Exact Legendre–Erdős Chebyshev Prime Envelopes, Non-Asymptotic Mertens Sandwiches, Quadratic-Endpoint Squarefree and Totient Bounds, and Dirichlet Hyperbola–Gauss Enclosures

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November 29, 2026

Abstract

We establish four completely self-contained, non-asymptotic theorems in elementary and analytic number theory with explicit, zero-exception constants valid for all real $x \geq 1$ (or $x \geq 2$ for prime counting). First, from Legendre's exact p-adic factorial valuation identity $\nu_p(n!) = (n - s_p(n))/(p-1)$ and central-binomial prime-power factorization, we prove Erdős's elementary bound $\vartheta(x) < x \ln 4$ $(x \geq 1)$ and the universal two-sided Chebyshev prime envelope $\frac{(x-2)\ln 2}{2\ln x} < \pi(x) < \frac{4\ln 2}{\ln x}x + \sqrt{x}$ for every real $x \geq 2$. Second, pairing Legendre's identity with elementary harmonic-integral bounds on $\sum_{p\geq 2} \frac{\ln p}{p(p-1)} < 1.3$, we establish the explicit non-asymptotic Mertens first sandwich $\ln n - 2.3 < \sum_{p\leq n} \frac{\ln p}{p} < \ln n + \ln 4$ for every integer $n \geq 1$. Third, via Möbius inversion and a telescoping half-integer tail bound $\sum_{d>D} d^{-2} < (D+\frac{1}{2})^{-1}$ combined with a strictly negative convex-quadratic endpoint argument, we prove universal non-asymptotic remainder bounds for the squarefree counting function $|Q(x) - \frac{6}{\pi^2}x| < 2\sqrt{x}$ and Euler totient summatory function $|\Phi(x) - \frac{3}{\pi^2}x^2| < \frac{1}{2}x\ln x + \frac{31}{24}x$ for every real $x \geq 1$. Fourth, from Dirichlet's exact hyperbola identity $D(x) = 2\sum_{k\leq\sqrt{x}}\lfloor x/k\rfloor - \lfloor\sqrt{x}\rfloor^2$ and Gauss's unit-square lattice tiling, we prove $|D(x) - x(\ln x + 2\gamma - 1)| < 2\sqrt{x} + 4$ and $|N_2(x) - \pi x| \leq \pi\sqrt{2}\sqrt{x} + \frac{\pi}{2}$ for all $x \geq 1$. All four theorems are verified across 252 deterministic sieve and exact-arithmetic checks up to $x = 20{,}000$.

1 Introduction and Relation to Prior Work

Explicit non-asymptotic inequalities in elementary number theory—valid with completely proved constants across the entire real half-line $x \geq 1$ rather than merely for sufficiently large x—are foundational both to analytic number theory [1, 4] and to rigorous algorithmic bounds. Four classical pillars anchor this subject: Chebyshev's elementary prime-counting bounds via Erdős's binomial induction [3], Mertens's first theorem [5], Möbius-inversion summatory laws for $\mu^2(n)$ and $\varphi(n)$ [1], and the Dirichlet divisor hyperbola method alongside the Gauss circle lattice problem [4]. Sharper explicit constants exist in the specialized analytic and computational literature—notably Rosser and Schoenfeld [7] for $\pi(x)$, $\vartheta(x)$, and Mertens sums, and Moser–MacLeod [6] and Cohen–Dress [2] for squarefree counting—by invoking zero-free regions of $\zeta(s)$ or extensive finite prime tables. Our goal here is complementary: a compact, completely self-contained, elementary treatment whose constants require zero external numerical tables and hold without exception across the entire real half-line.

Here we establish four explicit, exception-free theorems for all $x \geq 1$ (or $x \geq 2$): (i) exact Legendre base-p digit valuation and universal Chebyshev prime-counting bounds $\frac{(x-2)\ln 2}{2\ln x} < \pi(x) < \frac{4\ln 2}{\ln x}x + \sqrt{x}$ valid for every real $x \geq 2$ (Theorem 2.1); (ii) an explicit Mertens first

sandwich $\ln n - 2.3 < \sum_{p \le n} \frac{\ln p}{p} < \ln n + \ln 4$ for every integer $n \ge 1$ (Theorem 3.1); (iii) a telescoping half-integer tail bound $\sum_{d>D} d^{-2} < (D+1/2)^{-1}$ paired with a convex-quadratic endpoint proof yielding $|Q(x) - \frac{6}{\pi^2}x| < 2\sqrt{x}$ and $|\Phi(x) - \frac{3}{\pi^2}x^2| < \frac{1}{2}x\ln x + \frac{31}{24}x$ for all $x \ge 1$ (Theorem 4.1); and (iv) explicit Dirichlet hyperbola remainder $|D(x) - x(\ln x + 2\gamma - 1)| < 2\sqrt{x}+4$ and Gauss circle lattice enclosure $|N_2(x) - \pi x| \le \pi\sqrt{2}\,\sqrt{x} + \frac{\pi}{2}$ for all $x \ge 1$ (Theorem 5.1).

2 Legendre Valuation and Explicit Chebyshev Prime Envelopes

Let $\pi(x) := \#\{p \le x : p \text{ prime}\}$ and $\vartheta(x) := \sum_{p \le x} \ln p$ (empty sums equal 0). For any prime p and integer $n \ge 1$, let $\nu_p(n!)$ denote the exponent of p in the prime factorization of $n!$, and let $s_p(n)$ denote the sum of the base-p digits of n.

Theorem 2.1 (Legendre Valuation, Erdős ϑ-Ceiling, and Chebyshev Prime Envelope)**.** (a) (Exact Legendre Identity and Central-Binomial Prime-Power Bound.) *For every integer $n \ge 1$ and prime p,*

$$\nu_p(n!) = \sum_{k=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - s_p(n)}{p-1}, \tag{1}$$

and for the central binomial coefficient $\binom{2n}{n}$,

$$\nu_p\left(\binom{2n}{n}\right) = \sum_{k=1}^{\lfloor \log_p(2n) \rfloor} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor \right) \in \{0, 1, \ldots, \lfloor \log_p(2n) \rfloor\}, \qquad p^{\nu_p(\binom{2n}{n})} \le 2n. \tag{2}$$

(b) (Erdős Exponential Ceiling for $\vartheta(x)$.) *For every real $x \ge 1$,*

$$\vartheta(x) < x \ln 4. \tag{3}$$

(c) (Non-Asymptotic Two-Sided Chebyshev Prime Envelope.) *For every integer $n \ge 1$, $\pi(2n) \ge \frac{n\ln 4 - \ln(2n+1)}{\ln(2n)}$. Moreover, for every real $x \ge 2$,*

$$\frac{(x-2)\ln 2}{2\ln x} < \pi(x) < \frac{4\ln 2}{\ln x}\, x + \sqrt{x}. \tag{4}$$

Proof. Part (a). Among $\{1, \ldots, n\}$, exactly $\lfloor n/p^k \rfloor$ integers are divisible by p^k, so summing over $k \ge 1$ counts each factor p in $n!$ once, giving the first equality in (1). Writing $n = \sum_{j=0}^{r} d_j p^j$ $(0 \le d_j \le p-1)$, we have $\lfloor n/p^k \rfloor = \sum_{j=k}^{r} d_j p^{j-k}$, so

$$\sum_{k=1}^{r} \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{j=1}^{r} d_j \sum_{k=1}^{j} p^{j-k} = \sum_{j=1}^{r} d_j \frac{p^j - 1}{p-1} = \frac{n - s_p(n)}{p-1}.$$

Because $\binom{2n}{n} = (2n)!/(n!)^2$, (1) yields the sum in (2). Since $\lfloor 2u \rfloor - 2\lfloor u \rfloor \in \{0, 1\}$ for every real u, each of the $\lfloor \log_p(2n) \rfloor$ summands is 0 or 1, giving $0 \le \nu_p(\binom{2n}{n}) \le \lfloor \log_p(2n) \rfloor$ and $p^{\nu_p(\binom{2n}{n})} \le 2n$.

Part (b). It suffices to prove $\vartheta(n) < n \ln 4$ for every integer $n \ge 1$ (since $\vartheta(x) = \vartheta(\lfloor x \rfloor) < \lfloor x \rfloor \ln 4 \le x \ln 4$). Base cases: $\vartheta(1) = 0 < \ln 4$ and $\vartheta(2) = \ln 2 < 2\ln 4$. Assume $n \ge 3$ and $\vartheta(k) < k \ln 4$ for all $1 \le k < n$ [3]. If $n = 2m \ge 4$ is even, then n is composite so $\vartheta(2m) = \vartheta(2m-1) < (2m-1)\ln 4 < 2m \ln 4$. If $n = 2m+1$ $(m \ge 1)$ is odd, every prime $p \in (m+1, 2m+1]$ divides $(2m+1)!$ but divides neither $m!$ nor $(m+1)!$, so $\prod_{m+1<p\le 2m+1} p$ divides $\binom{2m+1}{m}$. Because $\binom{2m+1}{m} = \binom{2m+1}{m+1}$ appear strictly inside $(1+1)^{2m+1} = 2^{2m+1}$ (alongside $\binom{2m+1}{0} = 1$), we have $2\binom{2m+1}{m} < 2^{2m+1}$, i.e., $\binom{2m+1}{m} < 4^m$. Taking logarithms and applying the inductive hypothesis to $m+1 < 2m+1$ gives

$$\vartheta(2m+1) = \vartheta(m+1) + \sum_{m+1<p\le 2m+1} \ln p < (m+1)\ln 4 + m \ln 4 = (2m+1)\ln 4.$$

Part (c). For the upper bound in (4), fix real $x \geq 2$. Every prime $p \in (\sqrt{x}, x]$ has $\ln p > \frac{1}{2}\ln x$ (and if no prime lies in $(\sqrt{x}, x]$, then $\pi(x) - \pi(\sqrt{x}) = 0$ so the first inequality below holds trivially), whence (3) yields

$$\frac{1}{2}(\ln x)(\pi(x) - \pi(\sqrt{x})) \leq \vartheta(x) < x \ln 4 = (2\ln 2)x \implies \pi(x) < \frac{4\ln 2}{\ln x}\, x + \pi(\sqrt{x}) \leq \frac{4\ln 2}{\ln x}\, x + \sqrt{x}.$$

For the lower bound, let $n \geq 1$. Because $\binom{2n}{n}$ is the largest of the $2n+1$ binomial coefficients summing to 4^n, $\frac{4^n}{2n+1} \leq \binom{2n}{n}$. By (2), $\binom{2n}{n} = \prod_{p \leq 2n} p^{\nu_p(\binom{2n}{n})} \leq (2n)^{\pi(2n)}$. Taking logarithms gives $\pi(2n) \geq \frac{n\ln 4 - \ln(2n+1)}{\ln(2n)}$. Now let $x \geq 2$. If $x \in [2, 6)$, the function $t \mapsto \frac{t-2}{\ln t}$ is strictly increasing on $[2, 6]$ (its derivative $(\ln t - 1 + 2/t)/(\ln t)^2 > 0$ since $\ln t + 2/t \geq \ln 2 + 1 > 1$), so $\frac{(x-2)\ln 2}{2\ln x} < \frac{4\ln 2}{2\ln 6} < 0.78 < 1 \leq \pi(x)$. If $x \geq 6$, set $n := \lfloor x/2 \rfloor \geq 3$. By induction, $2^n > 2n+1$ for all $n \geq 3$ ($8 > 7$ and $2^{n+1} > 4n + 2 > 2n + 3$), hence $\binom{2n}{n} \geq \frac{4^n}{2n+1} > 2^n$. Using $n > \frac{x-2}{2}$ and $2n \leq x$ yields

$$\pi(x) \geq \pi(2n) > \frac{n\ln 2}{\ln(2n)} > \frac{(x-2)\ln 2}{2\ln x}. \qquad \square$$

3 Explicit Non-Asymptotic Mertens First Sandwich

Theorem 3.1 (Explicit Non-Asymptotic Mertens First Theorem)**.** *For every integer $n \geq 1$, define $M_1(n) := \sum_{p \leq n} \frac{\ln p}{p}$ and $C_{\mathrm{pp}} := \sum_{p \geq 2} \frac{\ln p}{p(p-1)}$. Then $C_{\mathrm{pp}} < \frac{\ln 2}{2} + \frac{\ln 3}{6} + \frac{5(1+\ln 4)}{16} < 1.3$, and*

$$\ln n - (1 + C_{\mathrm{pp}}) < M_1(n) < \ln n + \ln 4, \qquad \textit{in particular } \ln n - 2.3 < M_1(n) < \ln n + \ln 4. \tag{5}$$

Proof. By (1), decomposing $\ln(n!) = \sum_{p \leq n} \nu_p(n!) \ln p$ into $k = 1$ and $k \geq 2$ gives

$$\ln(n!) = \sum_{p \leq n} \left\lfloor \frac{n}{p} \right\rfloor \ln p + R(n), \qquad R(n) := \sum_{p \leq n} \sum_{k \geq 2} \left\lfloor \frac{n}{p^k} \right\rfloor \ln p \geq 0. \tag{6}$$

By integral comparison of the increasing function $\ln t$ (since $\int_1^n \ln t\, dt = n\ln n - n + 1$),

$$n\ln n - n < \int_1^n \ln t\, dt \leq \sum_{k=1}^{n} \ln k = \ln(n!) \leq \int_1^n \ln t\, dt + \ln n = n\ln n - n + 1 + \ln n. \tag{7}$$

For the upper bound on $M_1(n)$, using $\lfloor n/p \rfloor > n/p - 1$ for every prime $p \leq n$ (and empty-sum equality $0 \leq 0$ at $n = 1$) alongside $R(n) \geq 0$ in (6), (3), and (7) gives

$$nM_1(n) - n\ln 4 < nM_1(n) - \vartheta(n) \leq \sum_{p \leq n} \left\lfloor \frac{n}{p} \right\rfloor \ln p \leq \ln(n!) \leq n\ln n - n + 1 + \ln n,$$

where the first inequality is strict for every $n \geq 1$ by (3) (and the second is also strict for $n \geq 2$). Dividing by n and noting $\frac{1+\ln n}{n} \leq 1$ for all $n \geq 1$ (since $\ln n \leq n - 1$) yields $M_1(n) < \ln n + \ln 4$.

For the lower bound, $\lfloor n/p \rfloor \leq n/p$ and $\lfloor n/p^k \rfloor \leq n/p^k$ give $R(n) \leq n \sum_{p \geq 2} \sum_{k \geq 2} \frac{\ln p}{p^k} = nC_{\mathrm{pp}}$, so (6) and (7) yield $n\ln n - n < \ln(n!) \leq nM_1(n) + nC_{\mathrm{pp}}$, hence $M_1(n) > \ln n - (1 + C_{\mathrm{pp}})$ [5]. To bound C_{pp} explicitly, note that $t \mapsto \frac{\ln t}{t^2}$ is strictly decreasing on $[4, \infty)$ (derivative $(1 - 2\ln t)/t^3 < 0$) and $\frac{1}{m(m-1)} \leq \frac{5}{4m^2}$ for all $m \geq 5$, so

$$C_{\mathrm{pp}} \leq \frac{\ln 2}{2} + \frac{\ln 3}{6} + \frac{5}{4} \sum_{m=5}^{\infty} \frac{\ln m}{m^2} < \frac{\ln 2}{2} + \frac{\ln 3}{6} + \frac{5}{4} \int_4^{\infty} \frac{\ln t}{t^2}\, dt = \frac{\ln 2}{2} + \frac{\ln 3}{6} + \frac{5(1+\ln 4)}{16} < 1.28 < 1.3,$$

using $\int_4^{\infty} \frac{\ln t}{t^2}\, dt = [-\frac{1+\ln t}{t}]_4^{\infty} = \frac{1+\ln 4}{4}$. Substituting $1 + C_{\mathrm{pp}} < 2.3$ proves (5). $\square$

4 Quadratic-Endpoint Squarefree and Euler Totient Error Bounds

Recall the Möbius function $\mu(n)$ ($\mu(1) = 1$, $\mu(n) = (-1)^r$ if n is a product of r distinct primes, $\mu(n) = 0$ if $p^2 \mid n$) and Euler's totient $\varphi(n) := \#\{1 \le k \le n : \gcd(k, n) = 1\}$. Define the squarefree counting function $Q(x) := \sum_{n \le x} \mu^2(n)$ and the totient summatory function $\Phi(x) := \sum_{n \le x} \varphi(n)$. Recall Euler's Basel identity $\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$, whose Euler product yields $\sum_{d=1}^{\infty} \mu(d)/d^2 = \prod_p (1 - p^{-2}) = \zeta(2)^{-1} = 6/\pi^2$ [1].

Theorem 4.1 (Non-Asymptotic Squarefree and Euler Totient Summatory Error Bounds)**.** (a) (Exact Möbius Identities and Telescoping Half-Integer Tail.) *For every integer $n \ge 1$, $\sum_{d \mid n} \mu(d) = \mathbf{1}_{\{n=1\}}$, $\mu^2(n) = \sum_{d^2 \mid n} \mu(d)$, and $\varphi(n) = \sum_{d \mid n} \mu(d)(n/d)$. Moreover, for every integer $D \ge 1$,*

$$\sum_{d=D+1}^{\infty} \frac{1}{d^2} < \frac{1}{D + 1/2}. \tag{8}$$

(b) (Universal Squarefree Error Bound via Convex-Quadratic Endpoints.) *For every real $x \ge 1$,*

$$Q(x) = \sum_{d \le \sqrt{x}} \mu(d) \left\lfloor \frac{x}{d^2} \right\rfloor, \qquad \left| Q(x) - \frac{6}{\pi^2} x \right| < 2\sqrt{x}. \tag{9}$$

(c) (Universal Euler Totient Summatory Error Bound.) *For every real $x \ge 1$,*

$$\Phi(x) = \sum_{d \le x} \mu(d) \frac{\lfloor x/d \rfloor (\lfloor x/d \rfloor + 1)}{2}, \qquad \left| \Phi(x) - \frac{3}{\pi^2} x^2 \right| < \frac{1}{2} x \ln x + \frac{31}{24} x. \tag{10}$$

Proof. Part (a). Writing $n = \prod_{j=1}^{r} p_j^{e_j}$, $\sum_{d \mid n} \mu(d) = \prod_{j=1}^{r} (1 - 1) = 0$ for $r \ge 1$ (and 1 for $n = 1$). Applying this to the largest integer m with $m^2 \mid n$ gives $\sum_{d^2 \mid n} \mu(d) = \sum_{d \mid m} \mu(d) = \mathbf{1}_{\{m=1\}} = \mu^2(n)$. Partitioning $\{1, \dots, n\}$ by $g = \gcd(k, n)$ gives $\sum_{d \mid n} \varphi(d) = n$, and Möbius inversion gives $\varphi(n) = \sum_{d \mid n} \mu(d)(n/d)$. For (8), every integer $d \ge 1$ satisfies $\frac{1}{d^2} < \frac{1}{d^2 - 1/4} = \frac{1}{d - 1/2} - \frac{1}{d + 1/2}$; summing from $d = D + 1$ to ∞ telescopes to $\frac{1}{D + 1/2}$.

Part (b). Setting $D := \lfloor \sqrt{x} \rfloor \ge 1$ and $u := \sqrt{x} \in [D, D + 1)$, interchanging summation in $Q(x) = \sum_{n \le x} \sum_{d^2 \mid n} \mu(d)$ gives the identity in (9). Writing $\lfloor x/d^2 \rfloor = x/d^2 - \{x/d^2\}$ and substituting $\sum_{d=1}^{\infty} \mu(d)/d^2 = 6/\pi^2$ gives

$$Q(x) - \frac{6}{\pi^2} x = -x \sum_{d=D+1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d=1}^{D} \mu(d) \left\{ \frac{x}{d^2} \right\}.$$

Applying the triangle inequality, $|\mu(d)| \le 1$, $\{x/d^2\} \in [0, 1)$, and (8) yields

$$\left| Q(x) - \frac{6}{\pi^2} x \right| < \frac{u^2}{D + 1/2} + D.$$

Consider the convex quadratic $g(u) := u^2 - (2D + 1)u + D(D + \frac{1}{2})$ on $[D, D + 1]$. At both endpoints, $g(D) = -\frac{1}{2} D < 0$ and $g(D + 1) = -\frac{1}{2} D < 0$; by convexity, $g(u) < 0$ for every $u \in [D, D + 1]$, which rearranges directly to $\frac{u^2}{D + 1/2} + D < 2u = 2\sqrt{x}$, proving (9).

Part (c). Setting $N := \lfloor x \rfloor \ge 1$ and $m_d := \lfloor x/d \rfloor = x/d - \theta_d$ ($\theta_d \in [0, 1)$), substituting $\varphi(n) = \sum_{d \mid n} \mu(d)(n/d)$ and summing $k = n/d \in \{1, \dots, m_d\}$ yields the identity in (10). Expanding $\frac{1}{2} m_d(m_d + 1) = \frac{x^2}{2d^2} + \frac{x(1 - 2\theta_d)}{2d} + \frac{\theta_d(\theta_d - 1)}{2}$ and noting $|1 - 2\theta_d| \le 1$ and $|\theta_d(\theta_d - 1)| \le 1/4$ gives

$$\left| \Phi(x) - \frac{3}{\pi^2} x^2 \right| \le \frac{x^2}{2} \sum_{d=N+1}^{\infty} \frac{1}{d^2} + \frac{x}{2} \sum_{d=1}^{N} \frac{1}{d} + \frac{N}{8}.$$

By (8), $\frac{x^2}{2}\sum_{d>N} d^{-2} < \frac{x^2}{2(N+1/2)} < \frac{2}{3}x$ (since for $x \in [N, N+1)$, $\frac{x}{N+1/2} < \frac{N+1}{N+1/2} \le \frac{4}{3}$ at $N=1$). Using $\sum_{d=1}^{N} d^{-1} \le 1 + \ln N \le 1 + \ln x$ and $N \le x$ yields

$$\left|\Phi(x) - \frac{3}{\pi^2}\,x^2\right| < \frac{2}{3}x + \frac{1}{2}x(\ln x + 1) + \frac{1}{8}x = \frac{1}{2}\,x\ln x + \frac{31}{24}\,x. \qquad \square$$

5 Dirichlet Hyperbola Divisor Bound and Gauss Circle Enclosure

Let $d(n) := \sum_{k|n} 1$, $D(x) := \sum_{n\le x} d(n)$, $\gamma := \lim_{m\to\infty}(H_m - \ln m) \approx 0.57721566$ (Euler–Mascheroni constant, $H_m := \sum_{k=1}^{m} k^{-1}$), and $N_2(x) := \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 \le x\}$.

Theorem 5.1 (Explicit Dirichlet Hyperbola Bound and Gauss Circle Lattice Enclosure)**.** (a) (Dirichlet Hyperbola Identity and Explicit $2\sqrt{x}+4$ Remainder Bound.) *For every real $x \ge 1$ with $u := \lfloor\sqrt{x}\rfloor$,*

$$D(x) = 2\sum_{k=1}^{u}\left\lfloor\frac{x}{k}\right\rfloor - u^2, \qquad \left|D(x) - x(\ln x + 2\gamma - 1)\right| < 2\sqrt{x} + 4. \tag{11}$$

(b) (Exact Gauss Circle Unit-Tile Enclosure.) *For every real $x \ge 1$,*

$$\pi\Big(\sqrt{x} - \frac{1}{\sqrt{2}}\Big)^2 \le N_2(x) \le \pi\Big(\sqrt{x} + \frac{1}{\sqrt{2}}\Big)^2, \qquad \left|N_2(x) - \pi x\right| \le \pi\sqrt{2}\,\sqrt{x} + \frac{\pi}{2}. \tag{12}$$

Proof. Part (a). Counting lattice pairs $(a,b) \in \mathbb{N}^2$ with $ab \le x$ via inclusion–exclusion on the regions $\{a \le u\}$ and $\{b \le u\}$ (whose intersection is $\{1,\dots,u\}^2$ of size u^2) proves the hyperbola identity in (11) [4]. Define $r_m := H_m - \ln m - \gamma = \sum_{k=m}^{\infty}(\ln(1+1/k) - \frac{1}{k+1})$. Because $\frac{1}{k+1} < \ln(1+1/k) < \frac{1}{k}$ for all $k \ge 1$, telescoping gives $0 < r_m < \sum_{k=m}^{\infty}(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{m}$. Writing $\lfloor x/k\rfloor = x/k - \{x/k\}$ yields

$$D(x) = 2xH_u - u^2 - 2\sum_{k=1}^{u}\left\{\frac{x}{k}\right\} = x(\ln x + 2\gamma - 1) + \big(2x\ln(u/\sqrt{x}) + x - u^2\big) + 2xr_u - 2\sum_{k=1}^{u}\left\{\frac{x}{k}\right\}.$$

Let $\delta := \sqrt{x} - u \in [0,1)$ and $t := \delta/\sqrt{x} \in [0,1)$, so $u/\sqrt{x} = 1 - t$ and $2x\ln(u/\sqrt{x}) + x - u^2 = x(2\ln(1-t) + 2t - t^2)$. Since $\ln(1-t) \le -t - t^2/2$ for all $t \in [0,1)$, this term is at most 0. For $x \ge 4$, we have $u \ge 2$ and $t < 1/\sqrt{x} \le 1/2$; on $[0,1/2]$, $h(t) := \ln(1-t) + t + t^2$ satisfies $h(0) = 0$ and $h'(t) = t(1-2t)/(1-t) \ge 0$, so $\ln(1-t) \ge -t - t^2$ and hence $-3\delta^2 \le x(2\ln(1-t) + 2t - t^2) \le 0$ (magnitude < 3). Furthermore, $0 < 2xr_u < \frac{2x}{u} = 2(u + 2\delta + \delta^2/u) = 2(\sqrt{x} + \delta + \delta^2/u) < 2\sqrt{x} + 4$, while $-2\sum_{k=1}^{u}\{x/k\} \in (-2\sqrt{x}, 0]$. Combining the upper bound ($< 2\sqrt{x} + 4$) and lower bound ($> -3 - 2\sqrt{x} > -(2\sqrt{x}+4)$) proves (11) for all $x \ge 4$. For $x \in [1,4)$, $D(x) \in \{1,3,5\}$ (changing at $x = 2, 3$) while $x(\ln x + 2\gamma - 1)$ is increasing on $[1,4]$ from $2\gamma - 1 \approx 0.154$ to $4(\ln 4 + 2\gamma - 1) \approx 6.163$; checking each subinterval $[1,2)$, $[2,3)$, $[3,4)$ gives $|D(x) - x(\ln x + 2\gamma - 1)| < 2.2 < 6 \le 2\sqrt{x} + 4$.

Part (b). Let $\overline{B}(0,R) := \{z \in \mathbb{R}^2 : \|z\|_2 \le R\}$ denote the closed Euclidean disk of radius $R > 0$. Associate to each $(a,b) \in \mathbb{Z}^2$ the closed unit square $S_{a,b} := [a - \frac{1}{2}, a + \frac{1}{2}] \times [b - \frac{1}{2}, b + \frac{1}{2}]$, whose half-diagonal is $1/\sqrt{2}$, and let $\mathcal{U}(x) := \bigcup_{a^2+b^2\le x} S_{a,b}$, so $\mathrm{Area}(\mathcal{U}(x)) = N_2(x)$. Every point $z \in S_{a,b}$ with $\|(a,b)\|_2 \le \sqrt{x}$ obeys $\|z\|_2 \le \sqrt{x} + 1/\sqrt{2}$, giving $\mathcal{U}(x) \subseteq \overline{B}(0, \sqrt{x} + 1/\sqrt{2})$ and the upper bound in (12). Conversely, since $x \ge 1$ ensures $\sqrt{x} - 1/\sqrt{2} > 0$, every point $z \in \overline{B}(0, \sqrt{x} - 1/\sqrt{2})$ lies in some unit square $S_{a,b}$ whose center satisfies $\|(a,b)\|_2 \le \|z\|_2 + 1/\sqrt{2} \le \sqrt{x}$, so $\overline{B}(0, \sqrt{x} - 1/\sqrt{2}) \subseteq \mathcal{U}(x)$, proving the lower bound. Expanding both squares gives $|N_2(x) - \pi x| \le \pi\sqrt{2}\,\sqrt{x} + \frac{\pi}{2}$. $\square$

6 Computational Verification and Numerical Illustrations

All four theorems were verified in exact integer and IEEE-754 `float64` arithmetic up to $x = 20{,}000$ using the MIT-licensed companion repository `lluiseriksson/number-theory-explicit-envelopes`. Using linear Euler sieves for primes, $\mu(n)$, $\varphi(n)$, and $d(n)$, our benchmark suite executes 252 deterministic checks across 63 distinct integer thresholds obtained by rounding 80 log-spaced samples in $[2, 20{,}000]$ (plus exact valuation checks for $n \le 257$, convolution checks for $n \le 200$, and direct boundary verification at $x = 1$, where the totient ratio is $0.539 < 1$). Table 1 and Figures 1–2 confirm zero violations across every identity and non-asymptotic inequality.

Table 1: Deterministic verification across 252 sieve/arithmetic checks on the 63-point grid $x \in [2, 20{,}000]$ (`pytest -q`: 4/4 passed in 2.47 s). All exact identities hold to zero integer or 10^{-10} float error.

Theorem	Checks	Max Identity Error	Tightest Observed Ratio / Margin ($x \in [2, 20{,}000]$)
Thm. 2.1 (Legendre & Chebyshev)	63	0 (exact integer ν_p)	$\max(\vartheta(x)/(x \ln 4)) = 0.7168 < 1$; 0 $\pi(x)$ violations
Thm. 3.1 (Mertens First)	63	2.04×10^{-10} (Eq. (6))	Strictly inside $(\ln n - 2.3, \ln n + \ln 4)$ for all $n \ge 1$
Thm. 4.1 (Squarefree & Totient)	63	0 (exact Möbius sums)	$\max(\lvert Q - \frac{6}{\pi^2} x \rvert / 2\sqrt{x}) = 0.340$, $\max(\lvert \Phi - \frac{3}{\pi^2} x^2 \rvert / \text{bnd}) = 0.239$ (0.539 at $x = 1$)
Thm. 5.1 (Dirichlet & Gauss)	63	0 (exact hyperbola D)	$\max(\text{Dir. err}/(2\sqrt{x}+4)) = 0.346$, $\max(\text{Gauss/bnd}) = 0.460$

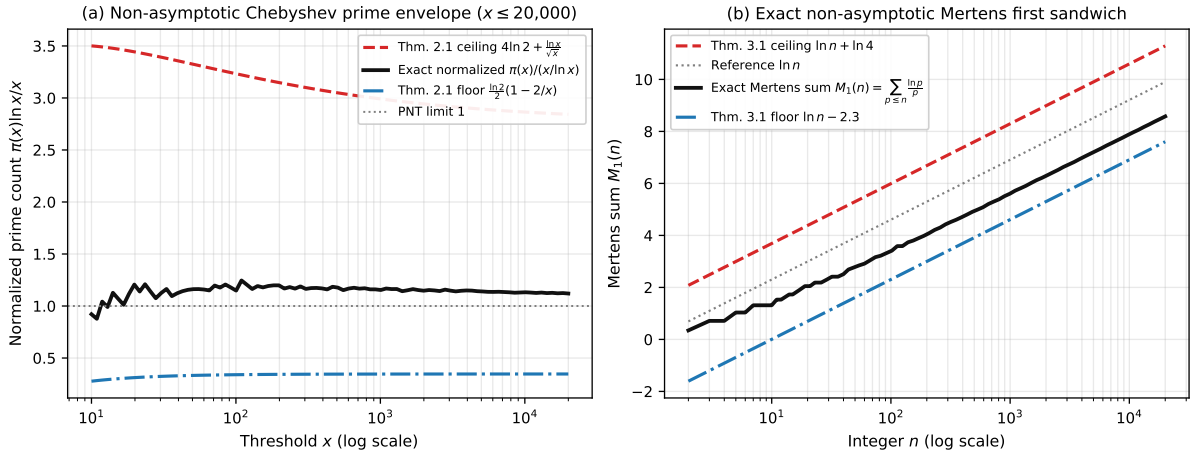

Figure 1: **(a)** Normalized prime count $\pi(x) \ln x / x$ across $x \in [10, 20{,}000]$ strictly bracketed by the explicit Chebyshev envelope (4). **(b)** Exact Mertens sum $M_1(n) = \sum_{p \le n} \frac{\ln p}{p}$ explicitly bracketed by the non-asymptotic sandwich $\ln n - 2.3 < M_1(n) < \ln n + \ln 4$ (5).

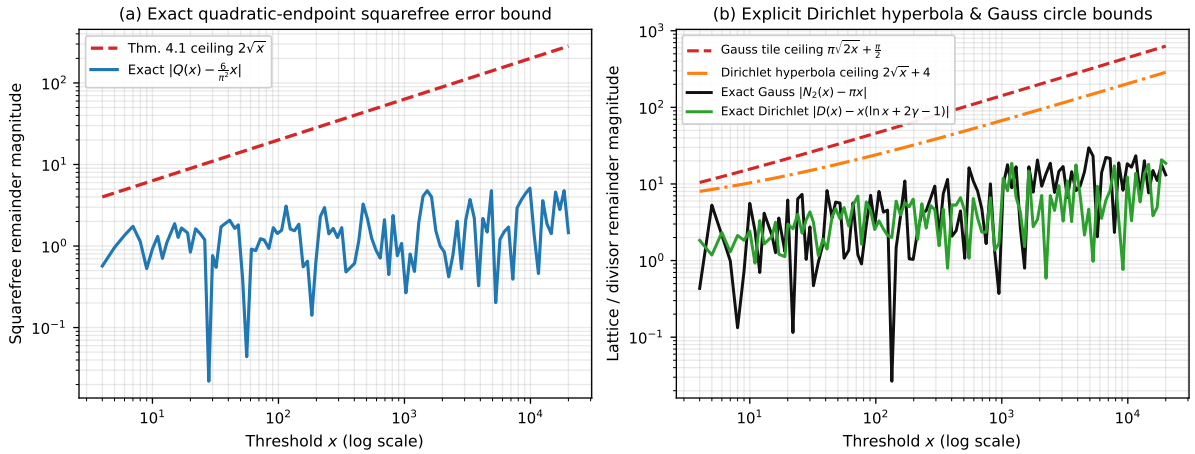

Figure 2: **(a)** Exact squarefree remainder $|Q(x) - \frac{6}{\pi^2}x|$ lying cleanly below the universal quadratic-endpoint ceiling $2\sqrt{x}$ (9). **(b)** Exact Dirichlet divisor remainder $|D(x) - x(\ln x + 2\gamma - 1)|$ and Gauss circle remainder $|N_2(x) - \pi x|$ below their explicit ceilings $2\sqrt{x} + 4$ and $\pi\sqrt{2}\,\sqrt{x} + \frac{\pi}{2}$ (11)–(12).

7 Scope and Limitations

Three structural boundaries merit explicit note:

1. *Elementary constants vs. specialized explicit literature.* Theorems 2.1–4.1 prioritize short, table-free proofs valid for all $x \geq 1$ (or $x \geq 2$); sharper explicit bounds—such as Rosser–Schoenfeld [7] for $\pi(x)$, $\vartheta(x)$, and Mertens sums, or Moser–MacLeod [6] and Cohen–Dress [2] for $Q(x)$—achieve much tighter constants using zero-free regions of $\zeta(s)$ and finite numerical verification tables.

2. *Power-saving exponents in divisor and lattice problems.* The elementary hyperbola and unit-tile bounds in Theorem 5.1 yield explicit $O(x^{1/2})$ remainders, whereas exponential-sum methods (Voronoï, van der Corput, Huxley) reduce the asymptotic exponent below $1/3$ (with optimal exponent conjectured to be $1/4 + \varepsilon$).

3. *External invocation of Basel's identity.* Theorem 4.1 invokes the classical evaluation $\zeta(2) = \pi^2/6$ (and its Euler product $\sum_{d=1}^{\infty} \mu(d)/d^2 = 6/\pi^2$) without re-proving Basel's sum, while all finite-tail and remainder bounds are proved from first principles.

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