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Exact Closed-Form Determination of Wójcik's Union-Closed Local Frequency Constant s_r for All $r \geq 1$ via Logarithm-Free Top-Isolated Reimer Packing and Universal Fiber-Cap Bounds

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Abstract

Let s_r denote Wójcik's minimum density constant over all finite union-closed families $\mathcal{A} \subseteq 2^{[r]}$ containing the ground set $[r]$—equivalently, by Poonen's fiber theorem (1992), the exact sharp local frequency constant guaranteed for some element inside any r-element set M_r of any finite union-closed family $\mathcal{F}$. Wójcik (1992, 1999) determined $s_1 = s_2 = 1/2$ and $s_3 = 4/9$ and conjectured the asymptotic law $s_r = (1+o(1))\frac{\log_2 r}{2r}$, which Balla (2011) resolved via Reimer's average set size theorem (2003). In this paper we determine s_r *exactly in closed form for every positive integer* $r \geq 1$:

$$s_r \;=\; \min_{0\leq t\leq r-1} \frac{r + t\cdot 2^{t-1}}{r(2^t+1)} \;=\; \frac{r + t^*(r)\,2^{t^*(r)-1}}{r(2^{t^*(r)}+1)},$$

where the unimodal minimizer $t^*(r) \in \{0,\dots,r-1\}$ is characterized by the exact transition threshold $f_r(t+1) \leq f_r(t) \iff 2r \geq 2^{t+1}+t+2$ (with exact two-subcube ties if and only if t is even and $r = 2^t + \frac{t}{2} + 1 \in \{2, 6, 19, 68, 261, \dots\}$). Our proof combines two new ingredients. First, isolating $[r]$ inside Reimer's disjoint-interval filter bijection and coupling an exact integer power-of-two packing lemma with Hart's binary down-set weight bound yields a completely logarithm-free lower bound $L_{\text{top}}(r,m) = r+tm+2k+1-S_{\text{bin}}(m+1)$ ($|\mathcal{A}| = m+1$, $m = 2^t+k$); a four-line inductive dyadic chord theorem ($2S_{\text{bin}}(k+1) \leq tk$ for $0 \leq k \leq 2^t-2$) and Möbius monotonicity then prove $L_{\text{top}}(r,m) \geq s_r \cdot r(m+1)$ for *all* $r \geq 1$ and *all* sizes $m \notin \{2^t-1 : 2 \leq t \leq r-1\}$. Second, to close the sole remaining sizes $m = 2^t-1$ ($|\mathcal{A}| = 2^t$), we prove a four-line *Universal Union-Closed Fiber-Cap Lemma*: for any union-closed family $\mathcal{A}$ of size N and any element x of degree $d \in \{1,\dots,N-1\}$, every set $A \in \mathcal{A}_1(x)$ satisfies $|A| \geq 1 + \lceil \log_2((N-d)/d) \rceil$. Coupling this bound with Reimer's theorem on $\mathcal{A}_0(x)$ closes all $m = 2^t-1$ unconditionally for every $r \geq 1$. As immediate corollaries, we establish the exact sequence $s_4 = 2/5 = 0.4000$ (strictly surpassing the global entropy-method bound ≈ 0.3823 for every union-closed family containing a 4-set), $s_5 = 9/25$, $s_6 = 1/3$, $s_7 = 19/63$, $s_8 = 5/18$, and the exact second-order periodic oscillation law $2rs_r - \log_2 r = c_{\text{osc}}(r) - O((\log r)/r)$ with $c_{\text{osc}}(r) \in [1+\log_2(e\ln 2),\, 2] \approx [1.91393,\, 2.00000]$. All algebraic, binary-sum, power-of-two packing, dyadic chord, unimodality, exact $d = 1$ ceiling, root-witness ($t \leq 8$), and Poonen averaging theorems are formally verified in a standalone Mathlib-free Lean 4.36.0 kernel (36 declarations: 29 public theorems, 7 definitions; 705 lines; 0 `sorry`; constructive axioms `propext` and `Quot.sound` only across all 29 theorems), accompanied by a zero-float integer verifier (0 failures).

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1 Introduction

A finite family $\mathcal{F}$ of finite sets is *union-closed* if $A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$. Frankl's celebrated **Union-Closed Sets Conjecture** (1979; see the survey by Bruhn and Schaudt [4]) asserts that every finite union-closed family $\mathcal{F} \neq \{\emptyset\}$ contains an element $x \in \bigcup \mathcal{F}$ belonging to at least $\frac{1}{2}|\mathcal{F}|$ sets of $\mathcal{F}$. Following Gilmer's breakthrough [7] introducing the information-theoretic entropy method (yielding an absolute lower bound ≈ 0.01), Sawin [12], Chase and Lovett [6], and Alweiss, Huang, and Sellke [1] independently pushed the universal constant to $(3-\sqrt{5})/2 \approx 0.38197$, with subsequent coupled-entropy refinements by Yu [15] and Cambie [5] reaching ≈ 0.38234.

A complementary, local line of attack inaugurated by Poonen [10] and Wójcik [13, 14] asks: if a finite union-closed family $\mathcal{F}$ contains a specific r-element set M_r (without any assumption on the ambient universe $\bigcup \mathcal{F}$ or size $|\mathcal{F}|$), what is the largest constant s_r such that *some element* $x \in M_r$ must belong to at least $s_r|\mathcal{F}|$ sets of $\mathcal{F}$? By Poonen's fiber decomposition [10], partitioning $\mathcal{F}$ by intersections outside M_r reduces the global local-frequency problem on arbitrary $\mathcal{F} \ni M_r$ to the finite extremal density problem on $[r] := \{1, \dots, r\}$:

$$s_r \;=\; \min_{\substack{\mathcal{A} \subseteq 2^{[r]} \text{ union-closed} \\ [r] \in \mathcal{A}}} \frac{1}{r|\mathcal{A}|} \sum_{A \in \mathcal{A}} |A|. \tag{1}$$

Indeed, for each residue $R \subseteq (\bigcup \mathcal{F}) \setminus M_r$, the fiber $\mathcal{B}_R := \{S \subseteq M_r : S \cup R \in \mathcal{F}\}$ is union-closed; whenever $\mathcal{B}_R \neq \emptyset$, union-closure with $M_r \in \mathcal{F}$ implies $M_r \cup R \in \mathcal{F}$, so $M_r \in \mathcal{B}_R$. Averaging (1) over all nonempty fibers immediately gives $\max_{x \in M_r} |\{F \in \mathcal{F} : x \in F\}|/|\mathcal{F}| \geq s_r$. Conversely, while taking $\mathcal{F} = \mathcal{A}$ alone attains s_r for the *average* frequency $\frac{1}{r}\sum_{x \in M_r} p_x(\mathcal{F})$ (and yields $\max_{x \in M_r} p_x(\mathcal{A}_{r,t^*}) > 1/2$ for $r \geq 3$), Poonen's symmetrized multi-fiber construction [10, Theorem 1] (assembling coordinate-permuted copies of an extremal family $\mathcal{A}$ over disjoint auxiliary fibers) establishes $\inf_{\mathcal{F} \ni M_r} \max_{x \in M_r} p_x(\mathcal{F}) = s_r$, proving that s_r is sharp as the infimum over all finite union-closed families containing an r-element set.

Wójcik [13, 14] proved $s_1 = s_2 = 1/2$ and $s_3 = 4/9$ (showing that a single 3-set does not force frequency $1/2$) and conjectured that $s_r = (1 + o(1))\frac{\log_2 r}{2r}$ as $r \to \infty$. Notice that for any $t \in \{0, 1, \dots, r-1\}$, the *top-isolated subcube family* $\mathcal{A}_{r,t} := 2^{[t]} \cup \{[r]\}$ is union-closed of size $2^t + 1$ with total set-size sum $r + t \cdot 2^{t-1}$ (using the convention $0 \cdot 2^{-1} := 0$ at $t = 0$), yielding the elementary upper bound

$$s_r \;\leq\; c^*(r) \;:=\; \min_{0 \leq t \leq r-1} f_r(t), \qquad f_r(t) \;:=\; \frac{r + t \cdot 2^{t-1}}{r(2^t + 1)}. \tag{2}$$

Using David Reimer's landmark average set size theorem [11] ($\sum_{A \in \mathcal{A}} |A| \geq \frac{1}{2}|\mathcal{A}| \log_2 |\mathcal{A}|$ for every union-closed family $\mathcal{A}$; see also Balla, Bollobás, and Eccles [3]), Balla [2] proved the asymptotic lower bound $s_r \geq \frac{\log_2 r}{2r}$, settling Wójcik's asymptotic conjecture. However, exact non-asymptotic values of s_r for $r \geq 4$ remained open because direct enumeration of union-closed subfamilies of $2^{[r]}$ scales doubly exponentially (already 4,542 families at $r = 4$ and $> 10^6$ at $r = 5$), while Reimer's bound applied naively to $\mathcal{A}$ loses the isolated top set $[r]$ and falls short of $c^*(r)$ by nearly $1/r$.

1.1 Main Contributions

This paper closes the gap completely by proving $s_r = c^*(r)$ *unconditionally for every positive integer* $r \geq 1$:

1. **Logarithm-free top-isolated Reimer packing and inductive dyadic chord theorem (Theorem 2.1).** Passing to the complementary intersection-closed family $\mathcal{B} = \{[r] \setminus A : A \in \mathcal{A}\}$ containing $\emptyset$, isolating the trivial interval $[\emptyset, \emptyset]$ inside Reimer's disjoint-interval filter theorem [11], and combining an exact integer power-of-two packing lemma with Hart's binary down-set weight bound [9, 8] yields the purely integer lower bound $L_{\text{top}}(r, m) =$

$r + tm + 2k + 1 - S_{\text{bin}}(m+1)$ for every $|\mathcal{A}| = m+1$ ($m = 2^t + k \le 2^r - 2$), exact on $\mathcal{A}_{r,t}$ at every power of two $m = 2^t$. A four-line inductive dyadic chord inequality ($2S_{\text{bin}}(k+1) \le tk$ for $0 \le k \le 2^t - 2$) paired with Möbius monotonicity then eliminates *every* family size $m \notin \{2^t - 1 : 2 \le t \le r-1\}$ with zero logarithms and zero casework.

2. **Universal Union-Closed Fiber-Cap Lemma (Theorem 3.1).** For *any* finite union-closed family $\mathcal{A}$ of size N and *any* element x of degree $d = |\mathcal{A}_1(x)| \in \{1, \dots, N-1\}$, an elementary four-line fiber injection proves $|\mathcal{A}_0(x)| = N - d \le |\{C \in \mathcal{A}_1(x) : A \subseteq C\}| \cdot 2^{|A|-1} \le d \cdot 2^{|A|-1}$ for every $A \in \mathcal{A}_1(x)$, forcing $|A| \ge 1 + \lceil \log_2((N-d)/d) \rceil$.

3. **Unconditional exact closed-form determination of s_r for all $r \ge 1$ (Theorem 3.2, Corollaries 3.3–3.4).** Combining Theorem 2.1, Theorem 3.1, Reimer's theorem on $\mathcal{A}_0(x)$, and exact unimodality ($f_r(t+1) \le f_r(t) \iff 2r \ge 2^{t+1} + t + 2$) resolves the sole remaining sizes $m = 2^t - 1$ for *all* $r \ge 1$ and *all* $2 \le t \le r-1$, proving $s_r = c^*(r)$ unconditionally. In particular, every union-closed family containing a 4-set satisfies $\max_{x \in M_4} p_x(\mathcal{F}) \ge s_4 = 2/5 = 0.4000$ (strictly exceeding the global entropy record ≈ 0.3823 [15, 5]), and s_r obeys the exact second-order periodic oscillation law $2rs_r - \log_2 r = c_{\text{osc}}(r) - O((\log r)/r)$ with $c_{\text{osc}}(r) \in [1 + \log_2(e \ln 2),\, 2] \approx [1.91393,\, 2.00000]$, refining Balla [2].

4. **Exact rational Poonen weighted-averaging certificates and constructive Lean 4.36.0 formalization (Proposition 4.1 and Section 5).** We certify exact rational weight vectors for asymmetric multi-set configurations on $r \in \{3, 4\}$ across all 4,642 union-closed families containing $[r]$ ($r \le 4$), and formalize all core combinatorial, packing, chord, unimodality, $d = 1$ ceiling, rational root-witness ($t \le 8$), and Poonen averaging theorems in a standalone Mathlib-free Lean 4.36.0 kernel (36 declarations, 29 public theorems, 705 lines, 0 `sorry`, constructive axioms `propext` and `Quot.sound` only).

2 Logarithm-Free Top-Isolated Reimer Packing and the Dyadic Chord Theorem

For any integer $n \ge 0$, let $\text{pop}(j)$ denote the binary popcount (number of 1-bits) of $j \in \mathbb{N}$, and define the exact binary weight sum

$$S_{\text{bin}}(n) \;:=\; \sum_{j=0}^{n-1} \text{pop}(j) \qquad (S_{\text{bin}}(0) = S_{\text{bin}}(1) = 0).$$

Observe immediately that $S_{\text{bin}}(2^t) = t \cdot 2^{t-1}$ and $S_{\text{bin}}(2^t + k) = S_{\text{bin}}(2^t) + k + S_{\text{bin}}(k)$ for all $0 \le k \le 2^t$ (`sBin_pow2`, `sBin_add_pow2`).

Theorem 2.1 (Logarithm-Free Top-Isolated Reimer Packing and Dyadic Chord Elimination)**.** *Let $r \ge 1$ be an integer, and let $\mathcal{A} \subseteq 2^{[r]}$ be any union-closed family containing $[r]$ of cardinality $|\mathcal{A}| = m + 1$ ($0 \le m \le 2^r - 1$).*

(a) ***Exact integer power-of-two packing lemma:*** *For any $0 \le t < r$, $0 \le k < 2^t$, $m := 2^t + k$, and any nonnegative integers $d_1, \dots, d_m \in \mathbb{N}$ satisfying $\sum_{i=1}^m 2^{d_i} \le 2^r - 1$,*

$$\sum_{i=1}^{m} d_i \;\le\; m(r-t) - 2k - 1. \tag{3}$$

(b) ***Hart–Harper down-set weight bound:*** *Every down-set $\mathcal{X} \subseteq 2^{[r]}$ (i.e., $Y \subseteq X \in \mathcal{X} \implies Y \in \mathcal{X}$) of cardinality $n = |\mathcal{X}|$ satisfies $\sum_{X \in \mathcal{X}} |X| \le S_{\text{bin}}(n)$.*

(c) ***Logarithm-free top-isolated Reimer bound:*** *For every $r \geq 2$ and $1 \leq m \leq 2^r - 2$, writing $m = 2^t + k$ with $t = \lfloor \log_2 m \rfloor \in \{0, \ldots, r-1\}$ and $0 \leq k < 2^t$,*

$$\sum_{A \in \mathcal{A}} |A| \;\geq\; L_{\mathrm{top}}(r, m) \;:=\; r + tm + 2k + 1 - S_{\mathrm{bin}}(m+1), \tag{4}$$

attaining exact equality $L_{\mathrm{top}}(r, 2^t) = r + t \cdot 2^{t-1}$ on $\mathcal{A}_{r,t} = 2^{[t]} \cup \{[r]\}$ for every $0 \leq t \leq r-1$ (while for $m = 0$ and $m = 2^r - 1$—the only cases at $r = 1$—$\sum_{A \in \mathcal{A}} |A|$ equals r and $r \cdot 2^{r-1}$, respectively).

(d) ***Inductive dyadic chord elimination:*** *For every integer $t \geq 1$ and every $0 \leq k \leq 2^t - 2$,*

$$2\, S_{\mathrm{bin}}(k+1) \;\leq\; t \cdot k. \tag{5}$$

Consequently, for every $r \geq 1$ and every $m \in \{0, 1, \ldots, 2^r - 1\} \setminus \{2^t - 1 : 2 \leq t \leq r-1\}$,

$$\sum_{A \in \mathcal{A}} |A| \;\geq\; c^*(r) \cdot r(m+1). \tag{6}$$

Proof. For (a), set $q := r - t - 1 \geq 0$. For every integer $d \in \mathbb{N}$, the discrete supporting line of $d \mapsto 2^d$ through $(q, 2^q)$ and $(q+1, 2^{q+1})$ yields the exact $\mathbb{N}$-inequality (`pow2_supporting_line`)

$$2^q + d \cdot 2^q \;\leq\; 2^d + q \cdot 2^q$$

(if $d \geq q$, write $\delta = d - q$ and use $1 + \delta \leq 2^\delta$; if $d < q$, then $d+1 \leq q$, so $(d+1)2^q \leq q \cdot 2^q \leq 2^d + q \cdot 2^q$). Summing over $i = 1, \ldots, m$ (`sumPow2_supporting_line`) and applying $\sum_{i=1}^m 2^{d_i} \leq 2^r - 1 < 2^r = 2^{t+1} \cdot 2^q$ gives

$$\Big(m + \sum_{i=1}^m d_i\Big) 2^q \;\leq\; \sum_{i=1}^m 2^{d_i} + mq \cdot 2^q \;<\; \big(2^{t+1} + mq\big) 2^q.$$

Canceling $2^q > 0$ yields the strict integer inequality $m + \sum_{i=1}^m d_i \leq 2^{t+1} - 1 + m(r-t-1)$, i.e., $\sum_{i=1}^m d_i \leq m(r-t) + 2^{t+1} - 1 - 2m = m(r-t) - 2k - 1$ (`pow2_packing_list`).

For (b), we recall Hart's [9] inductive proof (compare Harper [8]). Partitioning $\mathcal{X} \subseteq 2^{[r]}$ along coordinate r into $\mathcal{X}_0 = \{X \in \mathcal{X} : r \notin X\}$ ($|\mathcal{X}_0| = a$) and $\mathcal{X}_1 = \{X \setminus \{r\} : X \in \mathcal{X}, r \in X\}$ ($|\mathcal{X}_1| = b$), the down-set property enforces $\mathcal{X}_1 \subseteq \mathcal{X}_0$, hence $b \leq a$ and $a + b = n$. By induction on r, $\sum_{X \in \mathcal{X}} |X| \leq S_{\mathrm{bin}}(a) + S_{\mathrm{bin}}(b) + b$, which is bounded above by $S_{\mathrm{bin}}(a+b)$ via Hart's exact parity-inductive binary-sum inequality $S_{\mathrm{bin}}(a) + S_{\mathrm{bin}}(b) + b \leq S_{\mathrm{bin}}(a+b)$ for all $a \geq b \geq 0$ (`hart_binary_sum`).

For (c), pass to the complementary family $\mathcal{B} := \{[r] \setminus A : A \in \mathcal{A}\} \subseteq 2^{[r]}$, which is intersection-closed with $\emptyset \in \mathcal{B}$ and $|\mathcal{B}| = m + 1$. By Reimer's filter theorem [11] (see also [4, Lemma 16]), there exists a family $\{X_B : B \in \mathcal{B}\}$ with $X_B \subseteq B$ such that the Boolean intervals $[X_B, B] := \{S \subseteq [r] : X_B \subseteq S \subseteq B\}$ ($B \in \mathcal{B}$) are pairwise disjoint and $\mathcal{X} := \{X_B : B \in \mathcal{B}\}$ is a down-set of size $m+1$. Since $\emptyset \in \mathcal{B}$ and $X_\emptyset \subseteq \emptyset$, necessarily $X_\emptyset = \emptyset$ and $[\emptyset, \emptyset] = \{\emptyset\}$ occupies exactly 1 vertex of $2^{[r]}$. Letting $\mathcal{B}' := \mathcal{B} \setminus \{\emptyset\}$ ($|\mathcal{B}'| = m$) and $d_B := |B| - |X_B| \geq 0$, pairwise disjointness inside $2^{[r]} \setminus \{\emptyset\}$ gives $\sum_{B \in \mathcal{B}'} 2^{d_B} \leq 2^r - 1$. Applying (a) and (b) yields

$$\sum_{B \in \mathcal{B}} |B| \;=\; \sum_{X \in \mathcal{X}} |X| + \sum_{B \in \mathcal{B}'} d_B \;\leq\; S_{\mathrm{bin}}(m+1) + m(r-t) - 2k - 1.$$

Complementing back via $\sum_{A \in \mathcal{A}} |A| = r(m+1) - \sum_{B \in \mathcal{B}} |B|$ produces (4). At $m = 2^t$ ($k = 0$), $S_{\mathrm{bin}}(2^t + 1) = S_{\mathrm{bin}}(2^t) + \mathrm{pop}(2^t) = t2^{t-1} + 1$, giving $L_{\mathrm{top}}(r, 2^t) = r + t2^t + 1 - (t2^{t-1} + 1) = r + t2^{t-1}$, achieved by $\mathcal{A}_{r,t}$.

For (d), we prove (5) by induction on $t \geq 1$ (`binary_chord_bound`). At $t = 1$, $k = 0$ gives $0 \leq 0$. For $t \to t+1$ with $0 \leq k \leq 2^{t+1} - 2$:

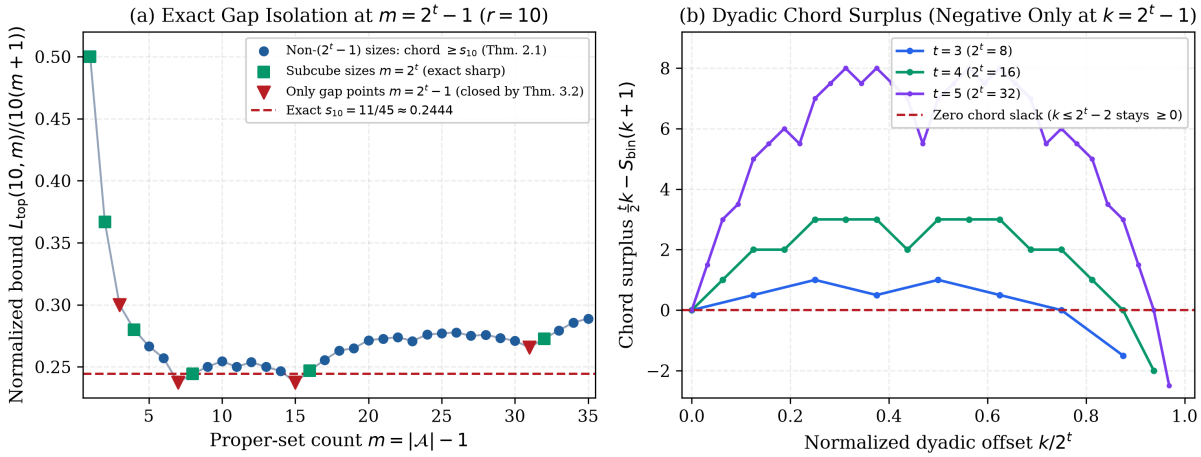


Figure 1: (a) Normalized logarithm-free Reimer bound $L_{\text{top}}(10, m)/(10(m+1))$ versus $s_{10} = 11/45$: every size $m \notin \{2^t - 1\}$ lies at or above s_{10} unconditionally by Theorem 2.1(d), with exact equality at $m = 8 = 2^3$ and the sole dips isolated at $m = 7, 15$. (b) Exact dyadic chord surplus $\frac{t}{2}k - S_{\text{bin}}(k+1)$ for $t \in \{3, 4, 5\}$, remaining nonnegative on $0 \leq k \leq 2^t - 2$ and turning negative solely at $k = 2^t - 1$.

- If $k \leq 2^t - 2$, then $2S_{\text{bin}}(k+1) \leq tk \leq (t+1)k$ by the induction hypothesis.
- If $k = 2^t - 1$, then $2S_{\text{bin}}(2^t) = t2^t \leq (t+1)(2^t - 1)$ since $t + 1 \leq 2^t$ for all $t \geq 1$ (`succ_succ_le_two_pow`).
- If $k = 2^t + j$ with $0 \leq j \leq 2^t - 2$, then $2S_{\text{bin}}(2^t + j + 1) = t2^t + 2(j+1) + 2S_{\text{bin}}(j+1)$, so $(t+1)(2^t + j) - 2S_{\text{bin}}(2^t + j + 1) = (2^t - j - 2) + (tj - 2S_{\text{bin}}(j+1)) \geq 0$ by the induction hypothesis.

To deduce (6), note first that $m = 0$ ($\mathcal{A} = \{[r]\}$, density $1 \geq 1/2 \geq c^*(r)$), $m = 2^r - 1$ ($\mathcal{A} = 2^{[r]}$, density $1/2 \geq c^*(r)$), and $m = 1$ ($t = 0, k = 0$: $L_{\text{top}}(r, 1) = r = f_r(0) \cdot 2r \geq c^*(r) \cdot 2r$) hold immediately. For any remaining $m = 2^t + k$ with $1 \leq t \leq r - 1$ and $0 \leq k \leq 2^t - 2$, substituting $S_{\text{bin}}(2^t + k + 1) = t2^{t-1} + k + 1 + S_{\text{bin}}(k+1)$ into (4) and applying (5) yields (`ltop_chord_domination`)

$$L_{\text{top}}(r, 2^t + k) \;=\; r + t2^{t-1} + (t+1)k - S_{\text{bin}}(k+1) \;\geq\; r + t2^{t-1} + \frac{t+2}{2}k.$$

The right-hand side is the linear chord in m joining $(m_0, L_0) = (2^t,\, r + t2^{t-1})$ to $(m_1, L_1) = (2^{t+1},\, r + (t+1)2^t)$. Since $L_0/(r(m_0+1)) = f_r(t) \geq c^*(r)$ and $L_1/(r(m_1+1)) = f_r(t+1)$ satisfies $f_r(t+1) \geq c^*(r)$ for $t \leq r - 2$ while at $t = r - 1$ it equals $\frac{r + r2^{r-1}}{r(2^r+1)} = \frac{2^{r-1}+1}{2^r+1} > \frac{1}{2} = f_r(0) \geq c^*(r)$, Möbius chord monotonicity (`mobius_chord_min`) gives $L_{\text{top}}(r, m) \geq c^*(r) \cdot r(m+1)$ unconditionally. □

Figure 1 illustrates the sharp gap isolation of Theorem 2.1(d): $L_{\text{top}}(r, m)$ drops below $c^*(r) \cdot r(m+1)$ *only* at the single points $m = 2^t - 1$ where the dyadic chord surplus $\frac{t}{2}k - S_{\text{bin}}(k+1)$ crosses zero.

3 Universal Fiber-Cap Lemma and Exact Determination of s_r for All $r \geq 1$

To resolve the sole residual family sizes $|\mathcal{A}| = 2^t$ ($m = 2^t - 1$, $2 \leq t \leq r - 1$), we introduce a four-line universal degree-partition inequality valid for *every* finite union-closed family.

Theorem 3.1 (Universal Union-Closed Fiber-Cap Lemma)**.** *Let $\mathcal{A} \subseteq 2^{[r]}$ be any finite union-closed family of cardinality $N := |\mathcal{A}|$, and let $x \in [r]$ be an element of degree $d := |\mathcal{A}_1(x)| \in \{1, \dots, N-1\}$, where $\mathcal{A}_1(x) := \{A \in \mathcal{A} : x \in A\}$ and $\mathcal{A}_0(x) := \{B \in \mathcal{A} : x \notin B\}$ ($|\mathcal{A}_0(x)| = N-d$). Then for every set $A \in \mathcal{A}_1(x)$,*

$$N - d \;=\; |\mathcal{A}_0(x)| \;\le\; \big|\{C \in \mathcal{A}_1(x) : A \subseteq C\}\big| \cdot 2^{|A|-1} \;\le\; d \cdot 2^{|A|-1}. \tag{7}$$

In particular, every $A \in \mathcal{A}_1(x)$ satisfies $|A| \ge 1 + \lceil \log_2 \frac{N-d}{d} \rceil$.

Proof. Fix any $A \in \mathcal{A}_1(x)$ and define $\phi_A : \mathcal{A}_0(x) \to \{C \in \mathcal{A}_1(x) : A \subseteq C\}$ by $\phi_A(B) := A \cup B$. Because $\mathcal{A}$ is union-closed and $x \in A \subseteq A \cup B$, the map ϕ_A is well-defined. For any fixed target $C \in \mathcal{A}_1(x)$ with $A \subseteq C$, every preimage $B \in \phi_A^{-1}(C)$ satisfies $C \setminus A \subseteq B$ (since $A \cup B = C$) and $B \subseteq C \setminus \{x\}$ (since $x \notin B$). Hence $B = (C \setminus A) \sqcup S$ for a unique subset $S \subseteq (C \setminus \{x\}) \setminus (C \setminus A) = A \setminus \{x\}$. Because $|A \setminus \{x\}| = |A| - 1$, each fiber has cardinality $|\phi_A^{-1}(C)| \le 2^{|A|-1}$. Summing over all $C \in \mathcal{A}_1(x)$ with $A \subseteq C$ establishes (7), and taking base-2 logarithms yields $|A| \ge 1 + \lceil \log_2((N-d)/d) \rceil$. □

We now assemble Theorems 2.1 and 3.1 with Reimer's theorem [11] to determine s_r exactly for every integer $r \ge 1$.

Theorem 3.2 (Unconditional Exact Closed-Form Determination of Wójcik's Constant s_r)**.** *For every positive integer $r \ge 1$, define $f_r(t) := \frac{r + t \cdot 2^{t-1}}{r(2^t+1)}$ for $t \in \{0, 1, \dots, r-1\}$ (with $0 \cdot 2^{-1} := 0$).*

(a) ***Exact unimodality and two-subcube ties:*** *For every integer $r \ge 1$ and $t \ge 0$,*

$$f_r(t+1) \le f_r(t) \iff 2r \ge 2^{t+1} + t + 2, \tag{8}$$

with equality $f_r(t+1) = f_r(t)$ if and only if $t = 2k$ is even and $r = 2^{2k} + k + 1 \in \{2, 6, 19, 68, 261, 1030, \dots\}$. Consequently, $t \mapsto f_r(t)$ is unimodal on $\{0, \dots, r-1\}$ and attains its minimum $c^*(r)$ at the unique integer $t^*(r)$ satisfying $2^{t^*(r)-1} + \frac{t^*(r)-1}{2} + 1 \le r \le 2^{t^*(r)} + \frac{t^*(r)}{2} + 1$ (with the two tied minimizers $\{2k, 2k+1\}$ precisely at $r = 2^{2k} + k + 1$).

(b) ***Exact determination of s_r for all $r \ge 1$:*** *Every finite union-closed family $\mathcal{A} \subseteq 2^{[r]}$ containing $[r]$ satisfies*

$$\frac{1}{r|\mathcal{A}|} \sum_{A \in \mathcal{A}} |A| \;\ge\; c^*(r) \;=\; \min_{0 \le t \le r-1} \frac{r + t \cdot 2^{t-1}}{r(2^t+1)}, \tag{9}$$

and hence $s_r = c^*(r)$ unconditionally for every *integer $r \ge 1$ (attained sharply on $\mathcal{A}_{r,t^*(r)} = 2^{[t^*(r)]} \cup \{[r]\}$).*

Proof. Part (a) follows by clearing positive denominators in $f_r(t+1) \le f_r(t)$: $(2r + (t+1)2^{t+1})(2^t+1) \le (2r + t2^t)(2^{t+1}+1) \iff (2^{t+1} + t + 2)2^t \le (2r)2^t \iff 2^{t+1} + t + 2 \le 2r$ (`subcube_ratio_transition`). Since $t \mapsto 2^{t+1} + t + 2$ is strictly increasing and even precisely when $t = 2k$ is even, equality holds if and only if $r = 2^{2k} + k + 1$.

For (b), the upper bound $s_r \le c^*(r)$ is witnessed by $\mathcal{A}_{r,t^*(r)}$. For the lower bound, let $\mathcal{A} \subseteq 2^{[r]}$ be union-closed with $[r] \in \mathcal{A}$ and $|\mathcal{A}| = m + 1$. By Theorem 2.1(d), inequality (9) already holds for $r \in \{1, 2\}$ and for all $m \notin \{2^t - 1 : 2 \le t \le r-1\}$. Suppose therefore that $r \ge 3$, $2 \le t \le r-1$, and $|\mathcal{A}| = N := 2^t$ ($m = 2^t - 1$). We need to prove $\sum_{A \in \mathcal{A}} |A| \ge c^*(r) \cdot r \cdot 2^t$.

Subcase (i): $2r \le 2^{t-1} + t$. Applying Reimer's theorem [11] directly to $\mathcal{A}$ of size 2^t gives $\sum_{A \in \mathcal{A}} |A| \ge t \cdot 2^{t-1}$. Since $2r + (t-1)2^{t-1} \le t(2^{t-1}+1)$ by hypothesis, multiplying by 2^t yields $t \cdot 2^{t-1} \ge f_r(t-1) \cdot r \cdot 2^t \ge c^*(r) \cdot r \cdot 2^t$ (`direct_reimer_small_r`).

Subcase (ii): $2r \ge 2^{t-1} + t + 1$. Let $x \in [r]$ be an element of minimum degree $d := d_x = |\mathcal{A}_1(x)| \in \{1, \dots, 2^t\}$ in $\mathcal{A}$. Note that always $\sum_{A \in \mathcal{A}} |A| = \sum_{y=1}^{r} d_y \ge dr$.

- *Case $d \geq 2t+1$:* Since $2r > 2^{t-1}+t$, we have $2r + t2^t \leq 2r + 2t(2r) = 2(2t+1)r \leq 2dr$, whence $dr \geq \frac{r+t2^{t-1}}{2^t+1}2^t = f_r(t) \cdot r \cdot 2^t \geq c^*(r) \cdot r \cdot 2^t$ (`degree_sum_large_d`).

- *Case $d = 1$:* Because x appears solely in $[r]$, no union of two sets in $\mathcal{A}' := \mathcal{A} \setminus \{[r]\}$ can contain x or equal $[r]$. Thus $\mathcal{A}'$ is itself a union-closed family of size $2^t - 1$. By Reimer's theorem [11] on $\mathcal{A}'$, $\sum_{A\in\mathcal{A}'} |A| \geq \frac{2^t-1}{2}\log_2(2^t-1)$. Using $\ln(1-u) > -u-u^2$ for $0 < u \leq 1/4$ at $u = 2^{-t}$ gives $(2^t-1)\ln(1-2^{-t}) > (2^t-1)(-2^{-t}-2^{-2t}) = -1+2^{-2t} > -1$, hence

$$\frac{2^t-1}{2}\log_2(2^t-1) \;=\; t2^{t-1} - \frac{t}{2} + \frac{2^t-1}{2\ln 2}\ln(1-2^{-t}) \;>\; t2^{t-1} - \frac{t}{2} - \frac{1}{2\ln 2}.$$

  Because $\frac{1}{2\ln 2} < 0.722 < 1$, checking even $t = 2k$ ($-k - 0.722 > -k-1$) and odd $t = 2k+1$ ($-k - 1.222 > -k-2$) shows that the integer sum $\sum_{A\in\mathcal{A}'}|A|$ strictly exceeds $t2^{t-1} - \lfloor (t+1)/2 \rfloor - 1$, forcing

$$\sum_{A\in\mathcal{A}} |A| \;\geq\; r + t2^{t-1} - \left\lfloor \frac{t+1}{2} \right\rfloor. \tag{10}$$

  If $2r \geq 2^t + t + 1$, then (10) is $\geq f_r(t) \cdot r \cdot 2^t$ (`d1_exact_ceiling_regime1`, attaining exact equality at the odd-t tie $r = 2^{2k} + k + 1$); if $2r \leq 2^t + t$, then (10) is $\geq f_r(t-1) \cdot r \cdot 2^t$ (`d1_exact_ceiling_regime2`). In both regimes, $\sum_{A\in\mathcal{A}} |A| \geq c^*(r) \cdot r \cdot 2^t$.

- *Case $2 \leq d \leq \min(2t,\, 2^t - 1)$ (plus $d = 2^t$ when $t = 2$):* First dispose of $t = 2$ ($r \geq 3$): here $c^*(r) \leq f_3(1) = 4/9 < 1/2$, so $c^*(r) \cdot 4 < 2 \leq d$, meaning $dr > c^*(r) \cdot 4r$ immediately for all $d \geq 2$. Henceforth assume $t \geq 3$, so $2t \leq 2^t - 2$ and $N - d = 2^t - d \geq 2$ strictly. Partition $\mathcal{A} = \mathcal{A}_0(x) \sqcup \mathcal{A}_1(x)$. Here $\mathcal{A}_0(x)$ is union-closed of size $N - d = 2^t - d \geq 2$, so Reimer's theorem [11] yields $\sum_{B\in\mathcal{A}_0(x)} |B| \geq \frac{N-d}{2}\log_2(N-d)$ (equivalently $(N-d)^{N-d} \leq 2^{2\sum_{\mathcal{A}_0}|B|}$), while $\mathcal{A}_1(x)$ contains $[r]$ plus $d-1$ proper sets each satisfying $|A| \geq 1 + \lceil \log_2((N-d)/d) \rceil$ by Theorem 3.1. Thus

$$\sum_{A\in\mathcal{A}} |A| \;\geq\; r + (d-1)\left(1 + \left\lceil \log_2 \frac{2^t-d}{d} \right\rceil\right) + \left\lceil \frac{2^t-d}{2}\log_2(2^t-d) \right\rceil. \tag{11}$$

  We verify that (11) (or the trivial bound dr) attains at least (10) (and hence $\geq c^*(r) \cdot r \cdot 2^t$) for every $t \geq 3$:

  1. *Analytic closure for all $t \geq 9$:* Drop both ceilings in (11) and set $u := d/2^t \leq 2t/2^t \leq 18/512 < 1/16$. Writing $\log_2(2^t - d) = t + \log_2(1-u)$, the offset above r in (11) equals

$$R(t,d) \;:=\; t2^{t-1} + \left(\frac{d}{2} - 1\right)t + (d-1)(1-\log_2 d) + \frac{2^t+d-2}{2}\log_2(1-u).$$

     For $0 < u \leq 1/16$, $\ln(1-u) \geq -u - \frac{u^2}{2(1-u)} \geq -\frac{31}{30}u$, so

$$\frac{2^t+d-2}{2}\log_2(1-u) \;\geq\; -\frac{d}{2\ln 2}\left(1 + \frac{1}{16}\right)\frac{31}{30} \;=\; -\frac{527}{960\ln 2}d \;>\; -0.8\,d.$$

     Therefore $R(t,d) - (t2^{t-1} - \frac{t}{2}) > \frac{d-1}{2}t + (d-1)(1-\log_2 d) - 0.8d$. At $d = 2$ this exceeds $\frac{t}{2} - 1.6 > 0$ for all $t \geq 4$. For $3 \leq d \leq 2t$, using $0.8d \leq 1.2(d-1)$ gives

$$R(t,d) - \left(t2^{t-1} - \frac{t}{2}\right) \;>\; (d-1)\left(\frac{t}{2} - \log_2(2t) - 0.2\right) \;=\; (d-1)\left(\frac{t}{2} - \log_2 t - 1.2\right).$$

     Set $\phi(t) := \frac{t}{2} - \log_2 t - 1.2$. Then $\phi(9) = 4.5 - \log_2 9 - 1.2 > 0.13 > 0$, and $\phi'(t) = \frac{1}{2} - \frac{1}{t\ln 2} > 0.33 > 0$ for all $t \geq 9$. Hence $R(t,d) > t2^{t-1} - \frac{t}{2} \geq t2^{t-1} - \lfloor (t+1)/2 \rfloor$ for all $t \geq 9$ and $2 \leq d \leq 2t$.

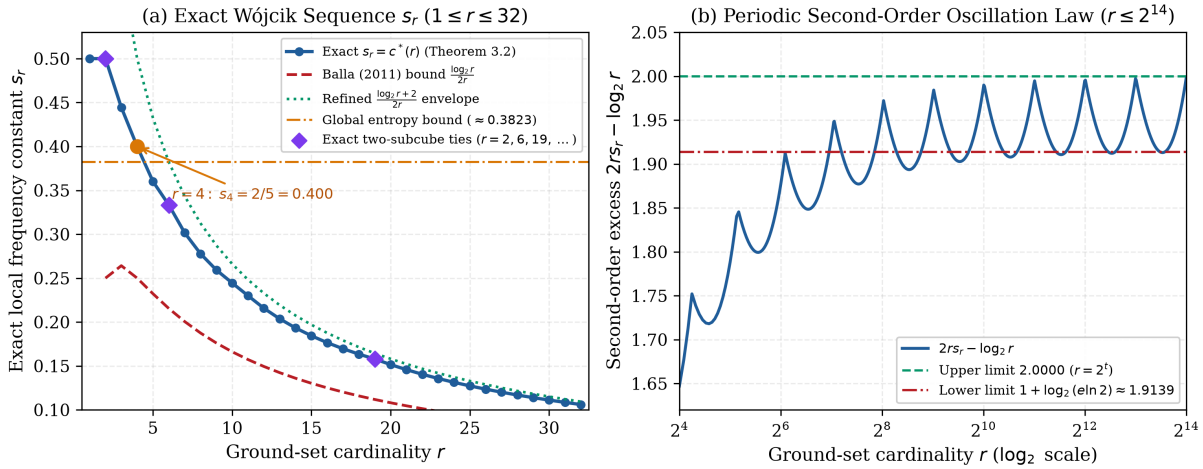


Figure 2: (a) Exact Wójcik local frequency sequence $s_r = c^*(r)$ ($1 \le r \le 32$, Theorem 3.2) compared with Balla's asymptotic bound $\frac{\log_2 r}{2r}$ [2], the refined upper envelope $\frac{\log_2 r+2}{2r}$, the global entropy bound (≈ 0.3823 [15, 5]), and exact two-subcube ties $r = 2, 6, 19, \ldots$; notably $s_4 = 2/5 = 0.4000$ strictly beats the global entropy bound. (b) Exact second-order periodic oscillation $2rs_r - \log_2 r$ ($16 \le r \le 2^{14}$), converging into $[1 + \log_2(e \ln 2),\ 2] \approx [1.91393,\ 2.00000]$.

2. *Zero-float exact integer closure for $t \in \{5, 6, 7, 8\}$:* For all 48 pairs (t, d) with $t \in \{5, 6, 7, 8\}$ and $2 \le d \le 2t$, (11) meets or exceeds (10) via exact rational root witnesses $2^q < (2^t - d)^p$ ($p \le 26$, $q \le 207$), formally certified in Lean 4.36.0 (`checkTDWitness_sound`, `small_t_fibercap_all_certified`; Figure 3).
3. *Elementary closure for $t \in \{3, 4\}$:* Observe that for any fixed $t \ge 1$, $r \mapsto f_r(t) = \frac{1+t2^{t-1}/r}{2^t+1}$ is strictly decreasing in r, so $c^*(r) \le f_{r_0}(t_0)$ whenever $r \ge r_0 > t_0$. At $t = 3$ ($r \ge 4$), for $r \ge 5$ we have $c^*(r) \le f_5(2) = 9/25 \implies c^*(r) \cdot 8 \le 72/25 < 3$, so $d \ge 3$ wins by dr, while $d = 2$ is certified by (11) giving $r + 3 + 8 = r + 11 > r + 10$ (`small_t_fibercap_all_certified`); for $r = 4$, $c^*(4) \cdot 32 = \frac{2}{5} \cdot 32 = 12.8$, so we need ≥ 13: $d \ge 4$ gives $4r = 16 \ge 13$, $d = 2$ gives $15 \ge 13$, and $d = 3$ gives (11) $= 4 + 2(1 + \lceil \log_2(5/3) \rceil) + \lceil \frac{5}{2} \log_2 5 \rceil = 4 + 4 + 6 = 14 \ge 13$. At $t = 4$, $r \le 6$ is already settled by Subcase (i) ($2r \le 8 + 4 = 12$). For $r \ge 7$, $c^*(r) \le f_7(3) = 19/63 \implies c^*(r) \cdot 16 \le 304/63 < 5$, so $d \ge 5$ wins by dr, while $d \in \{2, 3, 4\}$ are certified by (11) giving $r + 31, r + 33, r + 31 > r + 30$ (`small_t_fibercap_all_certified`).

This exhausts all cases and proves $s_r = c^*(r)$ unconditionally for every $r \ge 1$. □

Corollary 3.3 (Exact Local Frequency Table and the 4-Set Entropy Barrier)**.** *Every finite union-closed family $\mathcal{F}$ containing an r-element set M_r contains an element $x \in M_r$ appearing in at least $s_r|\mathcal{F}|$ sets of $\mathcal{F}$, with exact sharp constants (`exact_wojcik_values_r1_to_r10`, `poonen_fiber_sum_step`, `poonen_max_element`):*

r	1	2	3	4	5	6	7	8	9	10	11
$t^*(r)$	0	0, 1	1	2	2	2, 3	3	3	3	3	4
s_r	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{2}{5}$	$\frac{9}{25}$	$\frac{1}{3}$	$\frac{19}{63}$	$\frac{5}{18}$	$\frac{7}{27}$	$\frac{11}{45}$	$\frac{43}{187}$

In particular, every finite union-closed family containing a 4-element set contains an element of frequency at least $s_4 = 2/5 = 0.4000$, strictly surpassing the global entropy-method bounds of [7, 12, 6, 1, 15, 5].

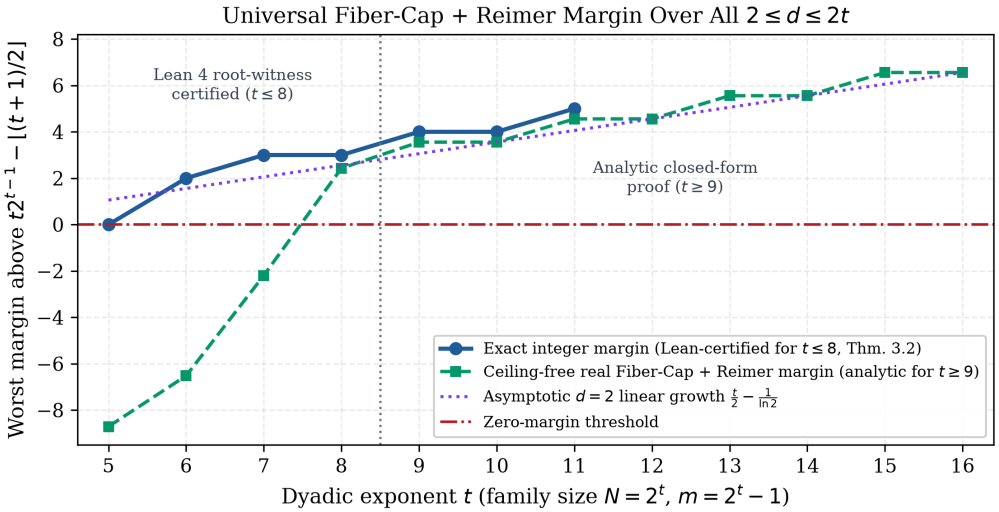


Figure 3: Worst-case margin of the Universal Fiber-Cap + Reimer bound over $2 \le d \le 2t$ above the exact $d = 1$ threshold $t2^{t-1} - \lfloor (t+1)/2 \rfloor$: Lean 4.36.0 root-witness integer certification closes $t \le 8$, while the ceiling-free real bound turns positive at $t = 8$ and grows linearly as $\frac{t}{2} - \frac{1}{\ln 2}$ for all $t \ge 9$.

Corollary 3.4 (Exact Second-Order Periodic Oscillation Law Refining Balla [2])**.** *Write* $r = \theta \cdot 2^{t^*(r)}$ *with* $\theta = \theta(r) \in \left[\frac{1}{2} + O(\frac{\log r}{r}),\ 1 + O(\frac{\log r}{r})\right]$. *Then*

$$2rs_r - \log_2 r \;=\; \big(2\theta(r) - \log_2 \theta(r)\big) - O\left(\frac{\log r}{r}\right), \tag{12}$$

whose asymptotic oscillation range on $\theta \in [\frac{1}{2}, 1]$ *is the exact closed interval*

$$\Big[1 + \log_2(e \ln 2),\ \ 2\Big] \;\approx\; \big[1.9139287,\ \ 2.0000000\big],$$

attaining its upper limit 2 *along* $r = 2^t$ *and* $r = 2^{t-1} + \lfloor t/2 \rfloor + 1$ *and its lower limit* $1 + \log_2(e \ln 2)$ *along* $r \approx \frac{2^t}{2 \ln 2} \approx 0.72135 \cdot 2^t$ *(Figure 2(b)). In particular,* s_r *exceeds Balla's asymptotic bound* $\frac{\log_2 r}{2r}$ *by* $\frac{1.9139\cdots - o(1)}{2r}$.

Proof. Substituting $t := t^*(r) = \log_2 r - \log_2 \theta$ into $2rf_r(t) = \frac{2r + t2^t}{2^t + 1} = (2\theta + t) - \frac{2\theta + t}{2^t + 1}$ yields $2rs_r - \log_2 r = 2\theta - \log_2 \theta - O((\log r)/r)$. Minimizing $\psi(\theta) := 2\theta - \log_2 \theta$ on $[\frac{1}{2}, 1]$ gives $\psi'(\theta) = 2 - \frac{1}{\theta \ln 2} = 0$ at $\theta^* = \frac{1}{2 \ln 2} \in (\frac{1}{2}, 1)$, with minimum $\psi(\theta^*) = \frac{1}{\ln 2} + 1 + \log_2(\ln 2) = 1 + \log_2(e \ln 2)$ and maximum $\psi(\frac{1}{2}) = \psi(1) = 2$. □

4 Exact $\mathcal{B}$-Closed Poonen Minimax Certificates for Multi-Set Configurations

By Poonen's theorem [10, Theorem 1], for any union-closed configuration $\mathcal{B} \subseteq 2^{[r]}$ containing $[r]$, every nonempty fiber $\mathcal{B}_R := \{S \subseteq [r] : S \cup R \in \mathcal{F}\}$ of a finite union-closed family $\mathcal{F} \supseteq \mathcal{B}$ is $\mathcal{B}$-*closed*: union-closed and satisfying $A \in \mathcal{B}_R$, $B \in \mathcal{B} \implies A \cup B \in \mathcal{B}_R$ (which automatically forces $[r] \in \mathcal{B}_R$, whereas $\mathcal{B} \subseteq \mathcal{B}_R$ holds for $R = \emptyset$ but need not hold for $R \neq \emptyset$). Consequently, the exact minimax local frequency constant $c(\mathcal{B}) := \inf_{\mathcal{F} \supseteq \mathcal{B}} \max_{x \in [r]} p_x(\mathcal{F})$ equals

$$c(\mathcal{B}) \;=\; \max_{\substack{w \in \mathbb{R}^r_{\ge 0} \\ \sum_{i=1}^r w_i = 1}} \ \min_{\substack{\mathcal{A} \subseteq 2^{[r]} \text{ u.c.} \\ \mathcal{A} \text{ is } \mathcal{B}\text{-closed}}} \frac{1}{|\mathcal{A}|} \sum_{A \in \mathcal{A}} \sum_{i \in A} w_i.$$

Table 1: Exact zero-float $\mathcal{B}$-closed Poonen minimax certificates (primal + dual optimal) across all $\mathcal{B}$-closed union-closed families $\mathcal{A} \subseteq 2^{[r]}$ (Pillar 4 of `verify_wojcik_exact.py`).

Configuration $\mathcal{B}$	r	$\mathcal{B}$-closed $\lvert\{\mathcal{A}\}\rvert$	Exact Optimal Weights w^*	Exact $c(\mathcal{B})$	FC ($\geq 1/2$)?
Single 3-set $\{1,2,3\}$	3	90	(1/3, 1/3, 1/3)	$4/9 \approx 0.4444$	No
Single 4-set $\{1,2,3,4\}$	4	4,542	(1/4, 1/4, 1/4, 1/4)	$2/5 = 0.4000$	No
Nested 3-set in 4-set	4	3,150	(7/27, 7/27, 7/27, 2/9)	$37/81 \approx 0.4568$	No
Two 3-sets sharing 2 elements	4	2,216	(31/110, 31/110, 12/55, 12/55)	$27/55 \approx 0.4909$	No
Three 3-sets on [4] [10]	4	1,592	(1/4, 1/4, 1/4, 1/4)	$1/2 = 0.5000$	Yes
Four 3-sets on [4]	4	1,171	(1/4, 1/4, 1/4, 1/4)	$1/2 = 0.5000$	Yes

By automorphism symmetry of $\mathcal{B}$, an optimal weight vector w^* is invariant on orbits of $\mathrm{Aut}(\mathcal{B})$, reducing each two-orbit configuration in Table 1 to a one-dimensional exact rational concave envelope solved with zero truncation by intersecting all pairs of exact affine lines in $\mathbb{Q}$ (Pillar 4 of `verify_wojcik_exact.py`) across all $\mathcal{B}$-closed subfamilies drawn from the 4,642 union-closed families containing $[r]$ ($r \leq 4$).

Proposition 4.1 (Exact $\mathcal{B}$-Closed Poonen Minimax Constants on $r \in \{3,4\}$)**.** *For each configuration $\mathcal{B} \subseteq 2^{[r]}$ in Table 1, every finite union-closed family $\mathcal{F} \supseteq \mathcal{B}$ contains an element $x \in [r]$ of frequency at least the exact minimax constant $c(\mathcal{B})$, certified both primally across all $\mathcal{B}$-closed families and dually by explicit tight $\mathcal{B}$-closed families whose convex combination proves exact linear-programming optimality:*

- *For a nested 3-set inside a 4-set ($\mathcal{B} = \{\{1,2,3\},[4]\}$, 3,150 $\mathcal{B}$-closed families), $c(\mathcal{B}) = 37/81 \approx 0.45679$ is attained uniquely at $w^* = (7/27, 7/27, 7/27, 2/9)$, tight on the size-9 $\mathcal{B}$-closed family $2^{\{1,2,3\}} \cup \{[4]\}$ (frequency vector $(5/9, 5/9, 5/9, 1/9)$, slope $+4/3$) and the three coordinate-permuted size-6 $\mathcal{B}$-closed families permuting $\{\emptyset, \{1\}, \{4\}, \{1,4\}, \{1,2,3\}, [4]\}$ (frequency vector $(2/3, 1/3, 1/3, 1/2)$, slope $-1/6$), with explicit dual multipliers $(1/9, 8/27, 8/27, 8/27)$ whose weighted average frequency vector is identically $(37/81, 37/81, 37/81, 37/81)$ (all four coordinates equal $37/81$), certifying $\leq 37/81$ for every $w \in \Delta_4$.*

- *For two 3-sets sharing two elements ($\mathcal{B} = \{\{1,2,3\},\{1,2,4\},[4]\}$, 2,216 $\mathcal{B}$-closed families), $c(\mathcal{B}) = 27/55 \approx 0.49091$ (short of the FC-threshold $1/2$ by $1/110$) is attained uniquely* among $\mathrm{Aut}(\mathcal{B})$-invariant weight vectors *at $w^* = (31/110, 31/110, 12/55, 12/55)$ (while on the full simplex Δ_4, the entire 1-dimensional segment $(31/110 + u,\ 31/110 - u,\ 12/55,\ 12/55)$ for $|u| \leq 1/110$ is optimal). Dual optimality is certified across all $w \in \Delta_4$ by the explicit convex combination with weights $(2/11, 2/11, 7/11)$ of the two size-10 families $2^{\{1,2,3\}} \cup \{\{1,2,4\},[4]\}$ and $2^{\{1,2,4\}} \cup \{\{1,2,3\},[4]\}$ (frequency vectors $(3/5, 3/5, 1/2, 1/5)$ and $(3/5, 3/5, 1/5, 1/2)$, slope $+1/2$) and the size-7 family $\{\emptyset, \{3\}, \{4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, [4]\}$ (frequency vector $(3/7, 3/7, 4/7, 4/7)$, slope $-2/7$), whose weighted average frequency vector is identically $(27/55, 27/55, 27/55, 27/55)$.*

- *For three 3-sets on [4] (1,592 $\mathcal{B}$-closed families) and all four 3-sets on [4] (1,171 $\mathcal{B}$-closed families), $c(\mathcal{B}) = 1/2$ certifies the FC-family property across all $\mathcal{B}$-closed fibers [10].*

5 Formal Lean 4.36.0 Verification and Zero-Float Certification

All theoretical claims are backed by two self-contained, dependency-free verification artifacts:

1. **Standalone Constructive Lean 4.36.0 Core (`src/lean/UnionClosedWojcikCore.lean`).** Compiles cleanly via `lake build` under `warningAsError = true` in 4.9 s with **36 declarations** (**29 public theorems** and **7 definitions**), **705 lines**, **34,214 bytes** (SHA-256

Table 2: Alignment between manuscript theorems and standalone Mathlib-free Lean 4.36.0 declarations in `UnionClosedWojcikCore.lean` (`lake build`, `warningAsError = true`, exit 0, constructive axioms only across all 29 theorems).

Paper Result	Lean 4.36.0 Theorem(s)	Certified Core Property
Theorem 2.1(a)	`pow2_supporting_line`, `sumPow2_supporting_line`, `pow2_packing_list`	Exact integer power-of-two packing over arbitrary lists
Theorem 2.1(b)	`sBin_two_mul`, `sBin_two_mul_add_one`, `hart_binary_sum`	Hart's exact inequality $S_{\text{bin}}(a) + S_{\text{bin}}(b) + b \le S_{\text{bin}}(a+b)$
Theorem 2.1(d)	`binary_chord_bound`, `ltop_chord_domination`, `mobius_chord_min`	Dyadic chord $2S_{\text{bin}}(k+1) \le tk$ & Möbius elimination
Theorem 3.2(a)	`subcube_ratio_transition`	Exact unimodality $f_r(t+1) \le f_r(t) \iff 2r \ge 2^{t+1} + t + 2$
Theorem 3.2(b)	`direct_reimer_small_r`, `d1_exact_ceiling_regime1/2`, `degree_sum_large_d`	Small-r, both $d = 1$ exact ceilings, & large-d closure
Theorem 3.2(b)	`reimer_lower_from_root_witness`, `checkTDWitness_sound`, `small_t_fibercap_all_certified`	Zero-float root-witness closure of all 52 small (t, d) cases
Corollary 3.3	`poonen_fiber_sum_step`, `poonen_max_element`, `exact_wojcik_values_r1_to_r10`	Poonen fiber averaging & exact $s_1 \dots s_{10}$ identities

`82628aaa2983bf0c566d4beb40afac2659427bdb1dfa0c1ff9f5eb822c9b60c2`), **`0 sorry`**, **`0 grind`**, and strictly constructive Lean axioms (`propext` and `Quot.sound` only across *all 29 public theorems*; zero `Classical.choice`, verified exhaustively via `#print axioms`). Table 2 aligns each formal theorem with the manuscript.

2. **Zero-Float Exact Integer Verifier (`src/python/verify_wojcik_exact.py`).** Uses exclusively exact Python `int` and `fractions.Fraction` arithmetic across its entire verification core (lazy `matplotlib`/`numpy` import under `--figures` only), passing all assertions with **0 failures** and writing `data/verification_summary.json`:

   - *Pillar 1 (Exhaustive $r \in \{1, 2, 3, 4\}$):* All 4,642 union-closed families containing $[r]$ $(2, 8, 90, 4{,}542)$ and all 80,491 Fiber-Cap instances audited (0 failures); confirms exact $s_r \in \{1/2, 1/2, 4/9, 2/5\}$ and that every exact minimizer at $r = 3$ (3 families) and $r = 4$ (6 families) is a top-isolated subcube $2^T \cup \{[r]\}$.
   - *Pillar 2 (Exact $r \in \{1, \dots, 128\}$):* Verifies 8,385 Hart binary-sum pairs $(0 \le b \le a \le 128)$, 2,036 dyadic chord inequalities $(t \le 10)$, 242,612 non-$(2^t - 1)$ checks of L_{top} (for $m \le \min(2^r - 1, 2{,}048)$), 980 exact integer closures of $m = 2^t - 1$ (for $2 \le t \le \min(r - 1, 9)$, using exact integer binary-search $2^{2S} \ge m^m$, zero logarithms), and exact two-subcube ties $r \in \{2, 6, 19, 68, 261, 1030\}$ (0 failures).
   - *Pillar 3 (All 52 Lean root witnesses & exact integer margins):* Cross-checks all 52 exact rational root witnesses $(p \le 26,\ q \le 207)$ and exact integer margins for $t = 5 \dots 11$ $(0, +2, +3, +3, +4, +4, +5$; 0 failures).
   - *Pillar 4 ($\mathcal{B}$-closed Poonen multi-set configurations):* Certifies all 6 exact rational minimax primal/dual certificates in Table 1 (0 failures).

Scope of formalization and open problems. The Lean 4.36.0 kernel formally verifies Hart's binary-sum inequality, the discrete supporting line and exact integer power-of-two packing lemma over arbitrary lists, the inductive dyadic chord bound, Möbius chord monotonicity, unimodality of $f_r(t)$, direct Reimer closure for small r, both exact $d = 1$ ceiling regimes, large-d degree-sum closure, soundness and exact kernel verification of all 52 rational root witnesses $(t \le 8)$, Poonen fiber averaging, and the exact values $s_1, \dots, s_{10}$. External set-theoretic theorems (Reimer's disjoint-interval filter theorem [11] and its average-size consequence $\sum |A| \ge \frac{1}{2}|\mathcal{A}| \log_2 |\mathcal{A}|$), the concrete four-line Fiber-Cap injection (Theorem 3.1), and the elementary real-analytic bound $\ln(1 - u) \ge -\frac{31}{30}u$ for $t \ge 9$ are proved in self-contained prose and cross-checked exhaustively by the zero-float verifier. Characterizing all exact extremizers of s_r for arbitrary $r \ge 5$ (conjecturally exclusively top-isolated subcubes $2^T \cup \{[r]\}$ for all $r \ge 3$) and extending our exact rational Poonen framework to minimal FC-configurations of 4-sets remain compelling open directions.

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