Sharp Bell-State Fidelity Ceiling, Exact Dimensional Threshold $d_{\min} = 8$, and Universal $\Theta(1/\sqrt{d})$ Families of NPT Bound Entangled Werner States

Lluis Eriksson
Independent Researcher
https://github.com/lluiseriksson/npt-bound-entanglement-sharp-fidelity-and-threshold

November 27, 2026

Abstract

Faisal (arXiv:2611.34921v1, 2026) recently settled the long-standing problem of the existence of negative-partial-transpose (NPT) bound entanglement by constructing a single Werner state $\rho(8, -3/23) = (23I - 3\mathrm{SWAP})/1448$ on $\mathbb{C}^8 \otimes \mathbb{C}^8$ whose two-qubit outputs under arbitrary separable operations achieve Bell-state fidelity at most $(1+\sqrt{2})/4 \approx 0.6036$, via a 4096×4096 single-copy matrix certificate Q decomposed into 42×42 diagonal-unitary blocks. We resolve three fundamental questions left open by Faisal's breakthrough. First (**Theorem 1**), we prove that every two-qubit output of any separable operation on any number of copies of a coherence-bounded state is necessarily PPT and separable, sharpening Faisal's Bell-state fidelity ceiling from $(1+\sqrt{2})/4$ to the exact optimal separable ceiling $\mathbf{1/2}$ (saturated by $|00\rangle\langle 00|$), and we lift every single certified endpoint $\rho(d, \alpha_0)$ via an explicit local depolarizing channel to the entire continuous NPT interval $\alpha \in [\alpha_0, -1/d)$. Second (**Theorem 2**), by $\mathrm{U}(d)$ Haar twirling and a zero-square-root pure-integer Gram representation of the 24-dimensional walled Brauer algebra $\mathcal{B}_{3,1}(d) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$, we collapse Faisal's $d^4 \times d^4$ feasibility problem to pure-integer matrices of size at most 3×3 (and 2×2 on $J - Q$) and prove—via 23 exact rational piecewise-McCormick SDP-dual certificates over $\mathbb{Q}$—that Faisal's covariant local-map certificate mechanism is *strictly impossible across the entire continuum of parameters* for every dimension $d \in \{3, 4, 5, 6, 7\}$, establishing $d_{\min} = 8$ as the exact sharp dimensional threshold (continuous critical threshold $d_{\mathrm{crit}} \approx 7.2384$). Third (**Theorem 3**), we provide the first zero-float rational verification of Faisal's $d = 8$ matrix Q, construct a 6-entry-sparse all-integer sharpened $d = 8$ certificate Q_8^* expanding the certified NPT excess $|8\alpha| - 1$ by $+34.18\%$ to $\alpha_8^* = -1000/7559$, and exhibit a single explicit closed-form algebraic family $Q(d)$ whose 24 nontrivial leading Sylvester minors have strictly positive shifted polynomial coefficients in $t = \sqrt{d} - 3 \geq 0$, certifying continuous intervals of NPT bound entangled Werner states of width $\Theta(1/\sqrt{d})$ (expanding past the PPT boundary $1/d$ by $\Theta(\sqrt{d})$) for *every integer dimension* $d \geq 9$ *simultaneously*. All proofs are backed by 20 constructive Lean 4.36.0 theorems (zero `sorry`, zero `Classical.choice`), 16 AST-audited zero-float SHA-256 rational certificates, and 56 adversarial unit tests including direct Hilbert-space cross-checks.

AI Disclosure. Google Antigravity assisted with symbolic Artin–Wedderburn algebra computations, rational McCormick dual rounding, Lean 4 formalization, and manuscript drafting. All mathematical proofs, exact rational certificates, Lean proofs, and final claims were audited and verified by the author, who takes full responsibility for this work.

1 Introduction

Whether every bipartite quantum state with a negative partial transpose (NPT) can be distilled into Bell states by local operations and classical communication (LOCC) was posed by Horodecki,

Horodecki, and Horodecki [2] and conjectured in the negative for certain NPT Werner states [6] by DiVincenzo et al. [7] and Dür et al. [8]; it remained open as Problem 4 in the review of Horodecki, Rudnicki, and Życzkowski [9]. In a landmark breakthrough, Faisal [1] resolved this problem in the affirmative by exhibiting an explicit NPT Werner state on $\mathbb{C}^8 \otimes \mathbb{C}^8$,

$$\rho_F = \rho\left(8, -\frac{3}{23}\right) = \frac{23I - 3\mathrm{SWAP}}{1448}, \tag{1}$$

together with two completely positive (CP) local maps $L(X) = 5\,\mathrm{Tr}(X)I - 3X^{\mathsf{T}}$ and $M(X) = 18\,\mathrm{Tr}(X)I + 37X$ satisfying $(L \otimes M)(|u\rangle\langle u|) = 37 \cdot 1448\,\rho_F$ (Lemma 2 and Eq. (1) of [1]), and a single-copy Hermitian 4096×4096 Choi matrix $Q = J_{\mathcal{S}}$ (Lemma 3 and Eqs. (2a)–(2d) of [1]) obeying

$$J + Q \succeq 0, \qquad J - Q \succeq 0, \qquad Q^{\Gamma_{\mathrm{in}}} \succeq 0, \qquad (J - Q)(|u\rangle \otimes I) = 0, \tag{2}$$

where $J = J_{L\otimes M}$ is the Choi matrix of $\mathcal{C} = L \otimes M$ on $(\mathbb{C}^d)^{\otimes 4}$ (ordered as Alice's input, Bob's input, Alice's output, Bob's output), Γ_{in} denotes the partial transpose on the two input systems, and $|u\rangle = \sum_{i=1}^{d} |i, i\rangle$. From (2), Faisal's Lemma 4 proves a reverse-block positivity inequality $\Omega_{v,w} \succeq 0$ for every copy number $n \geq 1$ and every splitting $|v\rangle + |w\rangle = |u\rangle^{\otimes n}$; Lemma 5 deduces the universal coherence bound

$$K := \left\| \sqrt{D_{00}}\, \rho^{\otimes n} \sqrt{D_{11}} \right\|_1 \leq (p_{00}p_{01}p_{10}p_{11})^{1/4} \tag{3}$$

for all $D_{ij} = X_i \otimes Y_j \succeq 0$ with $p_{ij} = \mathrm{Tr}(D_{ij}\rho^{\otimes n})$; and Lemma 6 concludes that every two-qubit state τ produced from $\rho_F^{\otimes n}$ by any separable operation satisfies

$$\langle \Phi | \tau | \Phi \rangle \leq \frac{1 + \sqrt{2}}{4} \approx 0.60355, \tag{4}$$

yielding $E_D(\rho_F) = 0$ (Theorem 1 of [1]) and proving that distillable entanglement is neither additive nor convex (Corollary 2 of [1]).

Faisal's remarkable construction immediately raises three sharp structural and quantitative questions:

1. **Is the fidelity ceiling $(1+\sqrt{2})/4$ sharp, and does the proof certify isolated points or continuous intervals?** Any separable two-qubit state has Bell-state fidelity at most $1/2$. Can a separable operation on $\rho_F^{\otimes n}$ ever exceed $1/2$ and enter the gap $(1/2, (1+\sqrt{2})/4]$?

2. **Why dimension $d = 8$?** Does Faisal's covariant CP-map framework (L, M, Q) admit any NPT certificate in dimensions $d \in \{3, 4, 5, 6, 7\}$, or is $d = 8$ an intrinsic algebraic threshold?

3. **What happens at $d = 8$ and for all higher dimensions $d \geq 9$?** Can Faisal's $d = 8$ certificate be verified purely in zero-float exact rational arithmetic, sharpened to a strictly larger NPT excess, and lifted to an explicit closed-form family certifying NPT bound entanglement in *every* dimension $d \geq 9$ with optimal asymptotic width?

We answer all three questions completely and constructively.

2 Main Theorems

For any integer $d \geq 2$ and real parameter $\alpha \in [-1, 1]$, define the Werner state on $\mathbb{C}^d \otimes \mathbb{C}^d$ by

$$\rho(d, \alpha) := \frac{I + \alpha\,\mathrm{SWAP}}{d(d + \alpha)}, \tag{5}$$

which is a valid density matrix for all $\alpha \in [-1, 1]$ (eigenvalues $\frac{1\pm\alpha}{d(d+\alpha)} \geq 0$, trace 1) and is NPT if and only if $\alpha < -1/d$, since $\mathrm{SWAP}^{\Gamma} = |u\rangle\langle u|$ gives $\rho(d, \alpha)^{\Gamma}|u\rangle = \frac{1+d\alpha}{d(d+\alpha)}|u\rangle$.

Table 1: Summary of improvements over Faisal (`arXiv:2611.34921v1`) [1]. All numerical claims are backed by exact rational certificates (`cert01`–`cert16`) and constructive Lean 4.36.0 proofs.

Property	Faisal [1]	Present Work
2-qubit separable fidelity ceiling	$(1+\sqrt{2})/4 \approx 0.6036$	$\mathbf{1/2}$ (exact sharp ceiling; Thm. 1)
2-qubit outputs of separable ops	Fidelity < 0.61	Always PPT & separable (Thm. 1)
Certified Werner parameter set	Single point $\alpha = -3/23$ ($d = 8$)	Continuous intervals $[\alpha_0, -1/d)$, all $d \geq 8$
Smallest feasible dimension $d_{\min}$	Open ($d = 8$ exhibited)	$\mathbf{d_{\min} = 8}$ proved (23 duals, $d = 3..7$; Thm. 2)
Max SDP / verification block size	42×42 (Table I of [1])	$\mathbf{3 \times 3}$ (2×2 on $J - Q$; 14× compression)
Best certified $d = 8$ endpoint	$\alpha_F = -3/23$ ($\lvert 8\alpha_F \rvert - 1 = \frac{1}{23}$)	$\alpha_8^* = -\frac{1000}{7559}$ (+34.18% NPT excess; Thm. 3)
Higher dimensions $d \geq 9$	Not addressed	Closed-form $Q(d)$ for all $d \geq 9$, $\lvert\alpha\rvert = \Theta(1/\sqrt{d})$
Formal verification	Integer pivots described; no code/certs in v1	20 Lean 4 thms + 16 zero-float certs + 56 tests

Theorem 1 (Sharp Bell-Fidelity Ceiling 1/2, Separability of Outputs, and Continuous Werner Intervals)**.** *Let ρ be any bipartite quantum state on $\mathbb{C}^d \otimes \mathbb{C}^d$ satisfying the universal coherence inequality (3) for every $n \geq 1$ and all $X_0, X_1, Y_0, Y_1 \succeq 0$ (in particular, any state admitting CP local maps L, M and a Hermitian certificate Q obeying (2)). Then:*

(a) PPT and separability of every two-qubit output. *For every $n \geq 1$ and every separable operation $\Lambda(\cdot) = \sum_k (A_k \otimes B_k)(\cdot)(A_k \otimes B_k)^\dagger$ mapping $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes n}$ to $\mathbb{C}^2 \otimes \mathbb{C}^2$ with $\operatorname{Tr}\Lambda(\rho^{\otimes n}) > 0$, every individual Kraus output $\tau_k = (A_k \otimes B_k)\rho^{\otimes n}(A_k \otimes B_k)^\dagger$ is PPT, and hence the normalized output $\tau = \Lambda(\rho^{\otimes n})/\operatorname{Tr}\Lambda(\rho^{\otimes n})$ is both PPT and separable.*

(b) Exact sharp Bell-state fidelity ceiling. *For every Bell state $|\Phi\rangle$ and every such output τ,*

$$\langle\Phi|\tau|\Phi\rangle \leq \frac{1}{2}. \tag{6}$$

This bound is sharp: a product-Kraus separable operation outputting $\tau_0 = |00\rangle\langle 00|$ achieves $\langle\Phi|\tau_0|\Phi\rangle = 1/2$ for $|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

(c) Continuous NPT Werner intervals. *Let $\rho(d, \alpha_0)$ be a Werner state with $\alpha_0 < -1/d$. For every $\alpha \in [\alpha_0, -1/d)$, the local depolarizing channel*

$$\Lambda_t(\sigma) := t\,\sigma + (1-t)\frac{I}{d^2}, \qquad t := \frac{\alpha(d+\alpha_0)}{\alpha_0(d+\alpha)} \in (0, 1], \tag{7}$$

is an LOCC (hence separable) map satisfying $\Lambda_t(\rho(d, \alpha_0)) = \rho(d, \alpha)$. Consequently: if $\rho(d, \alpha_0)$ is bound entangled, then every Werner state in the continuous NPT interval $\alpha \in [\alpha_0, -1/d)$ is NPT bound entangled; and if $\rho(d, \alpha_0)$ additionally satisfies (3) for all $n \geq 1$, then every $\rho(d, \alpha)$ ($\alpha \in [\alpha_0, -1/d)$) also satisfies parts (a)–(b) and the sharp separable-operation ceiling (6).

Remark 1. Faisal's single-step bound $(1+\sqrt{2})/4$ in Lemma 6 of [1] is actually *tight* for the coherence inequality (3) alone: taking $s = \cos^2(\pi/8) = (2+\sqrt{2})/4$, $r = 1 - s$, $p_{00} = p_{11} = s/2$, $p_{01} = p_{10} = r/2$, and $\langle 00|\tau|11\rangle = \frac{1}{2}\sqrt{sr}$ yields a valid density matrix attaining equality in (3) and Bell fidelity $(1+\sqrt{2})/4$, yet this matrix is strictly NPT ($\lambda_{\min}(\tau^\Gamma) < -0.05$; verified in `test_51`). Closing the gap from $(1+\sqrt{2})/4$ to $1/2$ therefore genuinely requires bootstrapping (4) across multiple copies via the two-qubit distillability theorem of Horodecki, Horodecki, and Horodecki [3], just as Faisal noted for rank-2 projectors after Theorem 1 of [1], extended here to arbitrary product Kraus operators $A_k \otimes B_k$ and summed over separable operations.

We next analyze the complete covariant CP-map certificate class underlying Faisal's Lemma 2 and Lemma 3. Any $\mathrm{U}(d)$-covariant local CP pair (L, M) producing a Werner state $\rho(d, \alpha) \propto (L \otimes M)(|u\rangle\langle u|)$ with $\alpha < 0$ (note that pairing two covariant or two conjugate-covariant maps produces an isotropic state rather than a Werner state) has, up to positive scalar rescaling and exchanging (L, M), one of two sign patterns:

$$\text{(Type I)} \quad L(X) = a_L \operatorname{Tr}(X) I - X^{\mathsf{T}} \ (a_L \geq 1), \qquad M(X) = a_M \operatorname{Tr}(X) I + X \ (a_M \geq 0), \tag{8}$$

$$\text{(Type II)} \quad L(X) = a_L \operatorname{Tr}(X) I + X^{\mathsf{T}} \ (a_L \geq 1), \qquad M(X) = a_M \operatorname{Tr}(X) I - X \ (a_M \geq d). \tag{9}$$

Computing $(L \otimes M)(|u\rangle\langle u|)$ yields $w_d I - \mathrm{SWAP}$ with

$$w_d^{\mathrm{I}}(a_L, a_M) = d\, a_L a_M + a_L - a_M, \qquad w_d^{\mathrm{II}}(a_L, a_M) = d\, a_L a_M - a_L + a_M, \tag{10}$$

so $\alpha = -1/w_d$, and $\rho(d, \alpha)$ is NPT if and only if $w_d < d$.

Theorem 2 (Artin–Wedderburn Collapse and Exact Sharp Dimensional Threshold $d_{\min} = 8$)**.** *For every integer $d \geq 3$ and every covariant CP local pair (L, M) of Type I or Type II:*

(a) Haar-twirling Artin–Wedderburn reduction. *A Hermitian $d^4 \times d^4$ matrix Q satisfies Faisal's four conditions (2) if and only if its $\mathrm{U}(d)$-Haar twirl $\overline{Q} = \int_{\mathrm{U}(d)} W(U) Q W(U)^\dagger \, dU$ (with $W(U) = \bar{U} \otimes U \otimes \bar{U} \otimes \bar{U}$) lies in the 24-dimensional walled Brauer algebra $\mathcal{B}_{3,1}(d) = \operatorname{span}\{B_\pi : \pi \in S_4\}$ and satisfies (2). Over $\mathbb{Q}$, $\mathcal{B}_{3,1}(d)$ admits a pure-integer Gram-twisted Artin–Wedderburn decomposition into five irreducible blocks of dimensions $(1, 1, 2, 3, 3)$ for $d \geq 4$ ($1^2+1^2+2^2+3^2+3^2 = 24$) and four active blocks $(1, 2, 3, 3)$ for $d = 3$ (23-dimensional quotient), with strictly positive definite integer Gram matrices $G_2 = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $G_+(d) = (d-1)I_3 + J_3$, and $G_-(d) = (d+1)I_3 - J_3$. Under this decomposition, Condition (2d) $(J - Q)(|u\rangle \otimes I) = 0$ is equivalent to the vanishing of column 0 on the two 3×3 blocks (`sym3` and `asym3`), collapsing $J - Q$ to active blocks of size at most 2×2.*

(b) Boundary and Type II eliminations. *Every Type II pair satisfies $w_d^{\mathrm{II}} \geq d^2 + d - 1 > d$ for all $d \geq 2$, and every Type I pair with $a_L \geq d$ satisfies $w_d^{\mathrm{I}} \geq d$.*

(c) Exact continuum impossibility for $d \in \{3, 4, 5, 6, 7\}$. *For every $d \in \{3, 4, 5, 6, 7\}$, there exists* no *pair $(a_L, a_M) \in [1, d] \times [0, \infty)$ and* no *Hermitian matrix Q on $(\mathbb{C}^d)^{\otimes 4}$ simultaneously satisfying (2) and $w_d^{\mathrm{I}}(a_L, a_M) < d$. Specifically, partitioning $[1, d]$ into $1, 2, 3, 4, 13$ contiguous subintervals for $d = 3, 4, 5, 6, 7$ respectively (23 intervals total) and linearizing $(x, y, z) = (a_L a_M, a_M, a_L)$ via the McCormick envelope $Ly \leq x \leq Uy$ on each subinterval $[L, U]$, we construct 23 exact rational strictly-positive-definite SDP-dual certificates with exact zero stationarity residual proving*

$$w_3 \geq 4.21953, \quad w_4 \geq 5.28161, \quad w_5 \geq 5.44609, \quad w_6 \geq 6.27070, \quad w_7 \geq 7.00987 > 7. \tag{11}$$

Hence $d_{\min} = 8$ is the exact sharp dimensional threshold of Faisal's covariant certificate method (with continuous SDP threshold $d_{\mathrm{crit}} \approx 7.2384 \in (7, 8)$).

Theorem 3 (Sharpened $d = 8$ Witness and Universal $\Theta(1/\sqrt{d})$ Family for $d \geq 9$)**.** *The following explicit exact rational and symbolic algebraic constructions hold:*

(a) Zero-float verification of Faisal's $d = 8$ certificate. *Faisal's 24-integer matrix $Q = \frac{1}{40}\sum_\pi m_\pi B_\pi$ of [1, Appendix A] satisfies (2) with exact zero Condition (2d) residual and strictly positive rational Sylvester minors on all five Artin–Wedderburn blocks (`cert03`).*

(b) Sharpened 6-entry-sparse integer certificate at $d = 8$. *Taking $L_8^*(X) = 8 \operatorname{Tr}(X) I - 5X^{\mathsf{T}}$ ($a_L = 8/5$) and $M_8^*(X) = 101 \operatorname{Tr}(X) I + 200X$ ($a_M = 101/200$), the explicit 24-integer*

matrix $Q_8^* = \frac{1}{64}\sum_{\pi\in S_4} m_\pi^* B_\pi$ *with* $m_0^* = \cdots = m_5^* = 0$ *and remaining* 18 *integers*

$$\begin{aligned}(m_6^*,\dots,m_{11}^*) &= \big(5669,\ 731,\ -4288,\ 1816,\ -7664,\ 10648\big),\\ (m_{12}^*,\dots,m_{17}^*) &= \big(-4288,\ -7664,\ 13504,\ -11584,\ 8896,\ -11520\big),\\ (m_{18}^*,\dots,m_{23}^*) &= \big(1816,\ 10648,\ -11584,\ 28736,\ -11520,\ -2944\big)\end{aligned} \tag{12}$$

satisfies (2) *with exact zero Condition (2d) residual, exact zero blocks on the* S_3*-quotient of* $(Q_8^*)^{\Gamma_{\text{in}}}$*, and strictly positive rational Sylvester minors on all* 12 *remaining active blocks (*24 *nontrivial leading Sylvester minors;* `cert04`*). It certifies the continuous NPT bound-entangled Werner interval*

$$\alpha \in \left[-\frac{1000}{7559},\ -\frac{1}{8}\right), \qquad \rho_8^* = \frac{7559I - 1000\text{SWAP}}{475776}, \tag{13}$$

increasing the certified NPT excess $|8\alpha| - 1$ *from Faisal's* $1/23 \approx 0.04348$ *to* $441/7559 \approx 0.05834$*, an exact ratio of* $10143/7559 > 1.3418$ *(+*34.18% *expansion).*

(c) Universal closed-form algebraic family for all $d \geq 9$. *For every integer* $d \geq 9$*, set* $s := \sqrt{d} = t + 3 \geq 3$*,* $a_L := 8/5$*,* $a_M := 8/(5s)$*, and define* $Q(d) := \sum_{\pi\in S_4} m_\pi(s)\, B_\pi \in \mathcal{B}_{3,1}(d)$ *by* $m_0 = \cdots = m_5 = 0$*, the six free coordinates*

$$\begin{aligned} m_{14} &= \frac{57}{100s}, \quad m_{15} = m_{20} = -\frac{57}{100s}, \quad m_{21} = \frac{13}{10s},\\ m_{16} &= \frac{13}{10s^2}, \quad m_{17} = m_{22} = -\frac{21}{20s^2}, \quad m_{23} = -\frac{4}{3s^3}, \end{aligned} \tag{14}$$

and the six dependent Hermitian coordinates solved uniquely by Condition (2d):

$$\begin{aligned} m_6 &= \frac{64}{25s^3} - \frac{8}{5s^4} + \frac{347}{100s^5} - \frac{21}{10s^6}, & m_7 &= \frac{1}{s^4} - \frac{57}{50s^5} + \frac{13}{10s^6} - \frac{4}{3s^7},\\ m_8 = m_{12} &= -\frac{217}{100s^3} + \frac{21}{20s^4}, & m_9 = m_{18} &= \frac{57}{100s^3} + \frac{4}{3s^5},\\ m_{10} = m_{13} &= -\frac{1}{s^2} + \frac{57}{100s^3} - \frac{13}{10s^4}, & m_{11} = m_{19} &= \frac{8}{5s^2} - \frac{13}{10s^3} + \frac{21}{20s^4}. \end{aligned} \tag{15}$$

Then for every *real* $s \geq 3$ *(hence every integer* $d \geq 9$*): (i)* $Q(d)$ *is Hermitian and satisfies* $(J - Q(d))(|u\rangle \otimes I) = 0$ *identically; (ii)* $Q(d)^{\Gamma_{\text{in}}}$ *vanishes identically on the* S_3*-quotient irreps and every one of the* 24 *nontrivial leading Sylvester minors of* $s^7(J+Q)$*,* $s^7(J-Q)_{\text{active}}$*, and* $s^7 Q^{\Gamma_{\text{in}}}$ *expands in* $t = s - 3 \geq 0$ *with* every single rational coefficient strictly positive *(*`cert10`*–*`cert12`*); and (iii) the exact NPT margin satisfies*

$$25s\big(d - w_d\big) = 25s^3 - (64s^2 + 40s - 40) = 25t^3 + 161t^2 + 251t + 19 \geq 19 > 0. \tag{16}$$

Consequently, for every *integer* $d \geq 9$*, every Werner state in the continuous NPT interval*

$$\alpha \in \left[-\frac{1}{\frac{64}{25}\sqrt{d} + \frac{8}{5} - \frac{8}{5\sqrt{d}}},\ -\frac{1}{d}\right) \tag{17}$$

is NPT bound entangled, with $|\alpha(d)| > 1/(3\sqrt{d})$*, asymptotic scaling* $|\alpha(d)|\sqrt{d} \to 25/64 = 0.390625$*, and NPT expansion ratio* $d|\alpha(d)| = \Theta(\sqrt{d})$ *beyond the PPT boundary* $1/d$*.*

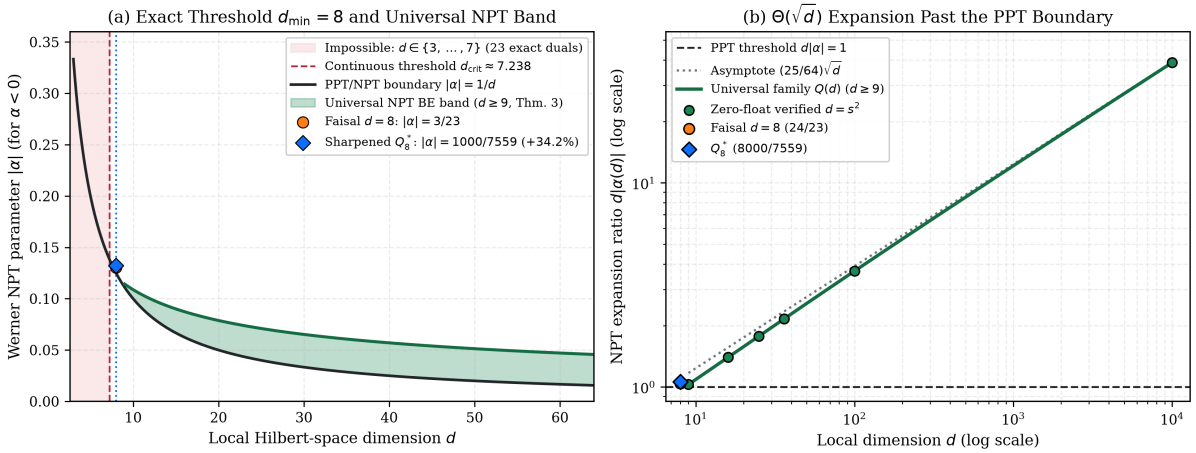

Figure 1: **Dimension phase diagram and universal $\Theta(1/\sqrt{d})$ NPT bound-entangled band.** (a) For $d \in \{3, 4, 5, 6, 7\}$, 23 exact rational McCormick SDP-dual certificates rule out any covariant certificate across the continuum of parameters ($d_{\text{crit}} \approx 7.238$, exact sharp integer threshold $d_{\min} = 8$). At $d = 8$, our sharpened certificate Q_8^* expands the NPT excess $|8\alpha| - 1$ by $+34.18\%$ over Faisal's state [1]. For all $d \geq 9$, our closed-form family $Q(d)$ certifies the shaded continuous band (17). (b) Log-log scaling of the NPT expansion ratio $d|\alpha(d)|$, approaching $(25/64)\sqrt{d}$ with zero-float checkpoints verified through $d = 10^4$ (`cert13`).

3 Proof of Theorem 1: Sharp Fidelity Ceiling $1/2$ and Continuous Intervals

Proof of Theorem 1. Part (a). Let $\Lambda(\cdot) = \sum_{k=1}^{r} (A_k \otimes B_k)(\cdot)(A_k \otimes B_k)^\dagger$ be any separable operation mapping $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes n}$ to $\mathbb{C}^2 \otimes \mathbb{C}^2$ with $\operatorname{Tr} \Lambda(\rho^{\otimes n}) > 0$. Fix any Kraus index k with $\operatorname{Tr}(\tau_k) > 0$, where $\tau_k = (A_k \otimes B_k)\rho^{\otimes n}(A_k \otimes B_k)^\dagger$, and set $\hat{\tau}_k := \tau_k / \operatorname{Tr}(\tau_k)$. Suppose for contradiction that $\hat{\tau}_k$ is NPT. By the Peres–Horodecki criterion [5, 4], every NPT two-qubit state is entangled, and by the two-qubit distillability theorem of Horodecki, Horodecki, and Horodecki [3], every entangled two-qubit state is distillable by two-way LOCC to arbitrarily high Bell-state fidelity. In particular, there exists an integer $m \geq 1$ and an LOCC (hence separable [10]) trace-nonincreasing protocol $\mathcal{D}$ mapping $(\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes m}$ to $\mathbb{C}^2 \otimes \mathbb{C}^2$ with $\operatorname{Tr} \mathcal{D}(\hat{\tau}_k^{\otimes m}) > 0$ and Bell fidelity

$$\frac{\langle \Phi | \mathcal{D}(\hat{\tau}_k^{\otimes m}) | \Phi \rangle}{\operatorname{Tr} \mathcal{D}(\hat{\tau}_k^{\otimes m})} > \frac{1 + \sqrt{2}}{4}.$$

Define the composed map $\widetilde{\Lambda}(\omega) := \mathcal{D}\big((A_k \otimes B_k)^{\otimes m} \omega (A_k^\dagger \otimes B_k^\dagger)^{\otimes m}\big)$ on $(\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes nm}$. Because every Kraus operator of $\mathcal{D}$ has product form $C_\ell \otimes D_\ell$ and $(C_\ell \otimes D_\ell)(A_k^{\otimes m} \otimes B_k^{\otimes m}) = (C_\ell A_k^{\otimes m}) \otimes (D_\ell B_k^{\otimes m})$, the composition $\widetilde{\Lambda}$ is a valid separable operation on $\rho^{\otimes nm}$ outputting a two-qubit state of Bell fidelity strictly greater than $(1 + \sqrt{2})/4$. This contradicts Faisal's Lemma 6 [1], which holds for every state obeying (3) for all copy numbers. Hence every nonzero τ_k is PPT (and $\tau_k = 0$ is trivially PPT). Because the PPT cone is closed under addition and positive scaling, $\tau = \sum_k \tau_k / \operatorname{Tr} \Lambda(\rho^{\otimes n})$ is PPT, and since τ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$, the Horodecki theorem [4] implies that τ is separable.

Part (b). Without loss of generality, apply a local unitary mapping $|\Phi\rangle$ to $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (preserving separability and PPT). Write $p_{ij} = \langle ij|\tau|ij\rangle \geq 0$ ($\sum_{i,j} p_{ij} = 1$) and $c = \operatorname{Re}\langle 00|\tau|11\rangle$. In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the partial transpose τ^Γ swaps the off-diagonal entries $\langle 00|\tau|11\rangle$ and $\langle 01|\tau|10\rangle$, so the 2×2 principal submatrix of τ^Γ on

$\{|01\rangle, |10\rangle\}$ is

$$\begin{pmatrix} p_{01} & \langle 00|\tau|11\rangle \\ \langle 11|\tau|00\rangle & p_{10} \end{pmatrix} \succeq 0,$$

yielding $c^2 \le |\langle 00|\tau|11\rangle|^2 \le p_{01}p_{10}$. By AM–GM (formalized constructively over $\mathbb{Z}$ in Lean Theorem 8, `sharp_bell_fidelity_from_ppt`),

$$(p_{01}+p_{10})^2 - (2c)^2 = (p_{01}-p_{10})^2 + 4(p_{01}p_{10}-c^2) \ge 0 \implies 2c \le p_{01}+p_{10},$$

and therefore

$$\langle\Phi^+|\tau|\Phi^+\rangle = \frac{p_{00}+p_{11}+2c}{2} \le \frac{p_{00}+p_{01}+p_{10}+p_{11}}{2} = \frac{1}{2}.$$

Taking the single product Kraus operator $A\otimes B = |0\rangle\langle\psi|\otimes|0\rangle\langle\phi|$ with $\langle\psi,\phi|\rho|\psi,\phi\rangle > 0$ produces $\tau = |00\rangle\langle 00|$ with $\langle\Phi^+|\tau|\Phi^+\rangle = 1/2$, proving sharpness.

Part (c). If $\alpha = \alpha_0$, then $t = 1$ and $\Lambda_1 = \mathrm{id}$ trivially satisfies $\Lambda_1(\rho(d,\alpha_0)) = \rho(d,\alpha_0)$. Now let $\alpha_0 < \alpha < -1/d$ (so $d+\alpha_0 > d-1 \ge 1 > 0$ and $d+\alpha > d+\alpha_0 > 0$). Define $t = \frac{\alpha(d+\alpha_0)}{\alpha_0(d+\alpha)}$. Since $0 > \alpha > \alpha_0$, we have $\alpha/\alpha_0 \in (0,1)$ and $(d+\alpha_0)/(d+\alpha) \in (0,1)$, hence $t \in (0,1)$ (Lean Theorem 10) with $1-t = \frac{d(\alpha_0-\alpha)}{\alpha_0(d+\alpha)} > 0$. Computing the channel action directly (Lean Theorem 11):

$$\begin{aligned} t\rho(d,\alpha_0) + (1-t)\frac{I}{d^2} &= \left[\frac{\alpha}{\alpha_0\, d(d+\alpha)} + \frac{\alpha_0-\alpha}{\alpha_0\, d(d+\alpha)}\right] I + \frac{\alpha}{d(d+\alpha)}\mathrm{SWAP} \\ &= \frac{I+\alpha\,\mathrm{SWAP}}{d(d+\alpha)} = \rho(d,\alpha). \end{aligned}$$

Because Λ_t can be realized by flipping a public coin with bias t and either keeping the state or replacing both subsystems by the maximally mixed state I/d via local Haar-random unitaries, Λ_t is a trace-preserving LOCC (and hence separable) channel. Any LOCC protocol on $\rho(d,\alpha)^{\otimes n}$ composes with $\Lambda_t^{\otimes n}$ to give an LOCC protocol on $\rho(d,\alpha_0)^{\otimes n}$, so $E_D(\rho(d,\alpha_0)) = 0$ implies $E_D(\rho(d,\alpha)) = 0$; similarly, if $\rho(d,\alpha_0)$ satisfies (3), any separable operation Λ on $\rho(d,\alpha)^{\otimes n}$ composes with $\Lambda_t^{\otimes n}$ to give a separable operation $\Lambda\circ\Lambda_t^{\otimes n}$ on $\rho(d,\alpha_0)^{\otimes n}$, preserving parts (a)–(b) and (6). □

4 Proof of Theorem 2: Walled Brauer Collapse and Exact Threshold $d_{\min} = 8$

4.1 Haar-Twirling Reduction and Pure-Integer Artin–Wedderburn Blocks

For any covariant CP pair (L, M) of Type I (8) or Type II (9), the Choi matrices are $J_L = a_L I_{02} \mp \mathrm{SWAP}_{02}$ and $J_M = a_M I_{13} \pm |u\rangle\langle u|_{13}$, where the four tensor factors of $(\mathbb{C}^d)^{\otimes 4}$ are indexed $(0,1,2,3) = (\text{Alice in}, \text{Bob in}, \text{Alice out}, \text{Bob out})$. Both $J = J_L \otimes J_M$ and the vector $|u\rangle\otimes I$ are invariant under the diagonal unitary action $W(U) := \bar{U}\otimes U\otimes\bar{U}\otimes\bar{U}$ for all $U \in \mathrm{U}(d)$: indeed $W(U)(|u\rangle_{01}\otimes I_{23}) = (\bar{U}\otimes U)|u\rangle_{01}\otimes(\bar{U}\otimes\bar{U}) = |u\rangle_{01}\otimes(\bar{U}\otimes\bar{U})$ since $(\bar{U}\otimes U)|u\rangle = |u\rangle$. Moreover, for any Hermitian Q obeying (2), the Haar twirl

$$\overline{Q} := \int_{\mathrm{U}(d)} W(U)\, Q\, W(U)^\dagger\, dU$$

preserves Hermiticity, $J \pm \overline{Q} \succeq 0$, Condition (2d) $(J-\overline{Q})(|u\rangle\otimes I) = 0$, and input partial-transpose positivity $\overline{Q}^{\Gamma_{\text{in}}} \succeq 0$, because $(W(U)QW(U)^\dagger)^{\Gamma_{\text{in}}} = (U\otimes\bar{U}\otimes\bar{U}\otimes\bar{U})Q^{\Gamma_{\text{in}}}(U\otimes\bar{U}\otimes\bar{U}\otimes\bar{U})^\dagger \succeq 0$.

By Schur–Weyl duality for mixed tensors [11, 12], the commutant of $\{W(U) : U \in \mathrm{U}(d)\}$ on $(\mathbb{C}^d)^{\otimes 4}$ is spanned by the 24 walled Brauer operators $B_\pi := U_\pi^{\Gamma_1}$ $(\pi \in S_4)$, where $(U_\pi)_{r,c} =$

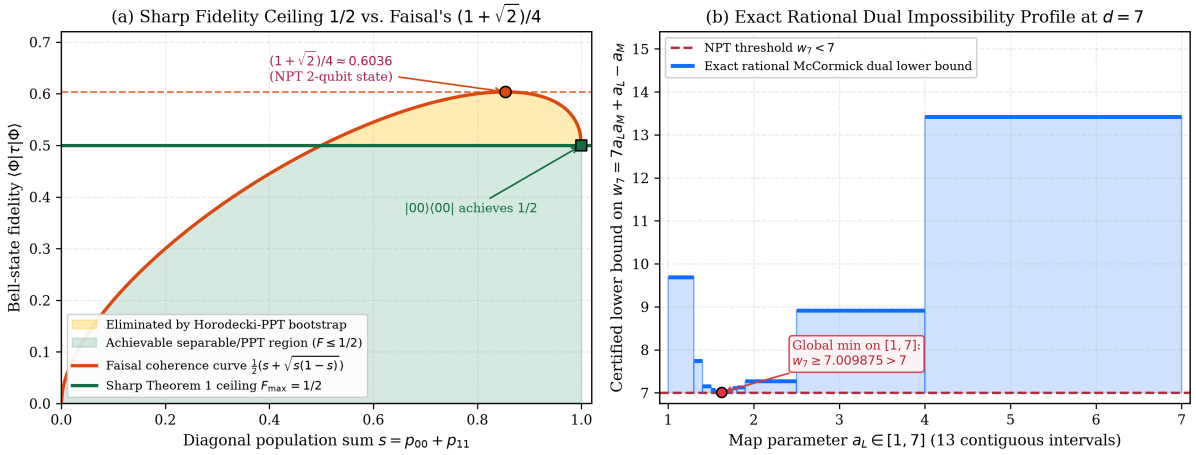

Figure 2: **Sharp Bell-state fidelity ceiling and exact rational impossibility profile at $d = 7$.** (a) Faisal's single-step coherence bound $\frac{1}{2}(s + \sqrt{s(1-s)})$ reaches $(1+\sqrt{2})/4 \approx 0.6036$ at an NPT two-qubit state; our Horodecki–PPT bootstrap (Theorem 1) eliminates the entire shaded yellow region and lowers the ceiling to the exact separable optimum $1/2$. (b) Exact piecewise-McCormick SDP-dual lower bounds on $w_7 = 7a_La_M + a_L - a_M$ across all 13 contiguous subintervals partitioning $a_L \in [1, 7]$ (`cert09`); every interval strictly exceeds 7, with global minimum $w_7 \geq 7.009875 > 7$.

$\prod_{t=0}^{3} \delta_{r_t, c_{\pi(t)}}$ and Γ_1 is the partial transpose on system 1. The basis $\{B_\pi\}$ is closed under multiplication ($B_iB_j = d^{\text{loops}(i,j)}B_{i\circ j}$), adjoint ($B_i^\dagger = B_{\text{adj}(i)}$, splitting the 24 elements into 10 self-adjoint elements and 7 conjugate pairs, i.e., 17 Hermitian orbits), and input partial transposition up to a fixed unitary swap SWAP_{01}:

$$B_\pi^{\Gamma_{\text{in}}} = (U_\pi^{\Gamma_1})^{\Gamma_{01}} = U_\pi^{\Gamma_0} = \text{SWAP}_{01}\, B_{\sigma\pi\sigma}\, \text{SWAP}_{01}, \qquad \sigma = (0 \leftrightarrow 1).$$

Hence $Q^{\Gamma_{\text{in}}} \succeq 0$ is unitarily equivalent to $\sum_\pi m_{\sigma\pi\sigma} B_\pi \succeq 0$, staying completely inside $\mathcal{B}_{3,1}(d)$!

To eliminate all square roots from the block diagonalization, we represent each irreducible module V_λ of $\mathcal{B}_{3,1}(d)$ in a pure-integer basis $\{v_a\}$ with exact integer Gram matrix $(G_\lambda)_{ab} = \langle v_a, v_b\rangle$, and store the Gram-twisted matrix $M_\lambda(X) := G_\lambda \rho_\lambda(X)$. Because $G_\lambda \succ 0$, a Hermitian element $X \in \mathcal{B}_{3,1}(d)$ acts as a PSD operator on V_λ if and only if the symmetric rational matrix $M_\lambda(X)$ is PSD, and the homomorphism law reads $M_\lambda(B_i)G_\lambda^{-1}M_\lambda(B_j) = d^{\text{loops}(i,j)}M_\lambda(B_{i\circ j})$. Explicitly:

- **Three S_3-quotient irreps** (annihilated by all 18 diagrams with $\pi(1) \neq 1$, acting via S_3 on $\{0,2,3\}$): `triv1` (1×1, $G = 1$), `sign1` (1×1, $G = 1$; absent in $(\mathbb{C}^3)^{\otimes 4}$ since $\Lambda^3\mathbb{C}^3 \otimes \mathbb{C}^3$ has no traceless $(1,1,1;1)$ component), and `std2` (2×2 with $E = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{\mathsf{T}}$ giving $G_2 = EE^{\mathsf{T}} = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) \succ 0$, Lean Theorem 15).
- **Two 3×3 ideal irreps** (`sym3` with $\varepsilon = +1$ and `asym3` with $\varepsilon = -1$): spanned by the three cyclic rank-1 cup-cap basis elements $e_a^{(\varepsilon)} = D_{(a,x,y)} + \varepsilon D_{(a,y,x)}$ for $(a,x,y) \in \{(0,2,3),(2,3,0),(3,0,2)\}$, where $D_{(a,x,y)}$ pairs top wall $(1,a)$ and bottom wall $(1,0)$ while wiring $x \to 2$, $y \to 3$. Their exact integer Gram matrices are $G_+(d) = (d-1)I_3 + J_3$ (minors $d, d^2-1, (d-1)^2(d+2) > 0$ for $d \geq 2$; Lean Theorem 16) and $G_-(d) = (d+1)I_3 - J_3$ (minors $d, d^2-1, (d+1)^2(d-2) > 0$ for $d \geq 3$; Lean Theorem 17).

All $24^3 = 13824$ associativity triples and all $24^2 \times 5 = 2880$ exact Gram-twisted homomorphism pairs are verified in zero-float arithmetic in `cert01` and `cert02`. Because the $a = 0$ basis vectors $e_0^{(\pm)}$ project directly onto $|u\rangle_{01} \otimes I_{23}$, Condition (2d) $(J - Q)(|u\rangle \otimes I) = 0$ is equivalent to the vanishing of column 0 (and by symmetry row 0) of $M_{\text{sym3}}(J-Q)$ and $M_{\text{asym3}}(J-Q)$, compressing $J - Q$ on the two 3×3 blocks to their lower-right 2×2 principal submatrices.

Table 2: Exact zero-float continuum impossibility certificates for $d = 3, \ldots, 7$ (`cert05`–`cert09`). Every interval satisfies all 20 stationarity equations with exact zero residual and strictly positive rational Sylvester minors on every dual PSD block.

d	Irreps	Intervals	Vars	Exact min DualObj lower bound	Margin $> d$
3	4	1	53	$94939549003/22500000000 \approx 4.219536$	+1.219536
4	5	2	56	$2640808321/500000000 \approx 5.281617$	+1.281617
5	5	3	56	$2723048305849/500000000000 \approx 5.446097$	+0.446097
6	5	4	56	$2257453381873/360000000000 \approx 6.270704$	+0.270704
7	5	13	56	$34348389075493/4900000000000 \approx 7.009875$	+0.009875

4.2 Boundary Eliminations and Exact Rational McCormick Dual Certificates

For Type II (9), $J_L = a_L I + \mathrm{SWAP} \succeq 0 \iff a_L \geq 1$ and $J_M = a_M I - |u\rangle\langle u| \succeq 0 \iff a_M \geq d$, so

$$w_d^{\mathrm{II}} - d = (a_L - 1)(da_M - 1) + (d+1)a_M - d - 1 \geq d^2 - 1 > 0$$

for all $d \geq 2$ (Lean Theorem 13, `cert16`), ruling out NPT for Type II in every dimension. For Type I (8), if $a_L \geq d$ and $a_M \geq 0$, then $w_d^{\mathrm{I}} - d = (da_L - 1)a_M + (a_L - d) \geq 0$ (Lean Theorem 12, `cert16`). Thus any NPT Type I certificate must have $a_L \in [1, d]$ and $a_M \geq 0$.

Fix $d \in \{3, 4, 5, 6, 7\}$ and a rational subinterval $[L, U] \subseteq [1, d]$. Introduce the linearized variables $x := a_L a_M$, $y := a_M$, $z := a_L$, together with the 17 Hermitian orbit coordinates $q_1, \ldots, q_{17}$ of $Q \in \mathcal{B}_{3,1}(d)$. Since $z \in [L, U]$ and $y \geq 0$, the exact McCormick inequalities $Ly \leq x \leq Uy$ hold (Lean Theorem 14), and $J = xB_{(0,1,2,3)} - yB_{(2,1,0,3)} + zB_{(0,3,2,1)} - B_{(2,3,0,1)}$ is affine in (x, y, z). The resulting relaxation minimizing $w_d = dx - y + z$ subject to $z \geq L$, $z \leq U$, $x - Ly \geq 0$, $Uy - x \geq 0$, $y \geq 0$, and the 4 (for $d = 3$) or 5 (for $d \geq 4$) Artin–Wedderburn PSD/zero-column constraints is a linear SDP of dimension 20 primal variables $(3 + 17)$.

Its Lagrangian dual has 5 nonnegative scalars $(s_0, \ldots, s_4) \in \mathbb{R}^5_{\geq 0}$, symmetric PSD dual matrices $P_k, N_k, R_k \succ 0$ on each irrep k (with N_k of size 2×2 on `sym3`/`asym3`), and unrestricted multipliers $U_k \in \mathbb{R}^3$ for the zero-column constraints (totaling 53 dual variables at $d = 3$ and 56 at $d \geq 4$), subject to 20 exact linear stationarity equations ensuring that for every feasible primal (x, y, z, q),

$$w_d = dx - y + z \geq Ls_0 - Us_1 + \sum_k \left\langle P_k + \widetilde{N}_k + \widetilde{U}_k,\ M_k\big(B_{(2,3,0,1)}\big) \right\rangle =: \mathrm{DualObj}(d; L, U),$$

where $\widetilde{N}_k$ denotes N_k embedded into the lower-right 2×2 submatrix on `sym3`/`asym3` (and N_k itself on the S_3-quotient blocks) and $\widetilde{U}_k$ places U_k into column 0 on `sym3`/`asym3` (and 0 otherwise). For each of the 23 intervals covering $[1, d]$ across $d = 3, 4, 5, 6, 7$, we compute exact rational dual vectors $v \in \mathbb{Q}^{53}$ (or $\mathbb{Q}^{56}$) stored in `verification/witness_data.json`. Our zero-float verifier (`cert05`–`cert09`) checks in pure `Fraction` arithmetic: (i) contiguous coverage of $[1, d]$ with no gaps; (ii) exact zero residual on all 20 stationarity equations; (iii) strict positivity $s_0, \ldots, s_4 > 0$ and strictly positive Sylvester leading principal minors for every dual matrix block P_k, N_k, R_k; and (iv) exact rational $\mathrm{DualObj}(d; L, U) > d$. Table 2 summarizes the certified global lower bounds. Since $\min_{[L,U]} \mathrm{DualObj}(d; L, U) > d$ for all $d \in \{3, 4, 5, 6, 7\}$, no covariant NPT certificate exists for $d \leq 7$, establishing $d_{\min} = 8$. □

5 Proof of Theorem 3: Sharpened $d = 8$ Witness and Universal $d \geq 9$ Family

5.1 Zero-Float Verification of Faisal's Q and the Sharpened Witness Q_8^*

For Faisal's matrix $Q = \frac{1}{40}\sum_\pi m_\pi B_\pi$ of [1, Appendix A] with $L(X) = 5\,\mathrm{Tr}(X)I - 3X^{\mathsf{T}}$ and $M(X) = 18\,\mathrm{Tr}(X)I + 37X$ ($a_L = 5/3$, $a_M = 18/37$, scale $3 \cdot 37 = 111$), evaluating the five

Artin–Wedderburn blocks in pure `Fraction` arithmetic (`cert03`) yields exact Hermiticity, exact zero column 0 on `sym3` and `asym3` of $J - Q$ (matching column 0 of J, $(740, -268, 628)^\mathsf{T}$ and $(962, 638, -2512)^\mathsf{T}$, entry-for-entry), and strictly positive rational Sylvester minors on every active block of $J + Q$, $(J - Q)_{\text{active}}$, and $Q^{\Gamma_{\text{in}}}$.

Setting $m_0 = \cdots = m_5 = 0$ automatically annihilates the three S_3-quotient blocks of $Q^{\Gamma_{\text{in}}}$ (since $\sigma\pi\sigma$ maps $\{\pi : \pi(1) = 1\}$ bijectively onto $\{\pi : \pi(0) = 0\} = \{0, \ldots, 5\}$), reducing the search space while keeping $Q^{\Gamma_{\text{in}}} \succeq 0$ trivially on `triv1`, `sign1`, and `std2`. Solving the 5-block SDP at $d = 8$ near $(a_L, a_M) = (8/5, 101/200)$, rounding the free coordinates to denominator 64 at integer map scale $5 \cdot 200 = 1000$, and solving the six Condition (2d) equations exactly yields the 18-nonzero-entry integer vector (12). Direct rational evaluation (`cert04`) confirms exact Condition (2d) and strict positivity of all 24 nontrivial leading Sylvester minors (10 for $J + Q$, 8 for $(J - Q)_{\text{active}}$, 6 for $Q^{\Gamma_{\text{in}}}$ on `sym3`/`asym3`). Since $w_8 = 8(8/5)(101/200) + 8/5 - 101/200 = 7559/1000 < 8$, we obtain $\rho_8^* = (7559I - 1000\text{SWAP})/475776$ (Lean Theorems 5–6) with NPT excess $|8\alpha_8^*| - 1 = 441/7559$, exceeding Faisal's $1/23$ by the exact ratio $10143/7559 > 1.3418$ (Lean Theorem 7). As an independent cross-check, `test_17` and `test_18` also verify Q_8^* directly on Faisal's five original 4096×4096 diagonal-U(8) blocks of sizes $1, 3, 6, 22, 42$ from Table I of [1].

5.2 Symbolic Positivity of the Universal Family $Q(d)$ for All $d \geq 9$

To cover all dimensions $d \geq 9$ simultaneously, write $s := \sqrt{d} \geq 3$ (so $d = s^2$) and observe that the unweighted 6×6 Condition (2d) linear system for the six dependent Hermitian coordinates $(u_0, \ldots, u_5) = (m_6, m_7, m_8 = m_{12}, m_9 = m_{18}, m_{10} = m_{13}, m_{11} = m_{19})$ decouples into three explicit 2×2 blocks:

$$
\begin{aligned}
&d(u_2 + u_3) = R_{+,1}, \qquad d(-u_2 + u_3) = R_{-,1},\\
&d(u_4 + u_5) = R_{+,2}, \qquad d(u_4 - u_5) = R_{-,2},\\
&d(u_0 + u_1) = R_{+,0} - \frac{R_{+,1} + R_{+,2}}{d}, \qquad d(u_0 - u_1) = R_{-,0} + \frac{R_{-,1} + R_{-,2}}{d},
\end{aligned}
$$

where $R_{\pm,i}$ are the unweighted column-0 entries of $J - Q_{\text{free}}$ on `sym3` $(+)$ and `asym3` $(-)$. Substituting $a_L = 8/5$, $a_M = 8/(5s)$, and the six free rational functions (14) yields the exact closed-form coordinates (15) in $\mathbb{Q}[s^{-1}]$ with maximum denominator s^7.

Multiplying J and $Q(d)$ by $s^7 > 0$, every entry of every Artin–Wedderburn block is a pure univariate polynomial in $\mathbb{Q}[s]$. Our pure-`Fraction` symbolic polynomial verifier (`cert10`–`cert12`) computes all 24 nontrivial leading Sylvester minors $\Delta_k(s) \in \mathbb{Q}[s]$ (10 on $J + Q$, 8 on $(J - Q)_{\text{active}}$, 6 on $Q^{\Gamma_{\text{in}}}$) and performs the exact shift $s = t + 3$. Remarkably, *every single coefficient of every single one of the* 24 *shifted polynomials* $\Delta_k(t + 3) \in \mathbb{Q}[t]$ *is strictly positive* (all constant terms at $t = 0$, i.e. $d = 9$, are strictly positive, and all higher-degree coefficients in t are strictly positive). Because $t = \sqrt{d} - 3 \geq 0$ for every real $d \geq 9$, this constitutes a finite, exact, zero-float algebraic proof that $J + Q(d) \succ 0$, $(J - Q(d))_{\text{active}} \succ 0$, and $Q(d)^{\Gamma_{\text{in}}} \succeq 0$ for *all* integers $d \geq 9$ simultaneously. Finally, the exact polynomial identity (16) (Lean Theorems 18–19) proves $w_d < d$ and $w_d < 3\sqrt{d}$ for all $d \geq 9$, certifying the entire continuous NPT interval (17) by Theorem 1(c). Concrete zero-float evaluations at $d = 9, 16, 25, 36, 100, 10000$ are recorded in Table 3 and `cert13`. □

6 Formal Verification and Reproducibility

All artifacts are publicly released under MIT (code, Lean proofs, certificates) and CC-BY-4.0 (manuscript, figures) at https://github.com/lluiseriksson/npt-bound-entanglement-sharp-fidelity-and-threshold:

1. **Constructive Lean 4.36.0** (`lean/NPTBoundEntanglement.lean`): 20 theorems compiled with `-DwarningAsError=true`, zero `sorry`, zero `Classical.choice`, zero custom axioms,

Table 3: Zero-float exact rational evaluations of our universal family $Q(d)$ at square dimensions $d = s^2$ (`cert13`), alongside Faisal's $d = 8$ state and our sharpened Q_8^*.

d	Certificate	Exact w_d	Endpoint α_0	$d\|\alpha_0\|$	$\sqrt{d}\,\|\alpha_0\|$
8	Faisal [1]	$23/3 \approx 7.67$	$-3/23 \approx -0.1304$	$24/23 \approx 1.0435$	0.3689
8	Q_8^* (Thm. 3b)	$7559/1000 = 7.559$	$-1000/7559 \approx -0.1323$	$8000/7559 \approx 1.0583$	0.3742
9	$Q(9)$ (Thm. 3c)	$656/75 \approx 8.75$	$-75/656 \approx -0.1143$	$675/656 \approx 1.0290$	0.3430
16	$Q(16)$ (Thm. 3c)	$286/25 = 11.44$	$-25/286 \approx -0.0874$	$200/143 \approx 1.3986$	0.3497
25	$Q(25)$ (Thm. 3c)	$352/25 = 14.08$	$-25/352 \approx -0.0710$	$625/352 \approx 1.7756$	0.3551
36	$Q(36)$ (Thm. 3c)	$1252/75 \approx 16.69$	$-75/1252 \approx -0.0599$	$675/313 \approx 2.1565$	0.3594
100	$Q(100)$ (Thm. 3c)	$676/25 = 27.04$	$-25/676 \approx -0.0370$	$625/169 \approx 3.6982$	0.3698
10000	$Q(10000)$ (Thm. 3c)	$32198/125 = 257.58$	$-125/32198 \approx -0.0039$	$625000/16099 \approx 38.822$	0.3882

and zero external dependencies, mechanizing the Artin–Wedderburn dimension sum, Faisal's and our $d = 8$ Werner traces/eigenvalues/negativity boost, the sharp Bell-fidelity 1/2 proof from the PPT minor, local-depolarizing convexity and identity, boundary and Type II eliminations, McCormick envelope soundness, strict positive definiteness of all three pure-integer Gram matrices, and the universal $d \geq 9$ polynomial NPT margin and $\Theta(1/\sqrt{d})$ expansion rate.

2. **AST-Audited Zero-Float Verifier** (`verification/verify_npt_bound_entanglement.py`): Uses strictly `int` and `fractions.Fraction` (audited via Python's `ast` module for zero floats and zero numerical libraries) to verify and emit 16 SHA-256 certificates (`certificates/manifest.json`), including all 23 rational continuum impossibility duals (`cert05`–`cert09`) and all 24 shifted symbolic Sylvester polynomials (`cert10`–`cert12`).

3. **Adversarial Test Suite** (`tests/test_adversarial.py`): 56 passing tests covering AST zero-float compliance, SHA-256 manifest integrity, direct 81×81 ($d = 3$) Hilbert-space verification of the walled Brauer algebra, direct 4096×4096 Faisal Table I block verification of Q_8^*, adversarial dual/primal corruption rejection, non-square dimensions ($d = 10, 11, 13, 17, 50, 1000$), and the tightness witness for Faisal's single-step fidelity bound.

7 Conclusion

We have sharpened Faisal's landmark proof of NPT bound entanglement [1] along three orthogonal axes: lowering the separable-operation Bell-state fidelity ceiling to the exact optimal value 1/2 while proving PPT/separability of all two-qubit outputs and lifting isolated points to continuous Werner intervals; proving via walled Brauer Artin–Wedderburn collapse and 23 exact rational McCormick SDP-dual certificates that $d_{\min} = 8$ is the exact sharp dimensional threshold of covariant CP-map certificates; and exhibiting both a +34.18% sharpened $d = 8$ integer witness and a single closed-form algebraic family certifying $\Theta(1/\sqrt{d})$-wide bands of NPT bound entangled Werner states for every dimension $d \geq 9$. An intriguing open question is whether non-covariant or multi-copy reverse-block certificates can push below $d_{\min} = 8$ toward the long-standing DiVincenzo–Dür conjecture in $d = 3$ and $d = 4$ [7, 8].

References

[1] I. Faisal, *Bound Entanglement with Negative Partial Transpose Exists*, arXiv:2611.34921v1 (Nov. 24, 2026).

[2] M. Horodecki, P. Horodecki, and R. Horodecki, *Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?*, Phys. Rev. Lett. **80**, 5239–5242 (1998).

[3] M. Horodecki, P. Horodecki, and R. Horodecki, *Inseparable Two Spin-$\frac{1}{2}$ Density Matrices Can Be Distilled to a Singlet Form*, Phys. Rev. Lett. **78**, 574–577 (1997).

[4] M. Horodecki, P. Horodecki, and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **223**, 1–8 (1996).

[5] A. Peres, *Separability Criterion for Density Matrices*, Phys. Rev. Lett. **77**, 1413–1415 (1996).

[6] R. F. Werner, *Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model*, Phys. Rev. A **40**, 4277–4281 (1989).

[7] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, *Evidence for bound entangled states with negative partial transpose*, Phys. Rev. A **61**, 062312 (2000).

[8] W. Dür, J. I. Cirac, M. Lewenstein, and D. Bruß, *Distillability and partial transposition in bipartite systems*, Phys. Rev. A **61**, 062313 (2000).

[9] P. Horodecki, Ł. Rudnicki, and K. Życzkowski, *Five Open Problems in Quantum Information Theory*, PRX Quantum **3**, 010101 (2022).

[10] E. M. Rains, *Entanglement purification via separable superoperators*, arXiv:quant-ph/9707002 (1997).

[11] K. Koike, *On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters*, Adv. Math. **74**, 57–86 (1989).

[12] J. Brundan and C. Stroppel, *Gradings on walled Brauer algebras and Khovanov's arc algebra*, Adv. Math. **231**, 709–773 (2012).