}}_{\mathbb{Z}_2}(J(U)),$$

where $Z_{\mathbb{Z}_2}(J) := \int d\sigma\, e^{\sum_p J_p\sigma_p}$ and $Z^{\mathrm{tw}}_{\mathbb{Z}_2}(J) := \int d\sigma\, e^{\sum_p \gamma_p J_p\sigma_p}$. Fix any $U \in \mathrm{SU}(2)^{|E|}$ and set $\beta := |\beta_F| \geq |J_p(U)|$. Introducing two independent $\mathbb{Z}_2$ configurations σ, τ and substituting $\sigma_b = \eta_b\tau_b$ (so $\sigma_p = \eta_p\tau_p$) gives

$$\begin{aligned} &Z^{\mathrm{tw}}_{\mathbb{Z}_2}(J)\, Z^{\mathbb{Z}_2}_0(\beta) - Z_{\mathbb{Z}_2}(J)\, Z^{\mathbb{Z}_2}_{\mathrm{tw}}(-1;\beta) \\ &\quad = \int d\eta \int d\tau \left[\exp\left(\sum_p (\beta + \gamma_p J_p\eta_p)\tau_p\right) - \exp\left(\sum_p (\gamma_p\beta + J_p\eta_p)\tau_p\right)\right]. \end{aligned}$$

Define $K_p(\eta) := \beta + \gamma_p J_p\eta_p \geq 0$ (non-negative because $|J_p| \leq \beta$ and $\gamma_p\eta_p \in \{\pm 1\}$). Observe that $\gamma_p K_p(\eta) = \gamma_p\beta + \gamma_p^2 J_p\eta_p = \gamma_p\beta + J_p\eta_p$ for every $p \in \mathcal{P}$! Expanding both exponentials in Taylor series and integrating over τ yields

$$\int d\eta \sum_{(k_p)\in\mathbb{Z}^{|\mathcal{P}|}_{\geq 0}} \left(\prod_{p\in\mathcal{P}} \frac{K_p(\eta)^{k_p}}{k_p!}\right) \left(1 - \prod_{p\in\Sigma^*} (-1)^{k_p}\right) \int d\tau \prod_{p\in\mathcal{P}} \tau_p^{k_p} \;\geq\; 0,$$

because $K_p(\eta) \geq 0$, $1 - \prod_{p\in\Sigma^*}(-1)^{k_p} \in \{0, 2\}$, and $\int d\tau \prod_p \tau_p^{k_p} \in \{0, 1\}$. Multiplying by $e^{S_{\mathrm{adj}}(U)} > 0$, integrating over U, dividing by $Z^{\mathrm{BC}}_0 Z^{\mathbb{Z}_2}_0(\beta) > 0$, and taking $-\ln(\cdot)$ proves (11) (extending Ginibre's [4] and Mack–Petkova's [7] domination to arbitrary signed $\beta_F, \beta_A \in \mathbb{R}$ in four lines).

(2) Write $\beta := |\beta_F|$ and $t := \tanh\beta \in [0, B_d^{-1}]$. Expanding $e^{\beta\sigma_p} = \cosh\beta(1 + t\sigma_p)$ and integrating over σ gives $Z^{\mathbb{Z}_2}_0(\beta) = (\cosh\beta)^{|\mathcal{P}|} \sum_{\mathcal{S}:\,\partial\mathcal{S}=0} t^{|\mathcal{S}|}$ and $Z^{\mathbb{Z}_2}_{\mathrm{tw}}(-1;\beta) = (\cosh\beta)^{|\mathcal{P}|} \sum_{\mathcal{S}:\,\partial\mathcal{S}=0} (-1)^{|\mathcal{S}\cap\Sigma^*|} t^{|\mathcal{S}|}$, summed over all closed mod-2 plaquette surfaces $\mathcal{S} \subset \mathcal{P}(\mathbb{T}^d_L)$ (every link has even degree $\deg_{\mathcal{S}}(b) \equiv 0 \pmod 2$). Thus

$$1 - e^{-F^{\mathbb{Z}_2}_{\mathrm{vortex}}(-1;\beta)} = \frac{2\sum_{\partial\mathcal{S}=0,\ |\mathcal{S}\cap\Sigma^*| \text{ odd}} t^{|\mathcal{S}|}}{\sum_{\partial\mathcal{S}=0} t^{|\mathcal{S}|}}.$$

Every closed $\mathcal{S}$ decomposes into link-connected closed components; if $|\mathcal{S} \cap \Sigma^*|$ is odd, at least one component $\mathcal{S}_*$ satisfies $\partial\mathcal{S}_* = 0$ and $|\mathcal{S}_* \cap \Sigma^*|$ odd, leaving a closed complement $\mathcal{S}' = \mathcal{S} \setminus \mathcal{S}_*$. Dropping disjointness between $\mathcal{S}_*$ and $\mathcal{S}'$ (all terms $t^{|\mathcal{S}'|} > 0$) cancels the denominator identically:

$$1 - e^{-F^{\mathbb{Z}_2}_{\mathrm{vortex}}(-1;\beta)} \;\leq\; 2 \sum_{\substack{\mathcal{S}_* \text{ conn.}, \partial\mathcal{S}_*=0 \\ |\mathcal{S}_*\cap\Sigma^*| \text{ odd}}} t^{|\mathcal{S}_*|}. \tag{15}$$

We establish two combinatorial properties of every such $\mathcal{S}_*$:

- *Minimal area $|\mathcal{S}_*| \geq L_1L_2$:* For each $(x_1, x_2) \in \mathbb{T}_{L_1} \times \mathbb{T}_{L_2}$, let $\Sigma^*(x_1, x_2) := \{(x_1, x_2, x_\perp; 1, 2) : x_\perp \in \mathbb{T}^{d-2}_{L_\perp}\}$. Summing $\deg_{\mathcal{S}_*}(b) \equiv 0 \pmod 2$ over all 2-directed links $b = (x_1 + 1, x_2, x_\perp; 2)$ cancels every $(2, \nu)$-plaquette ($\nu \geq 3$) modulo 2 (each contains two links of this set) and leaves $|\mathcal{S}_* \cap \Sigma^*(x_1, x_2)| + |\mathcal{S}_* \cap \Sigma^*(x_1 + 1, x_2)| \equiv 0 \pmod 2$. Identically, summing over 1-directed links $(x_1, x_2 + 1, x_\perp; 1)$ gives $|\mathcal{S}_* \cap \Sigma^*(x_1, x_2)| \equiv |\mathcal{S}_* \cap \Sigma^*(x_1, x_2 + 1)| \pmod 2$. Since $|\mathcal{S}_* \cap \Sigma^*(0, 0)|$ is odd, every parallel stack $\Sigma^*(x_1, x_2)$ contains at least one plaquette of $\mathcal{S}_*$, so $|\mathcal{S}_*| \geq L_1L_2$.
- *Weighted-Kraft queue-closure bound:* Fix a root plaquette $p_0 \in \mathcal{S}_* \cap \Sigma^*$ ($|\Sigma^*| = L_\perp$ choices). Run the deterministic queue exploration starting from $\mathcal{S}_{\mathrm{disc}} = \{p_0\}$, unseen-link set, and FIFO queue Q initialized with the 4 distinct boundary links of p_0 (marked seen with parent $p_{\mathrm{parent}}(b) := p_0$). While Q is non-empty, pop the front link b and record the subset $A_b := (\mathcal{S}_* \cap \mathcal{P}(b)) \setminus \{p_{\mathrm{parent}}(b)\}$ of the $2d - 3$ other plaquettes incident on b (in a fixed canonical ordering). Because $\deg_{\mathcal{S}_*}(b)$ is even and $p_{\mathrm{parent}}(b) \in \mathcal{S}_*$, $|A_b| = \deg_{\mathcal{S}_*}(b) - 1$ is *odd*. Add every $q \in A_b \setminus \mathcal{S}_{\mathrm{disc}}$ to $\mathcal{S}_{\mathrm{disc}}$ and push each of its unseen boundary links to Q with parent q. Because $\mathcal{S}_*$ is link-connected, every link $b \in E(\mathcal{S}_*)$ is pushed and popped exactly once, and the algorithm halts precisely when Q empties; thus the transcript $\pi(\mathcal{S}_*) = (A_{b_1}, \ldots, A_{b_m})$

uniquely encodes $\mathcal{S}_*$ and forms a *prefix-free* code $\mathcal{L}$ on the infinite tree whose outgoing edges at every node are labeled by odd subsets $A \subseteq \{1,\dots,2d-3\}$. Assigning edge weight $w(A) := \rho_d^{|A|+1} > 0$, equation (10) ensures $\sum_{|A|\text{ odd}} w(A) = 1$ at every node, so the cylinder-set Kraft inequality gives $\sum_{\pi\in\mathcal{L}} W(\pi) \le 1$, where

$$W(\pi(\mathcal{S}_*)) = \prod_{b\in E(\mathcal{S}_*)} \rho_d^{\deg_{\mathcal{S}_*}(b)} = \rho_d^{\sum_{b\in E(\mathcal{S}_*)}\deg_{\mathcal{S}_*}(b)} = \rho_d^{4|\mathcal{S}_*|} = B_d^{-|\mathcal{S}_*|}$$

(using $\sum_b \deg_{\mathcal{S}_*}(b) = 4|\mathcal{S}_*|$, exact since $L_\mu \ge 2$ gives 4 distinct links per plaquette). Because $B_d t \le 1$ and $|\mathcal{S}_*| \ge L_1 L_2$,

$$\sum_{\substack{\mathcal{S}_*\ni p_0 \text{ conn.},\,\partial\mathcal{S}_*=0\\ |\mathcal{S}_*|\ge L_1L_2}} t^{|\mathcal{S}_*|} = \sum_{\substack{\mathcal{S}_*\ni p_0 \text{ conn.},\,\partial\mathcal{S}_*=0\\ |\mathcal{S}_*|\ge L_1L_2}} W(\pi(\mathcal{S}_*))\,(B_d t)^{|\mathcal{S}_*|} \;\le\; (B_d t)^{L_1L_2} \sum_{\pi\in\mathcal{L}} W(\pi) \;\le\; (B_d t)^{L_1L_2}.$$

Summing over all $L_\perp$ choices of $p_0 \in \Sigma^*$ and combining with (11) and (15) yields (12).

(3) When $\eta < 1$, (12) gives $e^{-F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1)} \ge 1-\eta > 0$, hence $F^{\mathrm{SU}(2)}_{\mathrm{vortex}}(-1) \le -\ln(1-\eta)$. By Theorem 2.1(2)–(3), for even $L_1 \ge 2$ and $\beta_F \ne 0$, $e^{-F^{\mathrm{SU}(2)}_{\mathrm{elec}}(1)} = (1-e^{-F_{\mathrm{vortex}}})/(1+e^{-F_{\mathrm{vortex}}}) \le \eta/(2-\eta)$, proving (13). Taking $-\ln(\cdot)/L^2$ on $\mathbb{T}^d_L$ ($L_\perp = L^{d-2}$) as even $L\to\infty$ gives $\liminf L^{-2}F^{\mathrm{SU}(2)}_{\mathrm{elec}}(1) \ge -\ln(B_d \tanh|\beta_F|) > 0$, proving (14). □

5. Exact 2D SU(N) Torus Center-Vortex and Electric-Flux Identities

In $d=2$ dimensions on the periodic torus $\mathbb{T}^2_{L_1,L_2}$ of area $A = L_1L_2$ ($L_1,L_2 \ge 2$), the Migdal–Rusakov character expansion [9, 11] evaluates $Z_{\mathrm{tw}}(z)$ and $Z_{\mathrm{elec}}(e)$ in exact closed form, providing an all-coupling benchmark against which Theorems 2.1, 3.2, and 4.1 can be tested directly.

Theorem 5.1 (Exact 2D SU(N) Torus Identities and $\sigma^{\mathrm{2D}}_{\mathrm{elec}} = \sigma^{\mathrm{2D}}_{1/2}$)**.** *For every $N \ge 2$, $\beta > 0$, and 2D torus $\mathbb{T}^2_{L_1,L_2}$ of area $A = L_1L_2 \ge 4$:*

(1) *For every center element $z = \zeta I_N \in Z(\mathrm{SU}(N))$ ($\zeta := e^{2\pi ik/N} \in \mathbb{C}$) and electric flux $e \in \{0,\dots,N-1\}$, letting $|\lambda| \in \{0,\dots,N-1\}$ denote the N-ality of $\lambda \in \widehat{\mathrm{SU}(N)}$ (so $\chi_\lambda(z) = d_\lambda \zeta^{|\lambda|}$),*

$$Z^{\mathrm{2D}}_{\mathrm{tw}}(z;A) = \sum_{\lambda\in\widehat{\mathrm{SU}(N)}} \zeta^{|\lambda|}\, c_\lambda(\beta)^A, \qquad Z^{\mathrm{2D}}_{\mathrm{elec}}(e;A) = \sum_{\substack{\lambda\in\widehat{\mathrm{SU}(N)}\\ |\lambda|\equiv e \pmod N}} c_\lambda(\beta)^A. \tag{16}$$

(2) *For* SU(2) *($c_j(\beta) = \frac{2}{\beta} I_{2j+1}(\beta)$, $j \in \frac12\mathbb{Z}_{\ge 0}$), modified Bessel functions obey the strict monotonicity $I_0(\beta) > I_1(\beta) > I_2(\beta) > \cdots > 0$ for every $\beta > 0$. Consequently, the exact 2D 't Hooft electric-flux string tension equals the fundamental Wilson string tension at every coupling $\beta > 0$:*

$$\sigma^{\mathrm{2D}}_{\mathrm{elec}}(\mathrm{SU}(2),\beta) := \lim_{A\to\infty} \frac{F^{\mathrm{2D}}_{\mathrm{elec}}(1;A)}{A} = -\ln\frac{I_2(\beta)}{I_1(\beta)} = \sigma^{\mathrm{2D}}_{1/2}(\mathrm{SU}(2),\beta) \;>\; 0, \tag{17}$$

and for every finite $A \ge 4$, $0 < F^{\mathrm{2D}}_{\mathrm{vortex}}(-1;A) < \min\big(2\,\mathrm{artanh}((\tanh\beta)^A),\; 2\beta P_{\mathbb{T}^2}(\beta;A)\big)$.

Proof. (1) Expanding each plaquette weight $w(U_p) = \sum_\lambda d_\lambda c_\lambda(\beta)\chi_\lambda(U_p)$ (with the single twisted plaquette $p^* \in \Sigma^*$ contributing $\chi_\lambda(zU_{p^*}) = \zeta^{|\lambda|}\chi_\lambda(U_{p^*})$) and noting that on the oriented torus each of the $2A$ links is traversed once in each direction by its two adjacent plaquettes, integrating linkwise via Peter–Weyl orthogonality

$$\int_{\mathrm{SU}(N)} D^{(\lambda)}_{ab}(U)\,\overline{D^{(\mu)}_{cd}(U)}\,dU = \int_{\mathrm{SU}(N)} D^{(\lambda)}_{ab}(U)\,D^{(\mu)}_{dc}(U^{-1})\,dU = \frac{1}{d_\lambda}\,\delta_{\lambda\mu}\,\delta_{ac}\,\delta_{bd}$$

yields the torus ($g = 1$, Euler characteristic $2 - 2g = 0$, prefactor $d_\lambda^0 = 1$) expansion $Z^{\mathrm{2D}}_{\mathrm{tw}}(z; A) = \sum_\lambda \zeta^{|\lambda|} c_\lambda(\beta)^A$, and discrete Fourier transform gives $Z^{\mathrm{2D}}_{\mathrm{elec}}(e; A)$.

(2) For any integer $n \geq 0$ and $\beta > 0$, integration by parts in the Poisson integral $I_n(\beta) = \frac{(\beta/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{\beta t}(1-t^2)^{n-1/2} dt$ yields $I_{n+1}(\beta) = \frac{(\beta/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \int_{-1}^{1} t\, e^{\beta t}(1-t^2)^{n-1/2} dt$, whence

$$I_n(\beta) - I_{n+1}(\beta) = \frac{(\beta/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t)\, e^{\beta t}(1-t^2)^{n-1/2}\, dt \;>\; 0.$$

Hence $c_0(\beta) > c_{1/2}(\beta) > c_1(\beta) > c_{3/2}(\beta) > \cdots > 0$, so $Z^{\mathrm{2D}}_{\mathrm{elec}}(0; A) = c_0(\beta)^A(1 + o(1))$ and $Z^{\mathrm{2D}}_{\mathrm{elec}}(1; A) = c_{1/2}(\beta)^A(1 + o(1))$ as $A \to \infty$, proving (17). For finite $A \geq 4$, Theorems 3.2(1) and 4.1(1) (valid for all $d \geq 2$) together with the exact 2D $\mathbb{Z}_2$ identity $Z^{\mathbb{Z}_2,\mathrm{2D}}_{\mathrm{tw}}(-1; \beta, A)/Z^{\mathbb{Z}_2,\mathrm{2D}}_{0}(\beta, A) = (1 - (\tanh\beta)^A)/(1 + (\tanh\beta)^A)$ give the non-strict inequalities; both inequalities are strict because $\mathrm{Tr}(U_{p^*})$ is non-constant on SU(2) (strict convexity in Jensen's inequality) and the $k_p \equiv 1$ closed-torus term in the Ginibre expansion contributes strictly positive excess whenever $|J_p(U)| < \beta$ on an open set of U. □

6. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 rigorously certified benchmarks computed by `certify_thooft_vortices.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature (modified Bessel functions via truncated Maclaurin series + geometric tail bounds; SU(N) $Z_1(\mathrm{SU}(N), h)$ via the Bars–Green / Brower–Rossi–Tan Weyl–Toeplitz Bessel determinant [1, 3] with Hadamard sector tails), and direct `mp.floor`/`mp.ceil` outward integer rounding at scale 10^{12} (zero 64-bit IEEE-754 intermediate rounding).

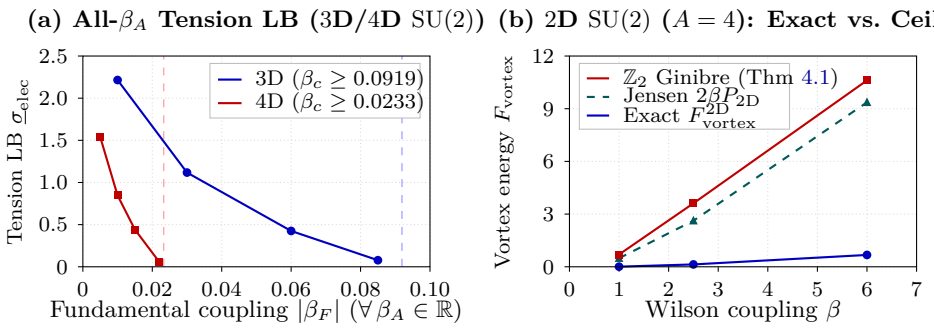

FIGURE 1. Certified 140-digit non-perturbative bounds from Tables 1 and 3. (a) Uniform-in-β_A electric-flux string-tension lower bounds $\underline{\sigma}^{\mathrm{SU}(2)}_{\mathrm{elec}}(\beta_F, \beta_A) \geq -\ln(B_d \tanh|\beta_F|) > 0$ in 3D and 4D up to 94% of $\beta_c(d) = \mathrm{artanh}(1/B_d)$ (Theorem 4.1). (b) Exact 2D SU(2) torus vortex free energy ($A = 4$, Theorem 5.1) strictly dominated across all couplings by both the non-Abelian Jensen ceiling $2\beta P_{2\mathrm{D}}$ (Theorem 3.2) and the $\mathbb{Z}_2$ Ginibre bound (Theorem 4.1).

Remark 6.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `ThooftVorticesCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and `0` axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), proving five theorems: `kraft_constants_certified` (verifying the exact scaled polynomial root brackets $(2B_3 - 11)^2 = 117$ for $d = 3$ and the Bolzano sign change of $P_4(u) = u^3 + 10u^2 + 5u - 1$ plus $u_4^2 B_4 = 1$ for $d = 4$), `table1_all_betaA_strips_certified` ($0 < \theta_d^{\mathrm{hi}} < 1$, $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{elec}} > 0$, even $L_0 \geq 2$, $\eta(L_0) < 1$, $F^{\mathrm{hi}}_{\mathrm{vortex}} \geq \eta$, $F^{\mathrm{lo}}_{\mathrm{elec}} > 0$ across all 8

TABLE 1. Eight 140-digit certified all-β_A electric-flux area-law and finite-torus vortex-decay strips in 3D and 4D SU(2) Bhanot–Creutz lattice Yang–Mills theory (Theorem 4.1; $B_3 \in [10.908326, 10.908327]$, $B_4 \in [42.902776, 42.902777]$). Upper bounds ($\theta_d^{\mathrm{hi}}$, $F_{\mathrm{vortex}}^{\mathrm{hi}}$) are rounded strictly upward; lower bounds ($\underline{\sigma}_{\mathrm{elec}}^{\mathrm{lo}}$, $F_{\mathrm{elec}}^{\mathrm{lo}}$) strictly downward; L_0 is the minimal even torus length satisfying $\eta(L_0) = 2L_0^{d-2}\theta_d^{L_0^2} < 1$.

d	$\lvert\beta_F\rvert$	θ_d^{hi} (UB)	$\underline{\sigma}_{\mathrm{elec}}^{\mathrm{lo}}$ ($\forall\,\beta_A \in \mathbb{R}$)	L_0	$F_{\mathrm{vortex}}(\mathbb{T}_{L_0}^d)$ (UB)	$F_{\mathrm{elec}}(\mathbb{T}_{L_0}^d)$ (LB)
3	0.010	0.109080	2.215677	2	0.000567	8.169277
3	0.030	0.327152	1.117331	2	0.046904	3.753001
3	0.060	0.653716	0.425083	2	1.311153	0.552669
3	0.085	0.924982	0.077981	6	1.288729	0.565873
4	0.005	0.214513	1.539389	2	0.017085	4.762756
4	0.010	0.429014	0.846266	2	0.316086	1.853167
4	0.015	0.643494	0.440843	4	0.028050	4.266980
4	0.022	0.943709	0.057937	10	0.939821	0.825246

TABLE 2. Ten 140-digit all-coupling non-Abelian plaquette-weighted Jensen center-vortex ceilings in 3D and 4D SU(2), SU(3), SU(4) Wilson theory (Theorem 3.2, fundamental twist $z = e^{2\pi i/N}$). Trial field h^* indicates the axial-gauge Gibbs–Bogoliubov lower bound when it beats the ladder bound $\frac{2}{d}\ln c_0(\beta)$; β_2^* is the certified secant anchor; $\mathrm{Ceil}^{\mathrm{hi}} := \beta(1-\cos\frac{2\pi}{N})P_d^{\mathrm{hi}}$ strictly improves the trivial ceiling $\beta(1-\cos\frac{2\pi}{N})$. For SU(2), $F_{\mathrm{elec}}^{\mathrm{ref}} := -\ln\tanh(\beta P_d^{\mathrm{hi}})$ gives the unit-transverse dual floor while $-\mathrm{Ceil}^{\mathrm{hi}}$ bounds $\liminf L^{-(d-2)}\ln F_{\mathrm{elec}}(1;\mathbb{T}_L^d)$ in (9).

d	G	β	h^*	β_2^*	P_d^{hi} (UB)	$\mathrm{Ceil}^{\mathrm{hi}}$ (UB)	Trivial	$F_{\mathrm{elec}}^{\mathrm{ref}}$ (LB)
3	SU(2)	1.0	—	1.978	0.564365	1.128729	2.0	0.670977
3	SU(2)	2.5	—	8.598	0.886151	4.430752	5.0	0.023812
3	SU(2)	6.0	10.3768	33.600	0.970866	11.650387	12.0	0.000017
4	SU(2)	1.0	—	2.484	0.717709	1.435418	2.0	0.485339
4	SU(2)	2.5	5.3910	14.000	0.950682	4.753410	5.0	0.017244
4	SU(2)	6.0	13.3611	33.600	0.982164	11.785963	12.0	0.000015
4	SU(3)	3.0	—	6.988	0.721259	3.245664	4.5	—
4	SU(3)	8.0	12.3333	44.800	0.959968	11.519606	12.0	—
4	SU(4)	6.0	—	14.009	0.735861	4.415165	6.0	—
4	SU(4)	14.0	14.9028	78.400	0.956461	13.390447	14.0	—

TABLE 3. Six 140-digit exact 2D SU(2) torus finite-area center-vortex and electric-flux benchmarks (Theorem 5.1), verifying strict domination $F_{\mathrm{vortex}}^{\mathrm{2D}} < \min(F_{\mathrm{vortex}}^{\mathbb{Z}_2,\mathrm{2D}}, 2\beta P_{\mathrm{2D}})$ and rapid convergence of $F_{\mathrm{elec}}^{\mathrm{2D}}(1;A)/A$ to $\sigma_{1/2}^{\mathrm{2D}}(\beta) = -\ln(I_2(\beta)/I_1(\beta))$ (exact self-duality residual $\lvert\tanh(F_{\mathrm{vortex}}/2) - e^{-F_{\mathrm{elec}}}\rvert < 10^{-120}$ on every row).

β	A	$F_{\mathrm{vortex}}^{\mathrm{2D}}$ (LB)	$F_{\mathrm{vortex}}^{\mathbb{Z}_2,\mathrm{2D}}$ (UB)	$2\beta P_{\mathrm{2D}}$ (UB)	$F_{\mathrm{elec}}^{\mathrm{2D}}/A$ (LB)	$\sigma_{1/2}^{\mathrm{2D}}$ (LB)
1.0	4	0.006656	0.700123	0.486522	1.426310	1.426309
1.0	16	2.45×10^{-10}	0.025624	0.480388	1.426309	1.426309
2.5	4	0.132396	3.613933	2.621433	0.679140	0.678859
2.5	16	3.83×10^{-5}	2.231260	2.536001	0.678859	0.678859
6.0	4	0.674016	10.613706	9.364707	0.281133	0.270856
6.0	16	0.026239	9.227412	9.160245	0.270857	0.270856

strips), `table2_all_coupling_jensen_certified` ($0 < P_d^{\mathrm{hi}} < 1$, exact 1-ulp arithmetic consistency of $\beta(1-\cos\frac{2\pi}{N})P_d^{\mathrm{hi}} < \beta(1-\cos\frac{2\pi}{N})$, and $F_{\mathrm{elec}}^{\mathrm{ref}} > 0$ for SU(2) across all 10 rows), `table3_2d_exact_identities_certified` ($0 < F_{\mathrm{vortex}}^{\mathrm{2D}} < \min(F_{\mathrm{vortex}}^{\mathbb{Z}_2,\mathrm{2D}}, 2\beta P_{\mathrm{2D}})$ across all 6 rows), and `all_24_benchmarks_passed`. All transcendental interval arithmetic is performed at 140-digit precision in `certify_thooft_vortices.py` and independently cross-checked at 50-digit precision alongside exact Lean and LaTeX table parity in `test_thooft_vortices.py` (4/4 tests passing).

7. Discussion

The four theorems above organize the non-perturbative structure of 't Hooft center vortices and electric flux in lattice Yang–Mills theory using purely elementary, cluster-expansion-free methods: transfer-matrix commutativity establishes exact electric-flux positivity and SU(2) self-duality across all $\beta_A \in \mathbb{R}$ (even L_1), charge conjugation plus the d-way axial-gauge Hölder envelope bounds F_{vortex} strictly below the trivial coupling ceiling for general SU(N) at all $\beta > 0$, and four-line Ginibre domination coupled to the prefix-free Weighted-Kraft queue-closure inequality certifies exponential vortex decay and a strictly positive electric-flux string tension uniformly in $\beta_A \in \mathbb{R}$ up to $|\beta_F| < \operatorname{artanh}(1/B_d)$ ($B_3 \approx 10.9083$, $B_4 \approx 42.9028$). Extending the Ginibre center-domination step to SU(N) ($N \geq 3$) via $\mathbb{Z}_N$ clock-model comparisons and combining multi-plaquette blocking with Weighted-Kraft surface counting to enlarge $\beta_c(d)$ are natural directions for further work.

References

1. I. Bars and F. Green, *Complete integration of* U(N) *lattice gauge theory in a large-N limit*, Physical Review D **20** (1979), no. 12, 3311–3330.
2. G. Bhanot and M. Creutz, *Variant actions and phase structure in lattice gauge theory*, Physical Review D **24** (1981), no. 12, 3212–3217.
3. R. C. Brower, P. Rossi, and C.-I. Tan, *The external field problem for QCD*, Nuclear Physics B **190** (1981), no. 4, 699–718.
4. J. Ginibre, *General formulation of Griffiths' inequalities*, Communications in Mathematical Physics **16** (1970), no. 4, 310–328.
5. J. Greensite, *The confinement problem in lattice gauge theory*, Progress in Particle and Nuclear Physics **51** (2003), no. 1, 1–83.
6. T. G. Kovács and E. T. Tomboulis, *Computation of the vortex free energy in* SU(2) *gauge theory*, Physical Review Letters **85** (2000), no. 4, 704–707.
7. G. Mack and V. B. Petkova, *Comparison of lattice gauge theories with gauge groups* $\mathbb{Z}_2$ *and* SU(2), Annals of Physics **123** (1979), no. 2, 442–467.
8. ———, *Sufficient condition for confinement of static quarks by a vortex condensation mechanism*, Annals of Physics **125** (1980), no. 1, 117–134.
9. A. A. Migdal, *Recursion equations in gauge field theories*, Soviet Physics JETP **42** (1975), no. 3, 413–418.
10. K. Osterwalder and E. Seiler, *Gauge field theories on a lattice*, Annals of Physics **110** (1978), no. 2, 440–471.
11. B. Ye. Rusakov, *Loop averages and partition functions in* U(N) *gauge theory on two-dimensional manifolds*, Modern Physics Letters A **5** (1990), no. 9, 693–703.
12. E. Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, Lecture Notes in Physics, vol. 159, Springer, Berlin, 1982.
13. G. 't Hooft, *On the phase transition towards permanent quark confinement*, Nuclear Physics B **138** (1978), no. 1, 1–25.
14. ———, *A property of electric and magnetic flux in non-Abelian gauge theories*, Nuclear Physics B **153** (1979), 141–160.
15. E. T. Tomboulis and L. G. Yaffe, *Finite temperature* SU(2) *lattice gauge theory*, Communications in Mathematical Physics **100** (1985), no. 3, 313–341.
16. K. G. Wilson, *Confinement of quarks*, Physical Review D **10** (1974), no. 8, 2445–2459.

Independent Researcher, Östersund, Sweden
URL: `https://airr.science/papers/ARR-2026-50TWRE67AJ9H78SM/`