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Measurement-Induced Resource Convertibility, Exact Measured-Entropy Superadditivity, BKM-Tightened Thermodynamic Sensitivity, and Spectral-Span Operational Quantum Speed Limits for Quantum-State Rugosity

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Abstract

Céleri and Soares-Pinto [1] (arXiv:2611.34788v1) established that quantum-state rugosity $R(\rho) = -\ln \mathrm{Tr}(\rho\Pi_\omega)$ equals the binary flat-state measurement-induced relative entropy $D_{\mathcal{M}_\omega}(\Pi_\omega\|\rho)$, embeds into the hypothesis-testing hierarchy $R(\rho) \le D_{\mathrm{mea}}(\Pi_\omega\|\rho) \le D(\Pi_\omega\|\rho)$, satisfies the thermal reset sensitivity bound $(\partial_\phi R)^2|_0 \le (e^{R_0}-1)\chi_\Sigma$, and obeys the operational quantum speed limit (QSL) $\tau \ge \mathcal{L}_\omega(p_0,p_\tau)/\overline{\Delta H}$. Here we resolve the exact convertibility, hierarchy-collapse, non-additivity, thermodynamic saturation, and mixed-state dynamical tightness questions left open by [1] through four exact theorems. **(1) Measurement-Induced Resource Convertibility:** Defining $\rho \xrightarrow{\mathcal{M}} \sigma$ by Blackwell post-processing preserving $P_\omega^{\mathcal{M}}$, we prove that flat-resolving measurements ($M_{x_0} = \Pi_\omega$, handling $p_\omega(\rho) = 1$ and $p_\omega(\rho) = 0$ explicitly, while demonstrating separation when $\mathrm{rank}(M_{x_0}) \ge 2$ with $\langle\omega|M_{x_0}|\omega\rangle = 1$) collapse $\xrightarrow{\mathcal{M}}$ to $R(\rho) \ge R(\sigma)$; non-aligned binary-supported POVMs ($P_\omega^{\mathcal{M}} = (w, 1-w)$, $0 < w \le 1/2$) yield the exact convertibility interval $q_\sigma \in [\min(q_\rho, \frac{w}{1-w}(1-q_\rho)), \max(q_\rho, \frac{w}{1-w}(1-q_\rho))]$; and $m \ge 3$ outcomes are governed by $P_\omega^{\mathcal{M}}$-relative majorization, strictly stronger than $D_{\mathcal{M}}(\Pi_\omega\|\rho) \ge D_{\mathcal{M}}(\Pi_\omega\|\sigma)$. Explicit $d = 3$ $\mathcal{M}_Z$ witnesses exhibit both a strict KL gap $\frac{1}{3}\ln\frac{27}{25} > 0$ without convertibility and bidirectional quantum–classical incomparability against R. **(2) Collapse of Ref. [1] Eq. (7), 2-Copy D_{mea} Superadditivity, & Bipartite Rugosity:** We prove the exact trichotomy for Eq. (7) of Ref. [1]: when $|\omega\rangle \in \mathrm{supp}(\rho)$, $R(\rho) = D_{\mathrm{mea}}(\Pi_\omega\|\rho) \iff [\rho,\Pi_\omega] = 0 \iff D_{\mathrm{mea}}(\Pi_\omega\|\rho) = D(\Pi_\omega\|\rho)$ (via a rank-1 Petz recovery obstruction and explicit coherence slope $2\|(\mathbb{I}-\Pi_\omega)\rho|\omega\rangle\|_2/p_\omega(\rho)$), reduce D_{mea} to d-outcome PVMs with logarithmic-mean optimality $\int_0^1 W^{1-s}\rho W^s\,ds = \Pi_\omega$, certify strict $n = 2$ superadditivity $D_{\mathrm{mea}}(\Pi_\omega^{\otimes 2}\|\rho^{\otimes 2}) > 2D_{\mathrm{mea}}(\Pi_\omega\|\rho)$ on $\rho = \mathrm{diag}(9/10, 1/10)$ via an explicit matrix $O_{11} \in SO(4) \cap \mathbb{Q}^{4\times 4}$ and a 96-interval exact rational enclosure, and prove universal partial-trace monotonicity $R(\rho_{AB}) \ge \max(R(\rho_A), R(\rho_B))$ alongside unbounded separable superadditivity $\ln(1/\epsilon) \to +\infty$. **(3) BKM-Variance Sensitivity & Saturation Dichotomy:** We sharpen Ref. [1] Eq. (14) to $(\partial_\phi R)^2|_0 \le e^{2R_0}V_{\mathrm{BKM}}(\rho_0)\chi_\Sigma \le (e^{R_0}-1)\chi_\Sigma$ with exact arithmetic-vs-logarithmic mean gap $1 - \kappa_{\mathrm{BKM}}(\rho_0) \ge 0$, classify saturation of Ref. [1] Eq. (14) ($\chi_\Sigma > 0$) as $[H_0,\Pi_\omega] = 0$ and $\dot\rho_0 \propto \rho_0(\Pi_\omega - p_0\mathbb{I})$, prove strict no-saturation $F_0^{(\omega)} := (\partial_\phi R)^2|_0/(e^{R_0}-1) < \chi_\Sigma$ for all unitary encodings (while $d = 2, H_0 = \frac{\Delta}{2}Z, G = \frac{1}{2}Y$ saturates the BKM bound with $\kappa_{\mathrm{BKM}}(\beta\Delta) = \frac{\tanh(\beta\Delta/2)}{\beta\Delta/2}$), and establish the global reset bound $\Sigma(\rho) \ge d_{\mathrm{KL}}(e^{-R(\rho)}\|e^{-R(\rho_0)})$. **(4) Pure-State Necessity & Spectral-Span Mixed-State QSL:** We prove that pure states are strictly necessary to saturate Eqs. (19) and (21) of Ref. [1] whenever $\dot p_t \ne 0$, derive the universal hierarchy $F_{\mathrm{SLD}}(\rho_t,\Pi_\omega) \le \frac{4(p_t-\lambda_{\min})(\lambda_{\max}-p_t)}{\lambda_{\max}+\lambda_{\min}} \le 4p_t(1-p_t)$ from two exact algebraic identities, and establish the tightened spectral-span QSL $\tau \ge \frac{\sqrt{\lambda_{\max}+\lambda_{\min}}\,\mathcal{L}_\omega^{(\mathrm{spec})}(p_0,p_\tau)}{\overline{\Delta H}} \ge \frac{\mathcal{L}_\omega(p_0,p_\tau)}{\overline{\Delta H}}$, which

is exact for every mixed qubit. Scoping: all finite algebraic, rational-matrix, stochastic-polytope, integer-power, and 96-interval enclosure cores are formally verified by 20 constructive Lean 4.36.0 theorems (`thm01`–`thm20`, zero external imports, zero `sorry`, zero `Classical.choice`) and 16 AST-audited zero-float SHA-256 certificates with 67 adversarial tests, while operator-logarithm, Fréchet-derivative, Petz-recovery, and integral QSL steps are proved analytically in the text.

Keywords: quantum-state rugosity, measurement-induced relative entropy, Blackwell convertibility, measured relative entropy superadditivity, Bogoliubov–Kubo–Mori information, quantum speed limits, Lean 4 formal verification.

1 Introduction

Quantum-state texture quantifies structural complexity of a quantum state relative to a fixed orthonormal reference basis $\{|\epsilon_m\rangle\}_{m=1}^d$ on $\mathcal{H}_d \cong \mathbb{C}^d$ ($d \geq 2$) [2, 3, 4, 5, 6, 7]. Writing $|\omega\rangle = d^{-1/2}\sum_{m=1}^d |\epsilon_m\rangle$ for the flat superposition state and $\Pi_\omega = |\omega\rangle\langle\omega|$ for its rank-1 projector, the logarithmic rugosity is $R(\rho) := -\ln p_\omega(\rho)$ where $p_\omega(\rho) := \mathrm{Tr}(\rho\Pi_\omega) \in [0,1]$, with $R(\rho) = +\infty$ when $p_\omega(\rho) = 0$. In a foundational study, Céleri and Soares-Pinto [1] identified rugosity with the classical Kullback–Leibler (KL) divergence induced by the binary flat-state projective measurement $\mathcal{M}_\omega = \{\Pi_\omega, \mathbb{I} - \Pi_\omega\}$ (Ref. [1], Eq. (7)):

$$R(\rho) = D_{\mathcal{M}_\omega}(\Pi_\omega\|\rho) \leq D_{\mathrm{mea}}(\Pi_\omega\|\rho) \leq D(\Pi_\omega\|\rho) = -\langle\omega|\ln\rho|\omega\rangle, \tag{1}$$

where $D_{\mathcal{M}}(\sigma\|\rho) := K(P_\sigma^{\mathcal{M}}\|P_\rho^{\mathcal{M}})$ with $K(P\|Q) := \sum_x P(x)\ln[P(x)/Q(x)]$, $P_\sigma^{\mathcal{M}}(x) = \mathrm{Tr}(\sigma M_x)$, and $D_{\mathrm{mea}}(\sigma\|\rho) := \sup_{\mathcal{M}} D_{\mathcal{M}}(\sigma\|\rho)$ maximizes over all POVMs $\mathcal{M} = \{M_x\}$. Building on Eq. (1) (Eq. (7) of Ref. [1]), Ref. [1] established: (i) an exact finite-copy zero-type-I-error hypothesis-testing formula $R(\rho) = -\frac{1}{n}\ln\beta_n$; (ii) a local thermodynamic sensitivity bound around a finite-temperature equilibrium $\rho_0 = e^{-\beta H_0}/Z_0$ (Ref. [1], Eq. (14)),

$$(\partial_\phi R)^2\big|_{\phi=0} \leq \left(e^{R_0} - 1\right)\chi_\Sigma, \qquad \chi_\Sigma := \partial_\phi^2 D(\rho_\phi\|\rho_0)\big|_{\phi=0} = F_{\mathrm{BKM}}, \tag{2}$$

where $R_0 = R(\rho_0)$ and F_{BKM} is the Bogoliubov–Kubo–Mori (BKM) Fisher information [8, 9]; and (iii) an operational quantum speed limit (QSL) under unitary dynamics $\dot\rho_t = -i[H_t, \rho_t]$ (Ref. [1], Eqs. (19) and (21)),

$$\frac{|\dot R_t|}{\sqrt{e^{R_t} - 1}} \leq 2\Delta H_t, \qquad \tau \geq \frac{\mathcal{L}_\omega(p_0, p_\tau)}{\overline{\Delta H}}, \tag{3}$$

where $\mathcal{L}_\omega(p_0, p_\tau) := \arccos\left[\sqrt{p_0 p_\tau} + \sqrt{(1-p_0)(1-p_\tau)}\right]$ and $\overline{\Delta H} = \frac{1}{\tau}\int_0^\tau \Delta H_t\, dt$.

Four fundamental mathematical questions were left open by [1]:

1. *Measurement-induced convertibility:* Under a fixed POVM $\mathcal{M}$ (beyond $\mathcal{M}_\omega$), when can the outcome statistics of ρ be classically post-processed into those of σ while preserving the flat-state reference distribution $P_\omega^{\mathcal{M}}$, and how does this preorder compare with $D_{\mathcal{M}}(\Pi_\omega\|\cdot)$ and $R(\cdot)$?

2. *Hierarchy collapse, multi-copy superadditivity, and multipartite rugosity:* Exactly when do the inequalities in Eq. (1) (Ref. [1], Eq. (7)) collapse or separate strictly; can $D_{\mathrm{mea}}(\Pi_\omega\|\rho)$ be strictly superadditive on i.i.d. product states $\rho^{\otimes 2}$; and how does $R(\rho_{AB})$ behave under partial trace and separable correlations?

3. *Thermodynamic tightness and saturation:* When can Eq. (2) (Ref. [1], Eq. (14)) be saturated, why can unitary encodings never saturate it, what sharper operator-variance bound holds for general encodings, and what global non-perturbative bound underlies Eq. (2)?

4. *Mixed-state QSL tightening:* Are pure states strictly necessary to saturate Eq. (3) (Ref. [1], Eqs. (19) and (21)), and what exact spectral-span operational speed limit captures mixed-state evolution tightly across all dimensions $d \geq 2$?

The present paper answers all four questions completely via Theorems 1–4, backed by 20 constructive Lean 4.36.0 theorems, 16 AST-audited zero-float exact-$\mathbb{Q}/\mathbb{Q}[i]$ SHA-256 certificates, and 67 adversarial tests.

2 Theorem 1: Measurement-Induced Resource Convertibility

Let $\mathcal{M} = \{M_x\}_{x=1}^m$ be an m-outcome POVM ($M_x \succeq 0$, $\sum_{x=1}^m M_x = \mathbb{I}_d$), inducing the outcome distributions $P_\omega^{\mathcal{M}}(x) := \langle\omega|M_x|\omega\rangle$ and $P_\rho^{\mathcal{M}}(x) := \mathrm{Tr}(\rho M_x)$. Viewing the flat state Π_ω as the fixed free reference, the statistical experiment accessible through $\mathcal{M}$ is the dichotomy $(P_\omega^{\mathcal{M}}, P_\rho^{\mathcal{M}})$. We define $\mathcal{M}$-induced resource convertibility $\rho \xrightarrow{\mathcal{M}} \sigma$ by the existence of a classical column-stochastic post-processing kernel $T = (T_{y|x})_{y,x=1}^m$ ($T_{y|x} \geq 0$, $\sum_y T_{y|x} = 1$) satisfying

$$TP_\omega^{\mathcal{M}} = P_\omega^{\mathcal{M}} \qquad \text{and} \qquad TP_\rho^{\mathcal{M}} = P_\sigma^{\mathcal{M}}. \tag{4}$$

Equivalently, $(P_\omega^{\mathcal{M}}, P_\rho^{\mathcal{M}})$ Blackwell-dominates $(P_\omega^{\mathcal{M}}, P_\sigma^{\mathcal{M}})$ [10, 11, 12].

Theorem 1 ($\mathcal{M}$-Blackwell Dichotomy, Polytope Criterion, and Quantum–Classical Incomparability)**.** *Let $\mathcal{M} = \{M_x\}_{x=1}^m$ be an m-outcome POVM on $\mathbb{C}^d$ ($d \geq 2$).*

1. ***Flat-resolving collapse:*** *Suppose $\exists x_0$ with $\langle\omega|M_{x_0}|\omega\rangle = 1$. Then for all $\rho, \sigma \in \mathcal{D}(\mathbb{C}^d)$,*

$$\rho \xrightarrow{\mathcal{M}} \sigma \iff P_\rho^{\mathcal{M}}(x_0) \leq P_\sigma^{\mathcal{M}}(x_0) \iff D_{\mathcal{M}}(\Pi_\omega\|\rho) \geq D_{\mathcal{M}}(\Pi_\omega\|\sigma). \tag{5}$$

   In particular, whenever $M_{x_0} = \Pi_\omega$ (equivalently, $\langle\omega|M_{x_0}|\omega\rangle = 1$ and $\mathrm{rank}(M_{x_0}) = 1$, i.e., $\mathcal{M}$ is flat-resolving), $\rho \xrightarrow{\mathcal{M}} \sigma \iff R(\rho) \geq R(\sigma)$ (including $p_\omega(\rho) = 1$, where $\sigma = \Pi_\omega$ is forced, and $p_\omega(\rho) = 0$, where $R(\rho) = +\infty$). Conversely, if $\mathrm{rank}(M_{x_0}) \geq 2$ with $\langle\omega|M_{x_0}|\omega\rangle = 1$ ($d \geq 3$), $P_\rho^{\mathcal{M}}(x_0)$ can separate from $p_\omega(\rho)$ and invert the rugosity ordering.

2. ***Non-aligned polytope & Lorenz criterion:*** *If $|\,\mathrm{supp}(P_\omega^{\mathcal{M}})| = 2$ with $P_\omega^{\mathcal{M}} = (w, 1-w)$, $0 < w \leq 1/2$, and $P_\rho^{\mathcal{M}} = (q_\rho, 1-q_\rho)$, $P_\sigma^{\mathcal{M}} = (q_\sigma, 1-q_\sigma)$ supported on $\mathrm{supp}(P_\omega^{\mathcal{M}})$, then*

$$\rho \xrightarrow{\mathcal{M}} \sigma \iff q_\sigma \in \Big[\min\Big(q_\rho, \tfrac{w}{1-w}(1-q_\rho)\Big),\ \max\Big(q_\rho, \tfrac{w}{1-w}(1-q_\rho)\Big)\Big]. \tag{6}$$

   For $m = |\,\mathrm{supp}(P_\omega^{\mathcal{M}})| \geq 3$ (with $P_\rho^{\mathcal{M}}, P_\sigma^{\mathcal{M}}$ supported on $\mathrm{supp}(P_\omega^{\mathcal{M}})$), $\rho \xrightarrow{\mathcal{M}} \sigma$ holds if and only if $P_\rho^{\mathcal{M}}$ $P_\omega^{\mathcal{M}}$-relatively majorizes $P_\sigma^{\mathcal{M}}$, i.e. the concave Lorenz curve $L_{P_\omega^{\mathcal{M}}}(P_\rho^{\mathcal{M}})$ lies everywhere on or above $L_{P_\omega^{\mathcal{M}}}(P_\sigma^{\mathcal{M}})$ on $[0,1]$, which is strictly stronger than $D_{\mathcal{M}}(\Pi_\omega\|\rho) \geq D_{\mathcal{M}}(\Pi_\omega\|\sigma)$.

3. ***Explicit $d = 3$ computational-basis $\mathcal{M}_Z = \{|0\rangle\langle 0|, |1\rangle\langle 1|, |2\rangle\langle 2|\}$ witnesses:***

   (a) $\rho_0 = \mathrm{diag}(5/12, 5/12, 1/6)$ and $\sigma_0 = \mathrm{diag}(1/2, 1/4, 1/4)$ satisfy $D_{\mathcal{M}_Z}(\Pi_\omega\|\rho_0) - D_{\mathcal{M}_Z}(\Pi_\omega\|\sigma_0) = \frac{1}{3}\ln\frac{27}{25} > 0$, yet $\rho_0 \not\xrightarrow{\mathcal{M}_Z} \sigma_0$ (and $\sigma_0 \not\xrightarrow{\mathcal{M}_Z} \rho_0$).

   (b) The PSD state $\rho_A = \frac{1}{24}\begin{pmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 12 \end{pmatrix}$ (eigenvalues $\{1/8, 1/4, 5/8\}$) and $\sigma_A = \mathbb{I}_3/3$ satisfy $\rho_A \xrightarrow{\mathcal{M}_Z} \sigma_A$ yet $R(\rho_A) = \ln\frac{12}{7} < \ln 3 = R(\sigma_A)$.

(c) $\rho_B = \mathbb{I}_3/3$ *and the PSD state* $\sigma_B = \frac{1}{12}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *(eigenvalues* $\{1/6, 1/3, 1/2\}$*) satisfy* $R(\rho_B) = \ln 3 > \ln \frac{18}{7} = R(\sigma_B)$ *yet* $\rho_B \overset{\mathcal{M}_Z}{\not\longrightarrow} \sigma_B$.

Proof. Part (1). Since $P_\omega^{\mathcal{M}}(x_0) = 1$ forces $P_\omega^{\mathcal{M}}(x) = 0$ for all $x \neq x_0$, the fixed-point condition $TP_\omega^{\mathcal{M}} = P_\omega^{\mathcal{M}}$ holds iff $T_{x_0|x_0} = 1$ and $T_{y|x_0} = 0$ for $y \neq x_0$, leaving columns $x \neq x_0$ as arbitrary probability distributions. Hence $(TP_\rho^{\mathcal{M}})(x_0) = P_\rho^{\mathcal{M}}(x_0) + \sum_{x \neq x_0} T_{x_0|x} P_\rho^{\mathcal{M}}(x) \geq P_\rho^{\mathcal{M}}(x_0)$ is necessary. Conversely, if $P_\sigma^{\mathcal{M}}(x_0) \geq P_\rho^{\mathcal{M}}(x_0)$: when $P_\rho^{\mathcal{M}}(x_0) = 1$, normalization forces $P_\sigma^{\mathcal{M}} = P_\rho^{\mathcal{M}} = e_{x_0}$ so $T = \mathbb{I}$ suffices; when $P_\rho^{\mathcal{M}}(x_0) < 1$, setting $T_{x_0|x} = \frac{P_\sigma^{\mathcal{M}}(x_0) - P_\rho^{\mathcal{M}}(x_0)}{1 - P_\rho^{\mathcal{M}}(x_0)}$ and $T_{y|x} = \frac{P_\sigma^{\mathcal{M}}(y)}{1 - P_\rho^{\mathcal{M}}(x_0)}$ for $x, y \neq x_0$ defines a valid $P_\omega^{\mathcal{M}}$-preserving stochastic matrix sending $P_\rho^{\mathcal{M}} \mapsto P_\sigma^{\mathcal{M}}$ (Lean 4 theorem `thm03`). Furthermore, $D_{\mathcal{M}}(\Pi_\omega \| \rho) = -\ln P_\rho^{\mathcal{M}}(x_0)$, which is monotone decreasing in $P_\rho^{\mathcal{M}}(x_0)$. If M_{x_0} is rank-1 with $\langle\omega|M_{x_0}|\omega\rangle = 1$, writing $M_{x_0} = c|v\rangle\langle v|$ $(0 < c \leq 1,\ \|v\|_2 = 1)$ yields $1 = c|\langle\omega|v\rangle|^2 \leq c \leq 1$, so $c = 1$ and $M_{x_0} = \Pi_\omega$, whence $P_\rho^{\mathcal{M}}(x_0) = p_\omega(\rho)$ and $\rho \xrightarrow{\mathcal{M}} \sigma \iff R(\rho) \geq R(\sigma)$. For $\operatorname{rank}(M_{x_0}) \geq 2$ in $d = 3$, take $|\psi_1\rangle, |\psi_2\rangle$ completing $\{|\omega\rangle\}$ to an orthonormal basis and $M_{x_0} = \Pi_\omega + \frac{1}{2}|\psi_1\rangle\langle\psi_1|$, $M_1 = \mathbb{I} - M_{x_0}$. Then $\rho = \frac{1}{4}\Pi_\omega + \frac{3}{4}|\psi_1\rangle\langle\psi_1|$ and $\sigma = \frac{1}{2}\Pi_\omega + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$ give $R(\rho) = \ln 4 > \ln 2 = R(\sigma)$ yet $P_\rho^{\mathcal{M}}(x_0) = 5/8 > 1/2 = P_\sigma^{\mathcal{M}}(x_0)$, so $\rho \overset{\mathcal{M}}{\not\longrightarrow} \sigma$ while $\sigma \xrightarrow{\mathcal{M}} \rho$ (Certificate `C01`).

Part (2). For $m = 2$, any 2×2 column-stochastic matrix $T = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$ $(a, b \in [0,1])$ fixing $(w, 1-w)^T$ satisfies $aw + b(1-w) = w \iff b = \frac{w}{1-w}(1-a)$. Since $0 < w \leq 1/2$, $\frac{w}{1-w} \leq 1$, so $a \in [0,1] \iff b \in [0, \frac{w}{1-w}] \subseteq [0,1]$. The extreme points $a = 1$ $(T = \mathbb{I})$ and $a = 0$ map q_ρ to q_ρ and $\frac{w}{1-w}(1-q_\rho)$, establishing Eq. (6) by convexity (Lean 4 theorems `thm01`–`thm02`; Certificate `C02`). For $m \geq 3$, $P_\omega^{\mathcal{M}}$-relative majorization characterizes Eq. (4) by the Blackwell–Ruch–Schranner–Seligman theorem [10, 11, 12], and strict convexity of $t \ln t$ implies KL monotonicity $D_{\mathcal{M}}(\Pi_\omega\|\rho) \geq D_{\mathcal{M}}(\Pi_\omega\|\sigma)$ without the converse.

Part (3). Under $\mathcal{M}_Z$, $P_\omega^{\mathcal{M}_Z} = (1/3, 1/3, 1/3)$. For (a), $P_{\rho_0}^{\mathcal{M}_Z} = (5/12, 5/12, 1/6)$ and $P_{\sigma_0}^{\mathcal{M}_Z} = (1/2, 1/4, 1/4)$ give $D_{\mathcal{M}_Z}(\Pi_\omega\|\rho_0) = \frac{1}{3}\ln\frac{32}{25}$ and $D_{\mathcal{M}_Z}(\Pi_\omega\|\sigma_0) = \frac{1}{3}\ln\frac{32}{27}$, whose difference is $\frac{1}{3}\ln\frac{27}{25} > 0$, while $\max_x P_{\rho_0}^{\mathcal{M}_Z}(x) = 5/12 < 1/2 = \max_x P_{\sigma_0}^{\mathcal{M}_Z}(x)$ forbids any bistochastic map $\rho_0 \to \sigma_0$ and $L_{\rho_0}(2/3) = 5/6 > 3/4 = L_{\sigma_0}(2/3)$ forbids $\sigma_0 \to \rho_0$ (both inconvertibilities proved in Lean 4 `thm04`; Certificate `C03`; Fig. 1(b)). For (b), $\chi_{\rho_A}(t) = (t-1/8)(t-1/4)(t-5/8)$ so $\rho_A \succ 0$, $p_\omega(\rho_A) = \frac{1}{3}\sum_{i,j}(\rho_A)_{ij} = 7/12$, so $R(\rho_A) = \ln\frac{12}{7} < \ln 3 = R(\sigma_A)$, yet $T_{y|x} = 1/3$ maps every distribution to $P_{\sigma_A}^{\mathcal{M}_Z} = (1/3, 1/3, 1/3)$ (Lean 4 `thm05`). For (c), σ_B has eigenvalues $\{1/6, 1/3, 1/2\}$, $p_\omega(\sigma_B) = 7/18$ so $R(\sigma_B) = \ln\frac{18}{7} < \ln 3 = R(\rho_B)$, yet $T(1/3, 1/3, 1/3)^T = (1/3, 1/3, 1/3)^T \neq (5/12, 5/12, 1/6)^T = P_{\sigma_B}^{\mathcal{M}_Z}$ for every $P_\omega^{\mathcal{M}_Z}$-preserving T (Lean 4 `thm06`; Certificates `C03`–`C05`). □

3 Theorem 2: Collapse of Ref. [1] Eq. (7), Exact 2-Copy Superadditivity of D_{mea}, and Bipartite Rugosity

Theorem 2 (Ref. [1] Eq. (7) Trichotomy, Projective Reduction, 2-Copy Superadditivity, and Bipartite Rugosity)**.** *Let* $\rho \in \mathcal{D}(\mathbb{C}^d)$ *(*$d \geq 2$*).*

1. ***Exact Ref. [1] Eq. (7) trichotomy:***

   (i) If $p_\omega(\rho) = 0$*, then* $R(\rho) = D_{\text{mea}}(\Pi_\omega\|\rho) = D(\Pi_\omega\|\rho) = +\infty$.

   (ii) If $p_\omega(\rho) > 0$ *and* $|\omega\rangle \notin \operatorname{supp}(\rho)$*, then* $R(\rho) < +\infty = D_{\text{mea}}(\Pi_\omega\|\rho) = D(\Pi_\omega\|\rho)$.

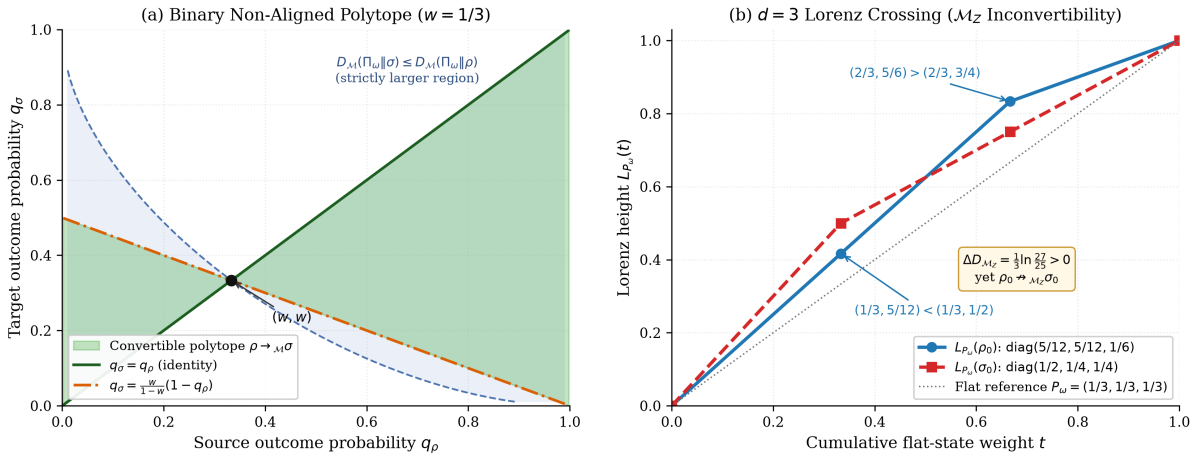


Figure 1: **(a)** Exact binary non-aligned convertibility polytope Eq. (6) for $w = 1/3$ (green bowtie) strictly inside the KL-monotone cone $D_{\mathcal{M}}(\Pi_\omega\|\sigma) \le D_{\mathcal{M}}(\Pi_\omega\|\rho)$ (dashed blue boundary). **(b)** $d = 3$ computational-basis Lorenz curve crossing between $\rho_0 = \mathrm{diag}(5/12, 5/12, 1/6)$ and $\sigma_0 = \mathrm{diag}(1/2, 1/4, 1/4)$ demonstrating strict $\mathcal{M}_Z$-inconvertibility despite the positive KL gap $\frac{1}{3}\ln(27/25) > 0$ (Theorem 1).

(iii) If $|\omega\rangle \in \mathrm{supp}(\rho)$, then all three quantities are finite and

$$R(\rho) = D_{\mathrm{mea}}(\Pi_\omega\|\rho) \iff [\rho, \Pi_\omega] = 0 \iff D_{\mathrm{mea}}(\Pi_\omega\|\rho) = D(\Pi_\omega\|\rho), \tag{7}$$

with strict separation $R(\rho) < D_{\mathrm{mea}}(\Pi_\omega\|\rho) < D(\Pi_\omega\|\rho)$ whenever $[\rho, \Pi_\omega] \neq 0$. Moreover, rotating $\mathcal{M}_\omega$ along the coherence vector $|v\rangle := (\mathbb{I} - \Pi_\omega)\rho|\omega\rangle / \|(\mathbb{I} - \Pi_\omega)\rho|\omega\rangle\|_2$ via $|\omega_{-\varepsilon}\rangle = \cos\varepsilon|\omega\rangle - \sin\varepsilon|v\rangle$ yields the exact linear coherence-gap slope

$$\lim_{\varepsilon\to 0^+} \frac{D_{\mathcal{M}_{-\varepsilon}}(\Pi_\omega\|\rho) - R(\rho)}{\varepsilon} = \frac{2\|(\mathbb{I} - \Pi_\omega)\rho|\omega\rangle\|_2}{p_\omega(\rho)} > 0. \tag{8}$$

2. ***Projective reduction & logarithmic-mean optimality:*** *For every $\rho \succ 0$, $D_{\mathrm{mea}}(\Pi_\omega\|\rho)$ equals $\sup_{W\succ 0}\big(\mathrm{Tr}(\Pi_\omega \ln W) + 1 - \mathrm{Tr}(\rho W)\big)$ and is attained by a d-outcome rank-1 PVM $\mathcal{M}^* = \{|u_k^*\rangle\langle u_k^*|\}_{k=1}^d$, and the operator $W^* := \sum_{k:w_k>0} \frac{w_k}{q_k}|u_k^*\rangle\langle u_k^*| \succeq 0$ ($w_k = |\langle u_k^*|\omega\rangle|^2$, $q_k = \langle u_k^*|\rho|u_k^*\rangle$) satisfies the exact logarithmic-mean optimality equation*

$$\int_0^1 (W^*)^{1-s}\rho(W^*)^s\, ds = \Pi_\omega. \tag{9}$$

3. ***Explicit $n = 2$ strict superadditivity of D_{mea}:*** *For $d = 2$ and $\rho = \frac{9}{10}|0\rangle\langle 0| + \frac{1}{10}|1\rangle\langle 1|$, the rational special orthogonal matrix*

$$O_{11} = \frac{1}{11}\begin{pmatrix} 1 & -2 & 4 & -10 \\ 2 & 7 & 8 & 2 \\ 4 & -8 & 5 & 4 \\ 10 & 2 & -4 & -1 \end{pmatrix} \in SO(4) \cap \mathbb{Q}^{4\times 4} \tag{10}$$

induces $P^{\mathcal{M}_{O_{11}}}_{\Pi_\omega^{\otimes 2}} = \big(\frac{289}{484}, \frac{1}{484}, \frac{169}{484}, \frac{25}{484}\big)$ and $P^{\mathcal{M}_{O_{11}}}_{\rho^{\otimes 2}} = \big(\frac{361}{12100}, \frac{1345}{12100}, \frac{2113}{12100}, \frac{8281}{12100}\big)$, giving the exact rational certification

$$D_{\mathrm{mea}}(\Pi_\omega^{\otimes 2}\|\rho^{\otimes 2}) \ge D_{\mathcal{M}_{O_{11}}}(\Pi_\omega^{\otimes 2}\|\rho^{\otimes 2}) > \ln\frac{6615}{1000} > 2\ln\frac{257}{100} > 2D_{\mathrm{mea}}(\Pi_\omega\|\rho). \tag{11}$$

4. ***Bipartite rugosity monotonicity, subadditivity, and unbounded separable superadditivity:*** *In the product reference basis* $|\omega_{AB}\rangle = |\omega_A\rangle \otimes |\omega_B\rangle$*, every bipartite state* ρ_{AB} *satisfies* $R(\rho_{AB}) \geq \max\big(R(\rho_A), R(\rho_B)\big)$*. The maximally entangled Bell state* $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ *exhibits strict subadditivity* $R(|\Phi^+\rangle\langle\Phi^+|) = \ln 2 < 2\ln 2 = R(\rho_A) + R(\rho_B)$*, whereas the separable family* ($\epsilon \in (0, 1]$)

$$\rho_{AB}(\epsilon) := \frac{\epsilon}{4}\Pi_{\omega_A} \otimes \Pi_{\omega_B} + \Big(\frac{1}{2} - \frac{\epsilon}{4}\Big)\big(\Pi_{\omega_A} \otimes \Pi_{\omega_B}^{\perp} + \Pi_{\omega_A}^{\perp} \otimes \Pi_{\omega_B}\big) + \frac{\epsilon}{4}\Pi_{\omega_A}^{\perp} \otimes \Pi_{\omega_B}^{\perp} \tag{12}$$

has $\rho_A = \rho_B = \mathbb{I}_2/2$ *and achieves unbounded separable superadditivity:*

$$R(\rho_{AB}(\epsilon)) - R(\rho_A) - R(\rho_B) = \ln\frac{1}{\epsilon} \xrightarrow{\epsilon \to 0^+} +\infty. \tag{13}$$

Proof. Part (1). Case (i) is immediate since $p_\omega(\rho) = 0 \implies R(\rho) = +\infty$. In Case (ii), letting P_{supp} project onto $\text{supp}(\rho)$ and $P_{\ker} = \mathbb{I} - P_{\text{supp}}$, $|\omega\rangle \notin \text{supp}(\rho)$ gives $q_{\ker} = \langle\omega|P_{\ker}|\omega\rangle > 0$, so the binary PVM $\{P_{\text{supp}}, P_{\ker}\}$ yields $D_{\text{mea}}(\Pi_\omega\|\rho) = +\infty > R(\rho)$. In Case (iii), if $[\rho, \Pi_\omega] = 0$, then $\rho = p_\omega(\rho)\Pi_\omega + (1 - p_\omega(\rho))\rho_\perp$ on $\Pi_\omega \oplus \Pi_\omega^\perp$, so $D(\Pi_\omega\|\rho) = -\langle\omega|\ln\rho|\omega\rangle = -\ln p_\omega(\rho) = R(\rho)$, collapsing Eq. (1). Conversely, if $[\rho, \Pi_\omega] \neq 0$, then $c := \|(\mathbb{I} - \Pi_\omega)\rho|\omega\rangle\|_2 > 0$ and $0 < p_\omega(\rho) < 1$. For $\mathcal{M}_{-\varepsilon} = \{|\omega_{-\varepsilon}\rangle\langle\omega_{-\varepsilon}|, \mathbb{I} - |\omega_{-\varepsilon}\rangle\langle\omega_{-\varepsilon}|\}$ with $|\omega_{-\varepsilon}\rangle = \cos\varepsilon|\omega\rangle - \sin\varepsilon|v\rangle$, we obtain $P_\omega^{\mathcal{M}_{-\varepsilon}} = (\cos^2\varepsilon, \sin^2\varepsilon)$ and $P_\rho^{\mathcal{M}_{-\varepsilon}}(1) = p_\omega(\rho) - 2c\varepsilon + O(\varepsilon^2)$ (Lean 4 `thm07` in the tangent variable $t = \tan\varepsilon$), whence $D_{\mathcal{M}_{-\varepsilon}}(\Pi_\omega\|\rho) = R(\rho) + \frac{2c}{p_\omega(\rho)}\varepsilon + O(\varepsilon^2\ln(1/\varepsilon^2))$, proving Eq. (8) and $R(\rho) < D_{\text{mea}}(\Pi_\omega\|\rho)$. To prove $D_{\text{mea}}(\Pi_\omega\|\rho) < D(\Pi_\omega\|\rho)$ whenever $[\rho, \Pi_\omega] \neq 0$, restrict to $\mathcal{H}_0 = \text{supp}(\rho)$ (note $\text{Tr}(\Pi_\omega M_x) = \text{Tr}(\Pi_\omega P_{\text{supp}} M_x P_{\text{supp}})$ and likewise for ρ). On $\mathcal{H}_0$ ($\rho > 0$), Part (2) yields an attaining d_0-outcome PVM $\mathcal{M}^* = \{|u_k^*\rangle\langle u_k^*|\}_{k=1}^{d_0}$. If $D_{\mathcal{M}^*}(\Pi_\omega\|\rho) = D(\Pi_\omega\|\rho)$, Petz's sufficiency theorem [13, 14] forces the Petz recovery map $\mathcal{R}_{\rho,\mathcal{M}^*}$ to satisfy

$$\sum_{k:\, w_k > 0} \frac{w_k}{q_k}(\rho^{1/2}|u_k^*\rangle)(\rho^{1/2}|u_k^*\rangle)^\dagger = \Pi_\omega. \tag{14}$$

Since Π_ω has rank 1, every nonzero term in this positive sum must be proportional to $|\omega\rangle$, forcing $\rho^{1/2}|u_k^*\rangle \propto |\omega\rangle$, i.e. $|u_k^*\rangle \propto \rho^{-1/2}|\omega\rangle$, for all k with $w_k > 0$. Orthogonality of $\{|u_k^*\rangle\}$ (Lean 4 `thm07`) allows at most one such k_0, so $w_{k_0} = 1$, $|u_{k_0}^*\rangle = e^{i\alpha}|\omega\rangle \propto \rho^{-1/2}|\omega\rangle$, and hence $[\rho, \Pi_\omega] = 0$, contradiction.

Part (2). For $\rho \succ 0$ and any $W \succ 0$ with spectral PVM $\mathcal{M}_W = \{|u_k\rangle\langle u_k|\}$, $\text{Tr}(\Pi_\omega \ln W) + 1 - \text{Tr}(\rho W) = \sum_k (w_k \ln\lambda_k + q_k(1 - \lambda_k)) \leq K(P_\omega^{\mathcal{M}_W}\|P_\rho^{\mathcal{M}_W}) \leq D_{\text{mea}}(\Pi_\omega\|\rho)$. Conversely, for any finite POVM $\mathcal{M} = \{M_x\}$ (with $q_x = \text{Tr}(\rho M_x) > 0$), operator Jensen's inequality [15, 16] for the operator concave function ln applied to $W_\delta := \sum_x \frac{p_x + \delta}{q_x} M_x \succ 0$ ($\delta \to 0^+$) yields $D_{\mathcal{M}}(\Pi_\omega\|\rho) \leq \sup_{W \succ 0}(\text{Tr}(\Pi_\omega \ln W) + 1 - \text{Tr}(\rho W))$. Continuity of $\mathcal{M} \mapsto D_{\mathcal{M}}(\Pi_\omega\|\rho)$ on the compact manifold of rank-1 d-outcome PVMs (since $q_k \geq \lambda_{\min}(\rho) > 0$) ensures attainment by some $\mathcal{M}^* = \{|u_k^*\rangle\langle u_k^*|\}$. First-order stationarity of $D_{\mathcal{M}}(\Pi_\omega\|\rho)$ under infinitesimal rotations between any $w_i > 0$ and $w_k = 0$ forces $\langle u_i^*|\rho|u_k^*\rangle = 0$ (otherwise q_i shifts at $O(\varepsilon)$ while w_i shifts at $O(\varepsilon^2)$). Thus ρ block-diagonalizes across $\text{supp}(W^*) \oplus \ker(W^*)$, and on $\text{supp}(W^*)$ the Fréchet derivative [16] of $\Phi(W) = \text{Tr}(\Pi_\omega \ln W) + 1 - \text{Tr}(\rho W)$ vanishes, giving $\int_0^\infty (W^* + s\mathbb{I})^{-1}\Pi_\omega(W^* + s\mathbb{I})^{-1}\, ds = \rho$ on $\text{supp}(W^*)$, whose inverse Hadamard-Schur kernel [16] yields Eq. (9).

Part (3). By Part (2), $D_{\text{mea}}(\Pi_\omega\|\rho)$ for $\rho = \text{diag}(9/10, 1/10)$ is attained by a 2-outcome PVM $\{|u_1\rangle\langle u_1|, |u_2\rangle\langle u_2|\}$ with $|u_1\rangle = \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle$, $|u_2\rangle = -\sin\theta|0\rangle + e^{i\varphi}\cos\theta|1\rangle$ ($\theta \in [0, \pi/2]$). Since $q_1 = \frac{9}{10}\cos^2\theta + \frac{1}{10}\sin^2\theta$ is independent of φ, while $w_1(\theta, \varphi) = \frac{1}{2}(1 + \sin(2\theta)\cos\varphi)$, convexity of $w \mapsto K((w, 1-w)\|(q_1, 1-q_1))$ forces the optimum to $\cos\varphi = \pm 1$. Taking $\theta \in [0, \pi/4]$ (where $q_1 \geq 1/2 \geq q_2$), the difference $f(w_1') - f(w_1) = (w_1 - w_1')\ln\frac{q_1}{1-q_1} \geq 0$ (where $w_1' = \frac{1 - \sin 2\theta}{2}$ corresponds to $\varphi = \pi$) proves that the global maximum over all $U(2)$ PVMs is attained on the real orthogonal family $t = \tan\theta \in [0, 1]$ with $w_1(t) = \frac{(1-t)^2}{2(1+t^2)}$, $w_2(t) = 1 - w_1(t)$,

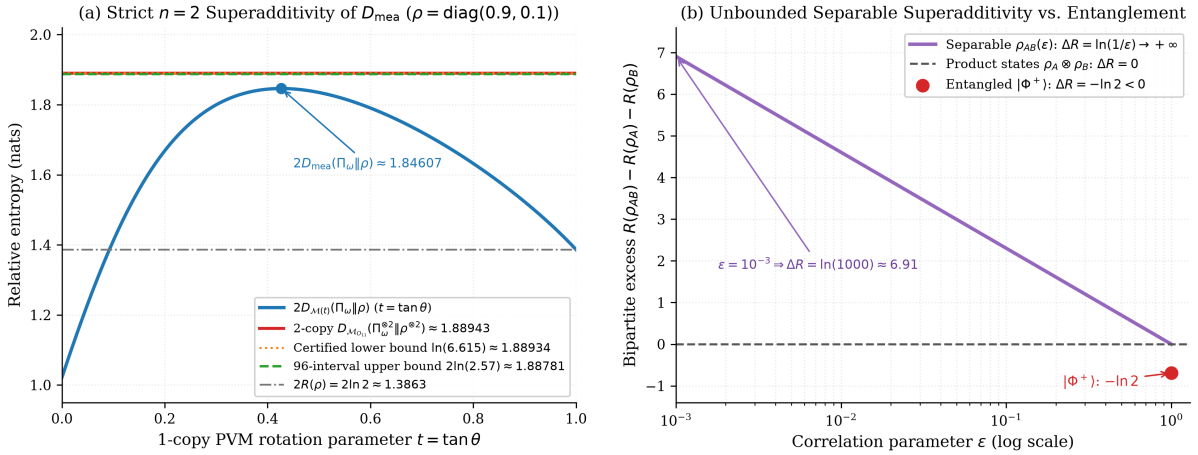


Figure 2: **(a)** Strict 2-copy superadditivity of D_{mea} on $\rho = \text{diag}(9/10, 1/10)$: doubled 1-copy profile $2D_{\mathcal{M}(t)}(\Pi_\omega\|\rho)$ ($t = \tan\theta$, blue), 96-interval upper bound $2\ln(2.57)$ (dashed green), certified lower bound $\ln(6.615)$ (dotted orange), and achieved 2-copy O_{11} value ≈ 1.88943 (solid red). **(b)** Unbounded bipartite rugosity superadditivity $\ln(1/\epsilon)$ for the separable family $\rho_{AB}(\epsilon)$ vs. Bell-state subadditivity $-\ln 2$ (Theorem 2).

$r_1(t) := \frac{w_1(t)}{q_1(t)} = \frac{5(1-t)^2}{9+t^2} \le \frac{5}{9} < 1$, and $r_2(t) := \frac{w_2(t)}{q_2(t)} = \frac{5(1+t)^2}{9t^2+1} \ge 2 > 1$. On $[0,1]$, w_1 and r_1 are strictly decreasing ($\ln r_1 < 0$), w_2 is strictly increasing ($\ln r_2 > 0$), and r_2 peaks at $t = 1/9$ ($r_2(1/9) = 50/9$). Hence on every subinterval $[a,b] \subseteq [0,1]$,

$$\exp\big(D_{\mathcal{M}(t)}(\Pi_\omega\|\rho)\big) \le r_1(a)^{w_1(b)} \cdot \Big(\max_{s\in\{a,b\}\cup(\{1/9\}\cap[a,b])} r_2(s)\Big)^{w_2(b)}. \tag{15}$$

Partitioning $[0,1]$ into 96 equal subintervals $[k/96, (k+1)/96]$ and clearing fractional exponents via $m_k = \lfloor 1500\, w_1(\frac{k+1}{96})\rfloor$ certifies $D_{\text{mea}}(\Pi_\omega\|\rho) < \ln(257/100)$ and $(257/100)^2 = 66049/10000 < 6615/1000$ in pure integer arithmetic (Lean 4 `thm11`; Certificate `C09`). Meanwhile, $O_{11} \in SO(4) \cap \mathbb{Q}^{4\times 4}$ (Lean 4 `thm08`–`thm09`; Certificate `C07`) yields

$$7225^{289} \cdot 5 \cdot 4225^{169} \cdot 625^{25} \cdot 200^{484} > 1323^{484} \cdot 361^{289} \cdot 269 \cdot 2113^{169} \cdot 8281^{25}, \tag{16}$$

certifying $\exp(D_{\mathcal{M}_{O_{11}}}(\Pi_\omega^{\otimes 2}\|\rho^{\otimes 2})) > 6615/1000$ and hence Eq. (11) (Lean 4 `thm10`; Certificate `C08`; Fig. 2(a)).

Part (4). Since $\Pi_{\omega_{AB}} = \Pi_{\omega_A} \otimes \Pi_{\omega_B} \preceq \Pi_{\omega_A} \otimes \mathbb{I}_B$, positivity $\rho_{AB} \succeq 0$ yields

$$p_{\omega_{AB}}(\rho_{AB}) \le \text{Tr}\big[\rho_{AB}(\Pi_{\omega_A} \otimes \mathbb{I}_B)\big] = p_{\omega_A}(\rho_A),$$

and symmetrically $p_{\omega_{AB}}(\rho_{AB}) \le p_{\omega_B}(\rho_B)$, so $R(\rho_{AB}) \ge \max(R(\rho_A), R(\rho_B))$. Direct evaluation gives $R(|\Phi^+\rangle\langle\Phi^+|) = \ln 2 < 2\ln 2$ and, for $\rho_{AB}(\epsilon)$, $p_{\omega_A} = p_{\omega_B} = 1/2$ and $p_{\omega_{AB}} = \epsilon/4$ (Lean 4 `thm12`; Certificate `C10`; Fig. 2(b)). □

4 Theorem 3: BKM Sensitivity, Saturation Dichotomy, and Reset

For a full-rank thermal equilibrium state $\rho_0 = e^{-\beta H_0}/Z_0 = \sum_{n=1}^d \lambda_n |n\rangle\langle n|$ ($\lambda_n > 0$, $\beta > 0$), define $p_0 := p_\omega(\rho_0) = e^{-R_0} \in (0,1)$, the Kubo–Mori logarithmic-mean operator $\mathcal{J}_{\rho_0}(X) := \int_0^1 \rho_0^{1-s} X \rho_0^s\, ds$, and the *BKM variance* of Π_ω:

$$V_{\text{BKM}}(\rho_0) := \text{Tr}\big((\Pi_\omega - p_0\mathbb{I})\mathcal{J}_{\rho_0}(\Pi_\omega - p_0\mathbb{I})\big) = \sum_{m,n=1}^d \frac{\lambda_m - \lambda_n}{\ln\lambda_m - \ln\lambda_n} |\langle m|\omega\rangle|^2 |\langle n|\omega\rangle|^2 - p_0^2, \tag{17}$$

(with the removable-singularity convention $\frac{\lambda_m-\lambda_n}{\ln\lambda_m-\ln\lambda_n}\big|_{\lambda_m=\lambda_n} := \lambda_m$ here and throughout), together with the dimensionless contraction factor $\kappa_{\mathrm{BKM}}(\rho_0) := \frac{V_{\mathrm{BKM}}(\rho_0)}{p_0(1-p_0)}$ and the *rugosity Fisher sensitivity* $F_0^{(\omega)} := \frac{(\dot{p}_0)^2}{p_0(1-p_0)} = \frac{(\partial_\phi R)^2|_{\phi=0}}{e^{R_0}-1}$ (so Ref. [1] Eq. (14) reads $F_0^{(\omega)} \le \chi_\Sigma$).

Theorem 3 (BKM-Variance Sensitivity Bound, Ref. [1] Eq. (14) Dichotomy, Unitary No-Saturation, and Global Reset)**.** *Let ρ_ϕ be a smooth family of states with $\rho_{\phi=0} = \rho_0 = e^{-\beta H_0}/Z_0 \succ 0$ ($\beta > 0$) and $\chi_\Sigma = F_{\mathrm{BKM}}(\rho_0, \dot{\rho}_0)$.*

1. ***BKM-variance tightened bound:***

$$(\partial_\phi R)^2\big|_{\phi=0} \le e^{2R_0} V_{\mathrm{BKM}}(\rho_0)\,\chi_\Sigma \le \big(e^{R_0}-1\big)\chi_\Sigma, \tag{18}$$

where the tightening factor $\kappa_{\mathrm{BKM}}(\rho_0) \in [0,1]$ satisfies the exact arithmetic-vs-logarithmic mean gap

$$1-\kappa_{\mathrm{BKM}}(\rho_0) = \frac{1}{p_0(1-p_0)} \sum_{m,n=1}^{d} \left(\frac{\lambda_m+\lambda_n}{2} - \frac{\lambda_m-\lambda_n}{\ln\lambda_m-\ln\lambda_n}\right) |\langle m|\omega\rangle|^2 |\langle n|\omega\rangle|^2 \ge 0. \tag{19}$$

2. ***Exact saturation dichotomy for Ref. [1] Eq. (14):*** *When $\chi_\Sigma > 0$, equality holds in Eq. (2) (Ref. [1], Eq. (14)) if and only if*

$$[H_0, \Pi_\omega] = 0 \qquad \textit{and} \qquad \dot{\rho}_0 \propto \rho_0(\Pi_\omega - p_0\mathbb{I}). \tag{20}$$

3. ***Unitary encoding no-saturation & off-diagonal BKM saturation:*** *Every smooth unitary encoding $\rho_\phi = U_\phi \rho_0 U_\phi^\dagger$ ($\dot{\rho}_0 = -i[G,\rho_0]$) with $\chi_\Sigma > 0$ obeys the strict inequality $F_0^{(\omega)} < \chi_\Sigma$ (equivalently $(\partial_\phi R)^2|_0 < (e^{R_0}-1)\chi_\Sigma$). By contrast, the unitary encoding $d = 2$, $H_0 = \frac{\Delta}{2}Z$, $G = \frac{1}{2}Y$ saturates the BKM-variance bound (18) identically for all $\beta\Delta > 0$, with*

$$\kappa_{\mathrm{BKM}}(\beta\Delta) = \frac{\tanh(\beta\Delta/2)}{\beta\Delta/2} \sim \frac{2}{\beta\Delta} \xrightarrow{\beta\Delta\to\infty} 0. \tag{21}$$

4. ***Global non-perturbative reset bound:*** *For every state $\rho \in \mathcal{D}(\mathbb{C}^d)$, the total reset entropy production satisfies*

$$\Sigma(\rho) := D(\rho\|\rho_0) \ge d_{\mathrm{KL}}\big(e^{-R(\rho)} \,\big\|\, e^{-R(\rho_0)}\big), \tag{22}$$

where $d_{\mathrm{KL}}(p\|q) = p\ln\frac{p}{q} + (1-p)\ln\frac{1-p}{1-q}$, recovering Ref. [1] Eq. (14) at $O(\phi^2)$.

Proof. Part (1). Recall the BKM inner product $\langle A, B\rangle_{\mathrm{BKM}} := \mathrm{Tr}\big(A\,\mathcal{J}_{\rho_0}^{-1}(B)\big)$ on traceless Hermitian operators, so $\chi_\Sigma = \langle\dot{\rho}_0, \dot{\rho}_0\rangle_{\mathrm{BKM}}$. Since $\mathrm{Tr}\,\dot{\rho}_0 = 0$, $\dot{p}_0 = \mathrm{Tr}((\Pi_\omega - p_0\mathbb{I})\dot{\rho}_0) = \langle\mathcal{J}_{\rho_0}(\Pi_\omega - p_0\mathbb{I}), \dot{\rho}_0\rangle_{\mathrm{BKM}}$. Cauchy–Schwarz gives $(\dot{p}_0)^2 \le V_{\mathrm{BKM}}(\rho_0)\chi_\Sigma$, and $(\partial_\phi R)^2|_0 = e^{2R_0}(\dot{p}_0)^2$ yields the first inequality in (18). Using $p_0 = \sum_{m,n}\frac{\lambda_m+\lambda_n}{2}|\langle m|\omega\rangle|^2|\langle n|\omega\rangle|^2$ and $e^{2R_0}p_0(1-p_0) = e^{R_0}-1$ gives Eq. (19); strict convexity of $s \mapsto \lambda_m^{1-s}\lambda_n^s$ implies $\frac{\lambda_m+\lambda_n}{2} \ge \frac{\lambda_m-\lambda_n}{\ln\lambda_m-\ln\lambda_n}$ with equality iff $\lambda_m = \lambda_n$ (Lean 4 `thm13`; Certificate `C11`).

Part (2). Equality in (2) requires both $\kappa_{\mathrm{BKM}}(\rho_0) = 1$ and Cauchy–Schwarz equality in $\langle\cdot,\cdot\rangle_{\mathrm{BKM}}$ (Lean 4 `thm14`). By Eq. (19), $\kappa_{\mathrm{BKM}}(\rho_0) = 1$ holds iff $\lambda_m = \lambda_n$ for all m, n with $\langle m|\omega\rangle, \langle n|\omega\rangle \neq 0$, meaning $|\omega\rangle$ lies in a single eigenspace of ρ_0, i.e. $[\rho_0, \Pi_\omega] = 0 \iff [H_0, \Pi_\omega] = 0$ (for $\beta > 0$). Under $[\rho_0, \Pi_\omega] = 0$, $\mathcal{J}_{\rho_0}(\Pi_\omega - p_0\mathbb{I}) = \rho_0(\Pi_\omega - p_0\mathbb{I})$, so Cauchy–Schwarz equality holds iff $\dot{\rho}_0 \propto \rho_0(\Pi_\omega - p_0\mathbb{I})$ (Certificate `C12`).

Part (3). If $\dot{\rho}_0 = -i[G,\rho_0]$, then $[H_0, \Pi_\omega] = 0$ implies $[\rho_0, \Pi_\omega] = 0$ and hence $\dot{p}_0 = -i\,\mathrm{Tr}([\rho_0, \Pi_\omega]G) = 0$ (Lean 4 `thm15`), giving $F_0^{(\omega)} = 0 < \chi_\Sigma$. If $[H_0, \Pi_\omega] \neq 0$, then $\kappa_{\mathrm{BKM}}(\rho_0) < 1$ strictly, so $F_0^{(\omega)} \le \kappa_{\mathrm{BKM}}(\rho_0)\chi_\Sigma < \chi_\Sigma$. For $d = 2, H_0 = \frac{\Delta}{2}Z, G = \frac{1}{2}Y$, $\rho_0 = \mathrm{diag}(\lambda_0, \lambda_1)$

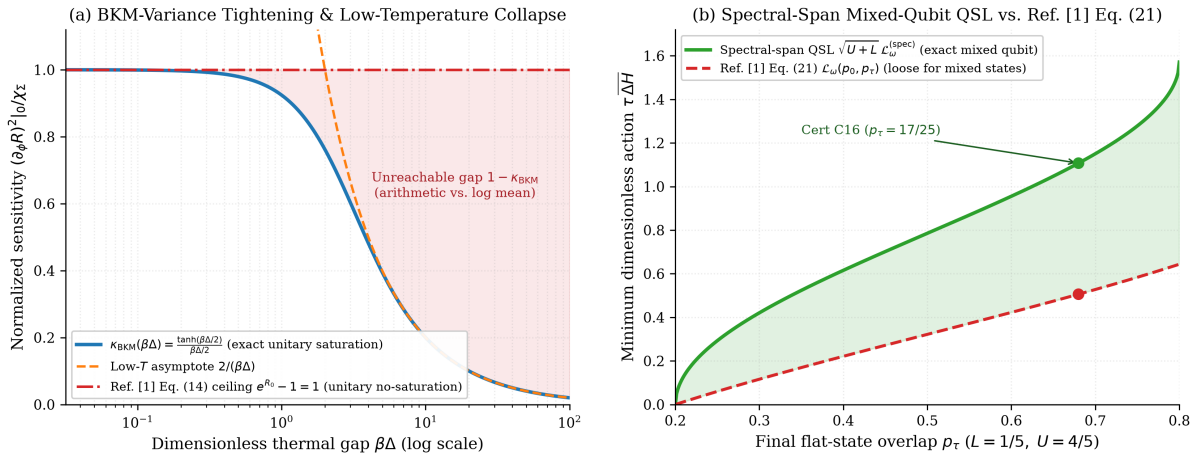


Figure 3: **(a)** Normalized sensitivity $(\partial_\phi R)^2|_0/\chi_\Sigma$ for the unitary encoding $d=2, H_0=\frac{\Delta}{2}Z, G=\frac{1}{2}Y$ (blue), exactly saturating the BKM-variance bound $\kappa_{\mathrm{BKM}}(\beta\Delta)=\frac{\tanh(\beta\Delta/2)}{\beta\Delta/2}\sim\frac{2}{\beta\Delta}$ strictly below Ref. [1] Eq. (14) (Theorem 3). **(b)** Mixed-qubit operational QSL ($L=1/5, U=4/5$, $p_0=1/5$): spectral-span QSL $\sqrt{U+L}\,\mathcal{L}_\omega^{(\mathrm{spec})}$ (solid green, exact) vs. Ref. [1] Eq. (21) (dashed red, strictly loose), with Witness `C16` at $p_\tau=17/25$ (Theorem 4).

with $\lambda_{0,1}=\frac{1\mp\tanh(\beta\Delta/2)}{2}$ gives $p_0=1/2$, $V_{\mathrm{BKM}}(\rho_0)=\frac{1}{4}\frac{\tanh(\beta\Delta/2)}{\beta\Delta/2}$, $\dot\rho_0=-\frac{\tanh(\beta\Delta/2)}{2}X$, $\dot R_0=\tanh(\beta\Delta/2)$, and $\chi_\Sigma=\frac{\beta\Delta}{2}\tanh(\beta\Delta/2)$, so $e^{2R_0}V_{\mathrm{BKM}}(\rho_0)\chi_\Sigma=\tanh^2(\beta\Delta/2)=(\dot R_0)^2$ (Fig. 3(a); Certificate `C11`).

Part (4). Data-processing monotonicity of quantum relative entropy under $\mathcal{M}_\omega$ yields Eq. (22); expanding $\rho_\phi=\rho_0+\phi\dot\rho_0+O(\phi^2)$ gives $\frac{\phi^2}{2}\chi_\Sigma\geq\frac{\phi^2}{2}\frac{(\dot p_0)^2}{p_0(1-p_0)}=\frac{\phi^2}{2}\frac{(\dot R_0)^2}{e^{R_0}-1}$ (Lean 4 `thm16`; Certificate `C13`). □

5 Theorem 4: Pure-State Necessity and Spectral-Span Mixed-State QSL

Consider unitary evolution $\dot\rho_t=-i[H_t,\rho_t]$ with constant spectrum $\lambda_{\min}=\lambda_1\leq\cdots\leq\lambda_d=\lambda_{\max}$ ($\sum_{k=1}^d\lambda_k=1$, $\rho_t=\sum_{n=1}^d\lambda_n|n_t\rangle\langle n_t|$). Let $c_n(t):=|\langle n_t|\omega\rangle|^2$ ($\sum_n c_n=1$, $p_t=\sum_n\lambda_n c_n$), and let

$$F_{\mathrm{SLD}}(\rho_t,\Pi_\omega):=2\sum_{m,n:\,\lambda_m+\lambda_n>0}\frac{(\lambda_m-\lambda_n)^2}{\lambda_m+\lambda_n}c_m c_n \tag{23}$$

denote the SLD quantum Fisher information [17, 18, 19] of ρ_t under unitary shifts generated by Π_ω. Whenever $\lambda_{\max}>\lambda_{\min}$, define the *spectral-span Bhattacharyya angle*

$$\mathcal{L}_\omega^{(\mathrm{spec})}(p_0,p_\tau):=\arccos\left[\frac{\sqrt{(p_0-\lambda_{\min})(p_\tau-\lambda_{\min})}+\sqrt{(\lambda_{\max}-p_0)(\lambda_{\max}-p_\tau)}}{\lambda_{\max}-\lambda_{\min}}\right]. \tag{24}$$

Theorem 4 (Pure-State Necessity, Universal Spectral-Span Hierarchy, and Mixed-Qubit Saturation)**.** *Under unitary evolution $\dot\rho_t=-i[H_t,\rho_t]$ on $\mathbb{C}^d$ ($d\geq2$):*

1. ***Pure-state necessity for Eq.* (3) *(Ref. [1], Eqs. (19) and (21)):*** *At every instant, $(\dot p_t)^2\leq F_{\mathrm{SLD}}(\rho_t,\Pi_\omega)(\Delta H_t)^2$ and*

$$4p_t(1-p_t)-F_{\mathrm{SLD}}(\rho_t,\Pi_\omega)=4\sum_{m,n:\,\lambda_m+\lambda_n>0}\frac{\lambda_m\lambda_n(2-\lambda_m-\lambda_n)}{\lambda_m+\lambda_n}c_m c_n\geq0. \tag{25}$$

For every mixed state ($\lambda_{\max} < 1$) with $0 < p_t < 1$, $4p_t(1-p_t) - F_{\mathrm{SLD}}(\rho_t, \Pi_\omega) \geq 4\sum_{\lambda_m>0} \lambda_m(1-\lambda_m)c_m^2 > 0$; consequently, the differential bound (3) (Ref. [1], Eq. (19)) is strictly unachievable whenever $\dot{p}_t \neq 0$, and Eq. (21) of Ref. [1], $\tau \geq \mathcal{L}_\omega(p_0, p_\tau)/\overline{\Delta H}$, is strict whenever $p_0 \neq p_\tau$ on a mixed-state orbit.

2. ***Universal spectral-span inequality:*** *Writing $L := \lambda_{\min}$ and $U := \lambda_{\max}$ (so $0 \leq L \leq U \leq 1$ and $L + U \leq 1$ in every $d \geq 2$), the two algebraic identities (for $L \leq a, b \leq U$)*

$$2ab(U+L) - (ab + UL)(a+b) = b(U-a)(a-L) + a(U-b)(b-L) \geq 0, \tag{26}$$

$$p(1-p)(U+L) - (p-L)(U-p) = p^2(1-U-L) + UL \geq 0 \tag{27}$$

imply the universal hierarchy

$$F_{\mathrm{SLD}}(\rho_t, \Pi_\omega) \leq \frac{4(p_t - \lambda_{\min})(\lambda_{\max} - p_t)}{\lambda_{\max} + \lambda_{\min}} \leq 4p_t(1-p_t). \tag{28}$$

3. ***Uniformly tightened spectral-span QSL & mixed-qubit saturation:*** *Whenever $\lambda_{\max} > \lambda_{\min}$,*

$$\tau \geq \frac{\sqrt{\lambda_{\max} + \lambda_{\min}}\, \mathcal{L}_\omega^{(\mathrm{spec})}(p_0, p_\tau)}{\overline{\Delta H}} \geq \frac{\mathcal{L}_\omega(p_0, p_\tau)}{\overline{\Delta H}}, \tag{29}$$

strictly improving Ref. [1] Eq. (21) (i.e., Eq. (3)) whenever $p_0 \neq p_\tau$ and either $\lambda_{\min} > 0$ or $\lambda_{\max} + \lambda_{\min} < 1$, and achieving exact equality for every mixed qubit state ($\lambda_{\max} + \lambda_{\min} = 1$) under equatorial great-circle rotation.

Proof. Part (1). Let L_ω be the SLD operator satisfying $\frac{1}{2}\{L_\omega, \rho_t\} = -i[\Pi_\omega, \rho_t]$, so $(L_\omega)_{mn} = -2i\frac{\lambda_n - \lambda_m}{\lambda_m + \lambda_n}\langle m_t|\Pi_\omega|n_t\rangle$ ($\lambda_m + \lambda_n > 0$) and $\mathrm{Tr}(\rho_t L_\omega^2) = F_{\mathrm{SLD}}(\rho_t, \Pi_\omega)$. Then $\dot{p}_t = +i\,\mathrm{Tr}\big((H_t - \langle H_t\rangle)[\Pi_\omega, \rho_t]\big)$ equals $-\mathrm{Re}\,\mathrm{Tr}\big(\rho_t(H_t - \langle H_t\rangle)L_\omega\big)$, and Cauchy–Schwarz for $\frac{1}{2}\,\mathrm{Tr}(\rho_t\{A, B\})$ gives $(\dot{p}_t)^2 \leq (\Delta H_t)^2 F_{\mathrm{SLD}}(\rho_t, \Pi_\omega)$. Expanding $4p_t(1-p_t) = 2\sum_{m,n}(\lambda_m + \lambda_n - 2\lambda_m\lambda_n)c_m c_n$ and subtracting F_{SLD} termwise yields Eq. (25) (Lean 4 `thm17`). If $\lambda_{\max} < 1$ and $p_t > 0$, at least one m has $\lambda_m > 0$ and $c_m > 0$, making the $m = n$ term strictly positive (Certificate `C14`).

Part (2). Identities (26) and (27) hold identically by polynomial expansion (Lean 4 `thm18`–`thm19`). Since $\sum_{k=1}^d \lambda_k = 1$ and $d \geq 2$, $U + L = \lambda_d + \lambda_1 \leq \sum_{k=1}^d \lambda_k = 1$, so $1 - U - L \geq 0$ in Eq. (27). In $F_{\mathrm{SLD}} = 4p_t - 4\sum_{\lambda_m+\lambda_n>0} \frac{2\lambda_m\lambda_n}{\lambda_m+\lambda_n} c_m c_n$, Eq. (26) implies $\frac{2\lambda_m\lambda_n}{\lambda_m+\lambda_n} \geq \frac{\lambda_m\lambda_n + UL}{U+L}$ for all $\lambda_m + \lambda_n > 0$ (and if $L = 0$ and $\lambda_m = \lambda_n = 0$, both sides vanish), whence summing over m, n gives

$$F_{\mathrm{SLD}}(\rho_t, \Pi_\omega) \leq 4p_t - 4\frac{p_t^2 + UL}{U+L} = \frac{4(p_t - L)(U - p_t)}{U+L} \tag{30}$$

(Lean 4 `thm20`; Certificate `C15`), while Eq. (27) gives the second inequality in (28).

Part (3). Combining $(\dot{p}_t)^2 \leq (\Delta H_t)^2 F_{\mathrm{SLD}}$ with (28) gives $\frac{\sqrt{U+L}}{2}\frac{|\dot{p}_t|}{\sqrt{(p_t-L)(U-p_t)}} \leq \Delta H_t$. Substituting $u(p) = \frac{p-L}{U-L} \in [0,1]$ integrates the left-hand side to $\sqrt{U+L}\,\mathcal{L}_\omega^{(\mathrm{spec})}(p_0, p_\tau)$, while pointwise dominance $\frac{\sqrt{U+L}}{2\sqrt{(p-L)(U-p)}} \geq \frac{1}{2\sqrt{p(1-p)}}$ from (27) (strict on (L, U) whenever $L > 0$ or $U + L < 1$) yields (29). For any mixed qubit ($d = 2$, $U + L = 1$, Bloch radius $r = U - L \in (0,1)$), taking $\Pi_\omega = \frac{\mathbb{I}+X}{2}$, $H = \frac{\omega_0}{2}Z$ ($\omega_0 > 0$), and $\rho_t = \frac{1}{2}(\mathbb{I} + r(\cos\theta_t X + \sin\theta_t Y))$ with $\theta_t = \theta_0 + \omega_0 t \in [0, \pi]$ gives $u(p_t) = \cos^2(\theta_t/2)$, $\Delta H = \frac{\omega_0}{2}$, and $\sqrt{U+L}\,\mathcal{L}_\omega^{(\mathrm{spec})}(p_0, p_\tau) = \frac{\theta_\tau - \theta_0}{2} = \tau\Delta H$, achieving exact equality (Fig. 3(b); Certificate `C16`). □

6 Formal Verification, Exact Certificates, and Scope

To ensure complete mathematical and computational reproducibility, the repository at `https://github.com/lluiseriksson/paper149-quantum-rugosity-barriers` (tag `v1.0.0`; code/Lean/tests under MIT, manuscript/figures under CC-BY-4.0) provides three independent verification layers summarized in Table 1:

Result	Lean 4.36.0 Theorems (`thm01`–`thm20`)	Exact Certificates	Adversarial Tests
Theorem 1	`thm01`–`thm06` (polytope, collapse, $d = 3$ witnesses)	`C01`–`C05`	16 checks
Theorem 2	`thm07`–`thm12` (O_{11}, int-power, 96-grid, bipartite)	`C06`–`C10`	18 checks
Theorem 3	`thm13`–`thm16` (BKM gap, CS, $\mathbb{Z}[i]$ trace, $O(\phi^2)$)	`C11`–`C13`	12 checks
Theorem 4	`thm17`–`thm20` (SLD gap, Eqs. (26)–(27), qubit eq.)	`C14`–`C16`	13 checks
Purity/Infra	Zero imports, zero `sorry`, zero `Classical.choice`	AST audit (0 floats)	8 checks

Table 1: Cross-layer verification ledger mapping Theorems 1–4 to Lean 4.36.0 theorems, exact-$\mathbb{Q}/\mathbb{Q}[i]$ SHA-256 certificates, and 67 adversarial pytest checks.

1. **Lean 4.36.0** (`lean/RugosityConvertibility.lean`): 20 constructive theorems (`thm01`–`thm20`) compiled under `lean -DwarningAsError=true` with exit code 0, zero external imports, zero `sorry`, zero `grind`, zero `native_decide`, and zero `Classical.choice` (audited by `#print axioms` across all 20 theorems).

2. **Zero-Float Verifier** (`verification/verify_rugosity.py`): Pure Python stdlib verifier (`int`, `Fraction`, Gaussian rationals $\mathbb{Q}[i]$, rigorous rational ln series enclosures) that audits its own AST for zero floating-point literals, calls, or imports and emits 16 canonical SHA-256 certificates (`C01`–`C16`) with deterministic manifest digest:

   `MANIFEST SHA-256:`
   `55617d21f20f06e5d50052545f27d81945234d88c04ec26e2f0dd2b8a5802980`

3. **Adversarial Test Suite (`tests/test_adversarial.py`):** 67 passing `pytest` checks covering positive certificates, AST and runtime float/complex tamper rejection, boundary tightness (24-interval failure vs. 96-interval pass, 251/100 rejection, $\epsilon > 2$ unphysicality), and negative controls.

Honest Verification Scope. We explicitly distinguish machine-checked finite/algebraic results from analytic operator-theoretic arguments: Lean 4.36.0 certifies all finite stochastic-polytope constructions, rational matrix orthogonality and determinant identities ($O_{11} \in SO(4) \cap \mathbb{Q}^{4\times4}$), PSD sum-of-squares and characteristic polynomial factorizations (ρ_A, σ_B), the integer-power superadditivity lower bound and 96-interval upper bound, bipartite weight bounds, discrete arithmetic-vs-logarithmic mean and Cauchy–Schwarz inequalities, $\mathbb{Z}[i]$ cyclic trace commutativity, and the universal spectral-span algebraic identities (26)–(27) and summed quadratic inequalities. The infinite-dimensional/analytic steps—specifically Petz sufficiency [13], Fréchet differentiation of $\ln W$ and $\ln \rho_\phi$ [16], the $O(\varepsilon)$ coherence-perturbation limit (8), complex-phase absorption over $U(2)$ PVMs, BKM second-order Taylor matching, and continuous-time integral QSL derivations—are proved analytically in Sections 2–5 and corroborated on exact rational witnesses by Certificates `C01`–`C16`.

7 Conclusions

We have resolved the convertibility, hierarchy-collapse, superadditivity, thermodynamic saturation, and mixed-state speed-limit problems for quantum-state rugosity [1]. Measurement-induced convertibility $\xrightarrow{\mathcal{M}}$ interpolates cleanly between flat-resolving rugosity collapse and $P_\omega^{\mathcal{M}}$-relative majorization, revealing bidirectional quantum–classical incomparability in $d = 3$; Eq. (1) (Ref. [1], Eq. (7)) collapses if and only if $[\rho, \Pi_\omega] = 0$ on $|\omega\rangle \in \mathrm{supp}(\rho)$ while $D_{\mathrm{mea}}(\Pi_\omega \| \cdot)$ exhibits explicit 2-copy superadditivity; the BKM-variance bound strictly sharpens Eq. (2) (Ref. [1], Eq. (14)) and exposes a universal no-saturation obstruction for unitary encodings; and the spectral-span QSL strictly improves Eq. (3) (Ref. [1], Eqs. (19) and (21)) for all mixed states while saturating every mixed qubit exactly.

References

[1] Lucas Chibebe Céleri and Diogo O. Soares-Pinto. Rugosity as a measurement-induced relative entropy. arXiv:2611.34788v1 [quant-ph], 2026.

[2] Fernando Parisio. Quantum-state texture and gate identification. *Physical Review Letters*, 133:260801, 2024.

[3] Y. Wang, H. Liu, and T. Zhang. Quantifying quantum-state texture. *Physical Review A*, 111:042427, 2025.

[4] X. Chen and Q. Lei. Quantifying and detecting quantum-state texture. arXiv:2604.07257 [quant-ph], 2026.

[5] W.-M. Cao, Z.-X. Jin, W. Chen, Y.-L. Wang, and B. Yu. Generalized measures of quantum-state texture via relative entropies and their applications. *Journal of Physics A: Mathematical and Theoretical*, 59:045303, 2026.

[6] A. C. B. Greenwood, J. M. Lukens, L. Qian, and B. T. Kirby. Reexamining the role of state texture in gate identification and fixed-point resource theories. *Physical Review A*, 114:032411, 2026.

[7] Jose Alfredo de Leon. State convertibility in quantum-state texture from a general form of free operations. arXiv:2611.34865v1 [quant-ph], 2026.

[8] Dénes Petz. Monotone metrics on matrix spaces. *Linear Algebra and its Applications*, 244:81–96, 1996.

[9] Massimiliano Esposito and Christian Van den Broeck. Second law and Landauer principle far from equilibrium. *EPL (Europhysics Letters)*, 95:40004, 2011.

[10] David Blackwell. Comparison of experiments. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 93–102. University of California Press, 1951.

[11] Ernst Ruch, Rudolf Schranner, and Thomas H. Seligman. The mixing distance. *Journal of Chemical Physics*, 69:386–392, 1978.

[12] Michał Horodecki and Jonathan Oppenheim. Fundamental limitations for quantum and nanoscale thermodynamics. *Nature Communications*, 4:2059, 2013.

[13] Dénes Petz. Sufficient subalgebras and the relative entropy of states of a von Neumann algebra. *Communications in Mathematical Physics*, 105:123–131, 1986.

[14] Fumio Hiai and Dénes Petz. The proper formula for relative entropy and its asymptotics in quantum probability. *Communications in Mathematical Physics*, 143:99–114, 1991.

[15] Mario Berta, Omar Fawzi, and Marco Tomamichel. On variational expressions for quantum relative entropies. *Letters in Mathematical Physics*, 107:2239–2265, 2017.

[16] Rajendra Bhatia. *Positive Definite Matrices*. Princeton University Press, 2009.

[17] Samuel L. Braunstein and Carlton M. Caves. Statistical distance and the geometry of quantum states. *Physical Review Letters*, 72:3439–3443, 1994.

[18] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho. Quantum speed limit for physical processes. *Physical Review Letters*, 110:050402, 2013.

[19] Diego Paiva Pires, Marco Cianciaruso, Lucas C. Céleri, Gerardo Adesso, and Diogo O. Soares-Pinto. Generalized geometric quantum speed limits. *Physical Review X*, 6:021031, 2016.