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EXACT ALL-COUPLING FRADKIN–SHENKER YANG–MILLS–HIGGS FREE-ENERGY ENCLOSURES, CONSTRUCTIVE HIGGS-PHASE MASS GAP, AND β-INDEPENDENT FUNDAMENTAL STRING BREAKING IN d-DIMENSIONAL SU(N) LATTICE GAUGE THEORY

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Abstract. The d-dimensional ($d \geq 2$) fundamental SU(N) lattice Yang–Mills–Higgs (Fradkin–Shenker / Osterwalder–Seiler) model couples Wilson plaquettes at inverse gauge coupling $\beta \geq 0$ to unit-length fundamental Higgs fields $V_x \in \mathrm{SU}(N)$ at hopping coupling $\kappa \geq 0$. We establish three self-contained, cluster-expansion-free non-perturbative theorems spanning the entire (β, κ) phase diagram:

(1) *Universal β-independent fundamental string breaking for all* SU(N)*, $d \geq 2$, $\beta \geq 0$, $\kappa > 0$:* Via a four-line Rayleigh–Ritz single-site Higgs-screening trial state on the strictly positive Osterwalder–Seiler transfer matrix, we prove the exact static-quark potential ceiling $V_{\mathrm{fund},L_s}(\mathrm{SU}(N), d, \beta, \kappa; R) \leq 2\ln(1/u_1(\mathrm{SU}(N), \kappa)) + 8(d-1)\kappa < \infty$ uniformly in separation R, torus size L_s, and *all* gauge couplings $\beta \in [0, \infty)$, proving $\sigma_{\mathrm{fund}}(\mathrm{SU}(N), d, \beta, \kappa) = 0$ identically whenever $\kappa > 0$.

(2) *Constructive Higgs-phase mass gap (all $\beta > 0$, large κ) and confinement-corner Bakry–Émery log-Sobolev inequality:* In unitary gauge for SU(2), an exact single-link Schwinger–Dyson integration-by-parts identity combined with an explicit antipodal tail bound certifies Dobrushin's contraction parameter $\alpha_{\mathrm{Dob}}(d, \beta, \kappa) < 1$ for *every* fixed $\beta > 0$ at sufficiently large κ, yielding a unique infinite-volume Gibbs state and constructive Higgs-phase mass gap $m_{\mathrm{Higgs}}(d, \beta, \kappa) \geq -\ln \alpha_{\mathrm{Dob}}(d, \beta, \kappa) > 0$. In the small-$(\beta, \kappa)$ confinement corner $16(d-1)\beta + 2\kappa < N^2$, exact Bakry–Émery curvature $\mathrm{Ric} + \mathrm{Hess}(-S_{\mathrm{unit}}) \geq (\frac{N}{2} - \frac{8(d-1)\beta+\kappa}{N}) g_{\mathrm{HS}} > 0$ yields a volume-uniform log-Sobolev inequality for all SU(N).

(3) *All-coupling free-energy enclosures with exact classical and one-loop matching:* A d-way axial-gauge Hölder decomposition and unitary-gauge Gibbs–Bogoliubov lower bound enclose $f^{\mathrm{YMH}}_{\mathbb{T}^d_L}(\mathrm{SU}(N), \beta, \kappa)$ across all $(\beta, \kappa) \in [0, \infty)^2$, matching *both* the exact classical energy $\beta + \frac{2}{d-1}\kappa$ and the exact one-loop $-\frac{N^2-1}{d-1}\ln\beta$ coefficient along every weak-coupling ray $\kappa = r\beta$ ($r > 0$). Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`HiggsFradkinShenkerCerts.lean`) verify all bounds.

1. Introduction and Honest Scope

In their landmark study of the Higgs mechanism on a lattice, Osterwalder and Seiler [9] and Fradkin and Shenker [7] introduced the fixed-length fundamental Yang–Mills–Higgs model, coupling SU(N) gauge link variables $U_b \in \mathrm{SU}(N)$ ($b \in E$) to site-based group-valued (N-flavor matrix) fundamental Higgs fields $V_x \in \mathrm{SU}(N)$ ($x \in V$; this completely breaks SU(N) in unitary gauge for all $N \geq 2$, and for $N = 2$ coincides identically via $\mathrm{SU}(2) \cong S^3$ with the single unit complex doublet $\phi_x \in \mathbb{C}^2$, $|\phi_x| = 1$, through $\Re(\phi_x^\dagger U_{xy} \phi_y) = \frac{1}{2}\mathrm{Re}\,\mathrm{Tr}(V_x^\dagger U_{xy} V_y)$). On a finite d-dimensional ($d \geq 2$) periodic torus $\mathbb{T}^d_L := (\mathbb{Z}/L\mathbb{Z})^d$ or free-boundary hypercube $\Lambda_{L-1} := \{0, \ldots, L-1\}^d$, the

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gauge-invariant action and partition function are

$$\begin{aligned} S_{\mathrm{YMH}}(U,V;\beta,\kappa) &:= \frac{\beta}{N}\sum_{p\in\mathcal{P}} \mathrm{Re}\,\mathrm{Tr}(U_p) + \frac{\kappa}{N}\sum_{b=(x,y)\in E} \mathrm{Re}\,\mathrm{Tr}(V_x^\dagger U_b V_y), \\ Z^{\mathrm{YMH}}(\beta,\kappa) &:= \int_{\mathrm{SU}(N)^{|E|}\times\mathrm{SU}(N)^{|V|}} e^{S_{\mathrm{YMH}}(U,V;\beta,\kappa)}\, dU\, dV, \end{aligned} \tag{1}$$

where $dU := \prod_{b\in E} dU_b$ and $dV := \prod_{x\in V} dV_x$ are products of normalized Haar measure on $\mathrm{SU}(N)$ and $\beta,\kappa \geq 0$. Under the exact Haar-preserving unitary-gauge change of variables $W_b := V_x^\dagger U_b V_y \in \mathrm{SU}(N)$ $(b=(x,y))$, every plaquette holonomy satisfies $W_p = V_{x_0}^\dagger U_p V_{x_0}$ (so $\mathrm{Re}\,\mathrm{Tr}\,W_p = \mathrm{Re}\,\mathrm{Tr}\,U_p$) and the V-integrals factor out identically as 1, giving the exact unitary-gauge representation

$$\begin{aligned} S_{\mathrm{unit}}(W;\beta,\kappa) &:= \frac{\beta}{N}\sum_{p\in\mathcal{P}} \mathrm{Re}\,\mathrm{Tr}(W_p) + \frac{\kappa}{N}\sum_{b\in E} \mathrm{Re}\,\mathrm{Tr}(W_b), \\ Z^{\mathrm{YMH}}(\beta,\kappa) &= \int_{\mathrm{SU}(N)^{|E|}} e^{S_{\mathrm{unit}}(W;\beta,\kappa)}\, dW. \end{aligned} \tag{2}$$

Historically, rigorous analyses of the confinement and Higgs regimes of (1) have relied on heavy polymer / cluster expansions [9, 7, 10]. In this paper, we establish three self-contained, cluster-expansion-free non-perturbative theorems covering all three regimes of the Fradkin–Shenker diagram:

- *Universal β-independent fundamental string breaking* (Section 2): A four-line Rayleigh–Ritz single-site Higgs-screening trial state proves the bound $V_{\mathrm{fund},L_s} \leq 2\ln(1/u_1(\mathrm{SU}(N),\kappa)) + 8(d-1)\kappa < \infty$ uniformly in R, L_s, and *every* $\beta \in [0,\infty)$, establishing $\sigma_{\mathrm{fund}} = 0$ for all $\kappa > 0$.
- *Constructive Higgs-phase mass gap and confinement-corner Bakry–Émery LSI* (Section 3): For SU(2) at *any* fixed $\beta > 0$ and large κ, an exact single-link Schwinger–Dyson integration-by-parts identity plus antipodal tail bound certifies Dobrushin's uniqueness condition $\alpha_{\mathrm{Dob}}(d,\beta,\kappa) < 1$ [5, 6, 11] and an explicit mass gap $m_{\mathrm{Higgs}} \geq -\ln\alpha_{\mathrm{Dob}} > 0$; in the small-$(\beta,\kappa)$ confinement corner $16(d-1)\beta + 2\kappa < N^2$, exact Bakry–Émery curvature [2, 3] proves a volume-uniform log-Sobolev inequality for all $\mathrm{SU}(N)$.
- *All-coupling free-energy enclosures with exact one-loop matching* (Section 4–5): A d-way axial-gauge Hölder upper bound and unitary-gauge Gibbs–Bogoliubov lower bound match both $\beta + \frac{2}{d-1}\kappa$ and $-\frac{N^2-1}{d-1}\ln\beta$ along every ray $\kappa = r\beta$ $(r > 0)$, verified across 24 zero-quadrature 140-digit certificates and a zero-axiom Lean 4.36.0 kernel.

Remark 1.1 (Honest Scope and Non-Claims). All results hold for fixed-lattice $(a = 1)$ $\mathrm{SU}(N)$ Yang–Mills–Higgs lattice gauge theory. Theorems 2.1 and 4.1 hold at *all* couplings $(\beta,\kappa) \in [0,\infty)^2$ ($\kappa > 0$ for Theorem 2.1), whereas Theorem 3.1 establishes a constructive transfer-matrix mass gap in the large-κ SU(2) Higgs region ($\alpha_{\mathrm{Dob}} < 1$) and a volume-uniform log-Sobolev inequality (Langevin spectral gap and single-plaquette variance ceiling) in the small-(β,κ) confinement corner ($16(d-1)\beta + 2\kappa < N^2$), without bridging the intermediate crossover corridor via analytic continuation or constructing a 4D continuum limit. This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Universal β-Independent Fundamental String Breaking in $d \geq 2$

Fix $d \geq 2$, $N \geq 2$, $\beta \geq 0$, $\kappa > 0$, and a spatial torus $\mathbb{T}^{d-1}_{L_s}$ ($L_s \geq 3$) with spatial link set E_s and site set V_s. Define the single-group partition function and mean character

$$c_0(\mathrm{SU}(N), \alpha) := \int_{\mathrm{SU}(N)} e^{\frac{\alpha}{N} \operatorname{Re}\operatorname{Tr} U}\, dU,$$

$$u_1(\mathrm{SU}(N), \alpha) := \partial_\alpha \ln c_0(\mathrm{SU}(N), \alpha) = \frac{1}{c_0(\mathrm{SU}(N), \alpha)} \int_{\mathrm{SU}(N)} \frac{1}{N} \operatorname{Re}\operatorname{Tr}(U)\, e^{\frac{\alpha}{N} \operatorname{Re}\operatorname{Tr} U}\, dU.$$

In temporal gauge $U_{(x;d)} = I_N$, the Osterwalder–Seiler transfer matrix [9] on $L^2(\mathrm{SU}(N)^{|E_s|} \times \mathrm{SU}(N)^{|V_s|})$ is the integral operator $\widetilde{\mathcal{T}}_d := \mathcal{V}^{1/2} \mathcal{K} \mathcal{V}^{1/2}$, where

$$\mathcal{V}(U, V) := \exp\left(\frac{\beta}{N} \sum_{p \in \mathcal{P}_s} \operatorname{Re}\operatorname{Tr}(U_p) + \frac{\kappa}{N} \sum_{b=(x,y) \in E_s} \operatorname{Re}\operatorname{Tr}(V_x^\dagger U_b V_y) \right),$$

$$\mathcal{K}\big((U, V), (U', V')\big) := \prod_{\ell \in E_s} e^{\frac{\beta}{N} \operatorname{Re}\operatorname{Tr}(U_\ell (U'_\ell)^\dagger)} \prod_{x \in V_s} e^{\frac{\kappa}{N} \operatorname{Re}\operatorname{Tr}(V_x^\dagger V'_x)}.$$

Since every Peter–Weyl coefficient $c_\lambda(\alpha) := \frac{1}{d_\lambda} \int_{\mathrm{SU}(N)} e^{\frac{\alpha}{N} \operatorname{Re}\operatorname{Tr} U} \overline{\chi_\lambda(U)}\, dU$ is strictly positive for $\alpha > 0$ (and $c_0(0) = 1$, $c_{\lambda \neq 0}(0) = 0$ at $\alpha = 0$), $\widetilde{\mathcal{T}}_d$ is a strictly positive-kernel, positive semi-definite compact self-adjoint operator. By Perron–Frobenius, its top eigenvalue $\Lambda_0 := \|\widetilde{\mathcal{T}}_d\| > 0$ is simple with a unique, strictly positive, unit-norm eigenfunction $\Omega_{L_s} > 0$, which is invariant under all time-independent spatial gauge transformations $(U_{(x,y)}, V_x) \mapsto (g_x U_{(x,y)} g_y^\dagger, g_x V_x)$.

For any pair of distinct sites 0 and Re_1 ($1 \leq R \leq L_s - 1$), let $\mathcal{H}_d^{\mathrm{fund}(0,R)} \subset L^2(\mathrm{SU}(N)^{|E_s|} \times \mathrm{SU}(N)^{|V_s|})$ denote the static fundamental quark–antiquark charge sector transforming in the fundamental representation at site 0, the antifundamental representation at site Re_1, and invariantly at all other sites, with orthogonal projection $P_{\mathrm{fund}(0,R)}$ and sector transfer operator $\mathcal{T}_d^{\mathrm{fund}(0,R)} := P_{\mathrm{fund}(0,R)} \widetilde{\mathcal{T}}_d P_{\mathrm{fund}(0,R)} \geq 0$. The exact static fundamental quark–antiquark ground-state potential is $V_{\mathrm{fund},L_s}(\mathrm{SU}(N), d, \beta, \kappa; R) := -\ln\big(\|\mathcal{T}_d^{\mathrm{fund}(0,R)}\| / \Lambda_0\big) \geq 0$.

Theorem 2.1 (Universal β-Independent Fundamental String Breaking for All $\mathrm{SU}(N)$)**.** *For every $d \geq 2$, every $N \geq 2$, every $\beta \in [0, \infty)$, every $\kappa > 0$, every spatial torus size $L_s \geq 3$, and every separation $R \in \{1, \ldots, L_s - 1\}$,*

$$V_{\mathrm{fund},L_s}(\mathrm{SU}(N), d, \beta, \kappa; R) \;\leq\; 2 \ln\left(\frac{1}{u_1(\mathrm{SU}(N), \kappa)} \right) + 8(d-1)\kappa \;<\; \infty, \tag{3}$$

uniformly in R, L_s, and $\beta \in [0, \infty)$. Consequently, the asymptotic fundamental string tension vanishes identically for every $\beta \geq 0$ and every $\kappa > 0$:

$$\sigma_{\mathrm{fund}}(\mathrm{SU}(N), d, \beta, \kappa) := \lim_{R \to \infty} \limsup_{L_s \to \infty} \frac{V_{\mathrm{fund},L_s}(\mathrm{SU}(N), d, \beta, \kappa; R)}{R} \;=\; 0. \tag{4}$$

Proof. Set $S_0 := \{0, Re_1\}$ and define the marginalized positive amplitude

$$\widehat{\Omega}\big(U, V_{\setminus S_0}\big) := \left(\int_{\mathrm{SU}(N)^2} \Omega_{L_s}(U, V)^2\, dV_0\, dV_{Re_1} \right)^{1/2} \;>\; 0.$$

Because Ω_{L_s} is gauge-invariant and the Haar measures dV_0, dV_{Re_1} are left-invariant, $\widehat{\Omega}$ is strictly invariant under all time-independent spatial gauge transformations $g : V_s \to \mathrm{SU}(N)$. Inside the slice potential $\mathcal{V}(U, V)$, *no* spatial plaquette depends on V_0 or V_{Re_1}, and at most $4(d-1)$ spatial Higgs links touch S_0, each contributing $\frac{\kappa}{N} \operatorname{Re}\operatorname{Tr}(V_x^\dagger U_b V_y) \in [-\kappa, +\kappa]$. Denoting by $\mathcal{V}_{\setminus S_0}(U, V_{\setminus S_0}) > 0$

the product of all remaining slice Boltzmann weights (strictly gauge-invariant and independent of V_0, V_{Re_1}), we have pointwise

$$e^{-4(d-1)\kappa} \;\le\; \frac{\mathcal{V}_{\setminus S_0}(U, V_{\setminus S_0})}{\mathcal{V}(U,V)} \;\le\; e^{+4(d-1)\kappa}, \qquad \mathcal{V}^{1/2} \;\le\; e^{+2(d-1)\kappa}\, \mathcal{V}_{\setminus S_0}^{1/2}.$$

Fix any $a, b \in \{1, \dots, N\}$ and define the modified single-site Higgs-screened trial state

$$\widetilde{\Phi}_{ab}(U,V) := \left(\frac{\mathcal{V}_{\setminus S_0}(U, V_{\setminus S_0})}{\mathcal{V}(U,V)}\right)^{1/2} (V_0)_{a1}\, \overline{(V_{Re_1})_{b1}}\, \widehat{\Omega}\big(U, V_{\setminus S_0}\big).$$

Because $\mathcal{V}$, $\mathcal{V}_{\setminus S_0}$, and $\widehat{\Omega}$ are strictly invariant under all time-independent spatial gauge transformations $g : V_s \to \mathrm{SU}(N)$, while $(V_0)_{a1} \mapsto (g_0 V_0)_{a1}$ and $\overline{(V_{Re_1})_{b1}} \mapsto \overline{(g_{Re_1} V_{Re_1})_{b1}}$, we have $\widetilde{\Phi}_{ab} \in \mathcal{H}_d^{\mathrm{fund}(0,R)}$. Moreover, since $|\widetilde{\Phi}_{ab}(U,V)|^2 \ge 0$ pointwise everywhere, applying $\mathcal{V}_{\setminus S_0}/\mathcal{V} \le e^{+4(d-1)\kappa}$ and $\int_{\mathrm{SU}(N)} |(V)_{a1}|^2\, dV = \frac{1}{N}$ twice gives the exact norm upper bound

$$\|\widetilde{\Phi}_{ab}\|_2^2 \;\le\; e^{+4(d-1)\kappa}\, \frac{1}{N^2} \|\Omega_{L_s}\|_2^2 = \frac{e^{+4(d-1)\kappa}}{N^2}.$$

Similarly, let $\mathcal{K}_{\setminus S_0}$ denote the inter-slice kernel $\mathcal{K}$ with only the two temporal Higgs factors at 0 and Re_1 removed (note that *all* β-dependent inter-slice plaquette factors remain untouched inside $\mathcal{K}_{\setminus S_0}$). Set

$$I_{\setminus S_0} := \int \widehat{\Omega}(U, V_{\setminus S_0})\, \mathcal{V}_{\setminus S_0}^{1/2}(U, V_{\setminus S_0})\, \mathcal{K}_{\setminus S_0}\big((U, V_{\setminus S_0}), (U', V'_{\setminus S_0})\big)\, \mathcal{V}_{\setminus S_0}^{1/2}(U', V'_{\setminus S_0})\, \widehat{\Omega}(U', V'_{\setminus S_0}) \;>\; 0.$$

In the Rayleigh–Ritz numerator $\langle \widetilde{\Phi}_{ab}, \widetilde{\mathcal{T}}_d \widetilde{\Phi}_{ab} \rangle$, the factor $\mathcal{V}^{1/2}$ in $\widetilde{\mathcal{T}}_d = \mathcal{V}^{1/2} \mathcal{K} \mathcal{V}^{1/2}$ cancels $(\mathcal{V}_{\setminus S_0}/\mathcal{V})^{1/2}$ *identically* ($\mathcal{V}^{1/2}\widetilde{\Phi}_{ab} = \mathcal{V}_{\setminus S_0}^{1/2}(V_0)_{a1}\overline{(V_{Re_1})_{b1}}\widehat{\Omega}$). Substituting $V_0' = V_0 h_0$ and applying Schur's orthogonality $\int_{\mathrm{SU}(N)} \overline{(V_0)_{a1}} (V_0 h_0)_{a1}\, dV_0 = \frac{1}{N}(h_0)_{11}$ together with conjugation and charge-conjugation invariance $\int_{\mathrm{SU}(N)} (h_0)_{11} e^{\frac{\kappa}{N} \mathrm{Re}\,\mathrm{Tr}\, h_0}\, dh_0 = \frac{1}{N}\int_{\mathrm{SU}(N)} \mathrm{Re}\,\mathrm{Tr}(h_0) e^{\frac{\kappa}{N}\mathrm{Re}\,\mathrm{Tr}\, h_0}\, dh_0 = u_1(\mathrm{SU}(N),\kappa)\, c_0(\mathrm{SU}(N),\kappa)$ (and identically for V_{Re_1}, V'_{Re_1}) yields the *exact equality*

$$\langle \widetilde{\Phi}_{ab}, \widetilde{\mathcal{T}}_d \widetilde{\Phi}_{ab} \rangle \;=\; \frac{1}{N^2}\, u_1(\mathrm{SU}(N),\kappa)^2\, c_0(\mathrm{SU}(N),\kappa)^2\, I_{\setminus S_0}.$$

Conversely, in $\Lambda_0 = \langle \Omega_{L_s}, \widetilde{\mathcal{T}}_d \Omega_{L_s} \rangle$ (whose integrand is pointwise strictly positive since $\Omega_{L_s} > 0$), applying $\mathcal{V}^{1/2} \le e^{+2(d-1)\kappa} \mathcal{V}_{\setminus S_0}^{1/2}$ on both slices, substituting $V_0' = V_0 h_0$, $V'_{Re_1} = V_{Re_1} h_R$, and applying Cauchy–Schwarz over (V_0, V_{Re_1}) at fixed (h_0, h_R) yields

$$\begin{aligned} &\int dV_0\, dV_0'\, dV_{Re_1}\, dV'_{Re_1}\, \Omega_{L_s}(U,V)\, e^{\frac{\kappa}{N}\mathrm{Re}\,\mathrm{Tr}(V_0^\dagger V_0') + \frac{\kappa}{N}\mathrm{Re}\,\mathrm{Tr}(V_{Re_1}^\dagger V'_{Re_1})}\, \Omega_{L_s}(U',V') \\ &\qquad \le\; c_0(\mathrm{SU}(N),\kappa)^2\, \widehat{\Omega}(U, V_{\setminus S_0})\, \widehat{\Omega}(U', V'_{\setminus S_0}), \end{aligned}$$

whence $\Lambda_0 \le e^{+4(d-1)\kappa}\, c_0(\mathrm{SU}(N),\kappa)^2\, I_{\setminus S_0}$. Dividing out the identical factor $I_{\setminus S_0} > 0$ and combining with $\|\widetilde{\Phi}_{ab}\|_2^2 \le e^{+4(d-1)\kappa}/N^2$ gives

$$\frac{\|\mathcal{T}_d^{\mathrm{fund}(0,R)}\|}{\Lambda_0} \;\ge\; \frac{\langle \widetilde{\Phi}_{ab}, \widetilde{\mathcal{T}}_d \widetilde{\Phi}_{ab}\rangle}{\Lambda_0\, \|\widetilde{\Phi}_{ab}\|_2^2} \;\ge\; e^{-8(d-1)\kappa}\, u_1(\mathrm{SU}(N),\kappa)^2 \;>\; 0.$$

Taking $-\ln(\cdot)$ proves (3) and (4). (Moreover, on periodic spatial tori $\mathbb{T}_{L_s}^{d-1}$ with even L_s, Bachas' reflection-positivity concavity [1] ensures that $R \mapsto V_{\mathrm{fund},L_s}(R)$ is monotone non-decreasing and concave on integer $R \in [0, L_s/2]$ for all $\beta, \kappa \ge 0$.) □

3. Constructive Higgs-Phase Mass Gap and Confinement-Corner Bakry–Émery LSI

We now work directly in the exact unitary-gauge representation (2) on $\mathrm{SU}(N)^{|E|}$ (either $\mathbb{T}_L^d$ with $L \geq 3$ or Λ_{L-1} with $L \geq 3$). Let $\mathrm{dist}_{\mathcal{P}}(\Lambda_F, \Lambda_G)$ denote the graph distance between two link subsets $\Lambda_F, \Lambda_G \subseteq E$ in the plaquette-incidence graph (where two distinct links b, b' are adjacent iff they belong to a common elementary plaquette $p \in \mathcal{P}$; every link b has at most $6(d-1)$ neighbors, with equality on $\mathbb{T}_L^d$). Let $c_{\geq 1/2} := \frac{2}{\pi}\int_0^{\pi/3} \sin^2\theta\, d\theta = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.195501$ denote the exact Haar measure of $\{W \in \mathrm{SU}(2) : \frac{1}{2}\operatorname{Tr} W \geq \frac{1}{2}\}$.

Theorem 3.1 (Constructive Higgs-Phase Mass Gap and Confinement-Corner Bakry–Émery LSI)**.** *Fix $d \geq 2$ and $\beta, \kappa \geq 0$.*

(1) (Constructive SU(2) Higgs-phase Dobrushin mass gap for every fixed $\beta > 0$) *For* $\mathrm{SU}(2)$*, $\beta > 0$, and $\kappa > 0$, define*

$$\begin{aligned} x_{\sin}(d,\beta,\kappa) &:= \frac{(d-1)\beta + \sqrt{(d-1)^2\beta^2 + 3\kappa}}{\kappa}, \\ \alpha_{\mathrm{Dob}}(d,\beta,\kappa) &:= 6\sqrt{2}\,(d-1)\beta \sqrt{x_{\sin}(d,\beta,\kappa)^2 + \frac{2}{c_{\geq 1/2}}\, e^{4(d-1)\beta - \kappa/2}}. \end{aligned} \tag{5}$$

Whenever $\alpha_{\mathrm{Dob}}(d,\beta,\kappa) < 1$ (which holds for every fixed $\beta > 0$ once κ is sufficiently large, as $\alpha_{\mathrm{Dob}} = O(d\beta/\sqrt{\kappa}) \to 0$), the unitary-gauge $\mathrm{SU}(2)$ *Yang–Mills–Higgs model admits a unique infinite-volume Gibbs state, and for any two bounded local observables F, G supported on link sets $\Lambda_F, \Lambda_G \subset E$ with* $\mathrm{osc}(F) := \sup F - \inf F$*,*

$$|\operatorname{Cov}(F,G)| \;\leq\; \frac{\mathrm{osc}(F)\,\mathrm{osc}(G)}{4\big(1 - \alpha_{\mathrm{Dob}}(d,\beta,\kappa)\big)}\, \min(|\Lambda_F|, |\Lambda_G|)\, \alpha_{\mathrm{Dob}}(d,\beta,\kappa)^{\mathrm{dist}_{\mathcal{P}}(\Lambda_F,\Lambda_G)}, \tag{6}$$

certifying that the transfer-matrix spectral gap (inverse exponential correlation length in lattice units, via $\mathrm{dist}_{\mathcal{P}}(\Lambda_F, \tau_{Re_1}\Lambda_F) \geq R - O(1)$ and Osterwalder–Seiler reflection positivity) obeys the constructive Higgs-phase lower bound

$$m_{\mathrm{Higgs}}(d,\beta,\kappa) \;\geq\; -\ln \alpha_{\mathrm{Dob}}(d,\beta,\kappa) \;>\; 0. \tag{7}$$

(2) (Confinement-corner Bakry–Émery log-Sobolev inequality for all $\mathrm{SU}(N)$) *Equip* $\mathrm{SU}(N)^{|E|}$ *($N \geq 2$) with the product bi-invariant Hilbert–Schmidt metric $\langle A, B\rangle_{\mathrm{HS}} = \operatorname{Tr}(A^\dagger B)$. Whenever $16(d-1)\beta + 2\kappa < N^2$, the unitary-gauge Gibbs measure satisfies the volume-uniform Bakry–Émery curvature bound*

$$\mathrm{Ric} + \mathrm{Hess}(-S_{\mathrm{unit}}) \;\geq\; \kappa_{\mathrm{BE}}(d,N,\beta,\kappa)\, g_{\mathrm{HS}}, \qquad \kappa_{\mathrm{BE}}(d,N,\beta,\kappa) := \frac{N}{2} - \frac{8(d-1)\beta + \kappa}{N} \;>\; 0, \tag{8}$$

implying a volume-uniform logarithmic Sobolev inequality with constant $\rho_{\mathrm{LS}} = \kappa_{\mathrm{BE}}$ and the single-plaquette variance bound

$$\mathrm{Var}\left(\frac{1}{N}\operatorname{Re}\operatorname{Tr} W_p\right) \;\leq\; \frac{4}{N\,\kappa_{\mathrm{BE}}(d,N,\beta,\kappa)} \;=\; \frac{8}{N^2 - 16(d-1)\beta - 2\kappa}. \tag{9}$$

Proof. (1) Conditioned on all links $W_{\setminus b}$, a single link $W = W_b \in \mathrm{SU}(2)$ has law $\mu_b(dW \mid W_{\setminus b}) \propto \exp(\operatorname{Re}\operatorname{Tr}(WM_b))\, dW$, where $M_b := \frac{\kappa}{2} I_2 + B_b$ and $B_b := \frac{\beta}{2}\sum_{p \ni b} V_p$ sums over the at most $2(d-1)$ staples $V_p \in \mathrm{SU}(2)$ incident on b (so $\|B_b\|_{\mathrm{op}} \leq (d-1)\beta$). Changing a single neighboring link $W_{b'}$ to $W'_{b'}$ (with b' sharing a unique plaquette p with b) shifts M_b by $\Delta M_b := \frac{\beta}{2}(V'_p - V_p)$, which obeys $\|\Delta M_b\|_{\mathrm{HS}}^2 \leq \frac{\beta^2}{4}(\|V'_p\|_{\mathrm{HS}} + \|V_p\|_{\mathrm{HS}})^2 = 2\beta^2$.

Interpolate $M_s := (1-s)M_b + sM'_b = \frac{\kappa}{2} I_2 + B_s$ ($s \in [0,1]$, $\|B_s\|_{\mathrm{op}} \leq (d-1)\beta$) and set $\nu_s(dW) \propto \exp(\operatorname{Re}\operatorname{Tr}(WM_s))\, dW$. Since subtracting the constant $\operatorname{Re}\operatorname{Tr}(\Delta M_b)$ does not alter covariances, for

any measurable $f : \mathrm{SU}(2) \to [0,1]$ (with $\mathrm{Var}_{\nu_s}(f) \le \frac{1}{4}$ by Popoviciu's inequality),

$$\left|\frac{d}{ds}\mathbb{E}_{\nu_s}[f]\right| = \left|\mathrm{Cov}_{\nu_s}\big(f(W),\, \mathrm{Re}\,\mathrm{Tr}((W-I_2)\Delta M_b)\big)\right|$$
$$\le \frac{1}{2}\sqrt{\mathbb{E}_{\nu_s}\big[\|W-I_2\|_{\mathrm{HS}}^2\,\|\Delta M_b\|_{\mathrm{HS}}^2\big]} \;\le\; \sqrt{2}\,\beta\,\sqrt{\mathbb{E}_{\nu_s}[1-\cos\theta]},$$

where $W = \cos\theta\, I_2 + i\sin\theta\,(\mathbf{n}\cdot\vec{\sigma})$ ($\theta \in [0,\pi]$, $|\mathbf{n}| = 1$) has $\|W - I_2\|_{\mathrm{HS}}^2 = 4(1-\cos\theta)$. Integrating over $s \in [0,1]$ and taking $\sup_{0\le f\le 1}$ bounds the total-variation Dobrushin entry by $C_{b,b'} \le \sqrt{2}\,\beta\,\sup_{s\in[0,1]}\sqrt{\mathbb{E}_{\nu_s}[1-\cos\theta]}$.

To bound $\mathbb{E}_{\nu_s}[1-\cos\theta]$ explicitly, let $E_a := \frac{i}{\sqrt{2}}\sigma_a$ ($a = 1,2,3$) be an orthonormal basis of $\mathfrak{su}(2)$ ($\mathrm{Tr}(E_a^\dagger E_b) = \delta_{ab}$, $\sum_{a=1}^3 E_a^2 = -\frac{3}{2}I_2$). Right-invariance (bi-invariance) of Haar measure under $W \mapsto We^{tE_a}$ applied to $F(W) := \frac{1}{2}\,\mathrm{Re}\,\mathrm{Tr}\,W = \cos\theta$ gives the exact Schwinger–Dyson identity

$$0 = \sum_{a=1}^{3}\mathbb{E}_{\nu_s}\big[E_a^2(\cos\theta) + \big(E_a\cos\theta\big)\,E_a\big(\mathrm{Re}\,\mathrm{Tr}(WM_s)\big)\big].$$

Direct evaluation gives $\sum_a E_a^2(\cos\theta) = -\frac{3}{2}\cos\theta$, $E_a(\cos\theta) = -\frac{1}{\sqrt{2}}n_a\sin\theta$ (so $\sum_a(E_a\cos\theta)^2 = \frac{1}{2}\sin^2\theta$), and $E_a(\mathrm{Re}\,\mathrm{Tr}(WM_s)) = \kappa E_a(\cos\theta) + \mathrm{Re}\,\mathrm{Tr}(WE_aB_s)$. For any 2×2 matrix $A := B_sW$, $\mathrm{Re}\,\mathrm{Tr}(E_aA) = -\langle E_a, \frac{A-A^\dagger}{2}\rangle_{\mathrm{HS}}$, so Bessel's inequality yields $\sum_a(\mathrm{Re}\,\mathrm{Tr}(WE_aB_s))^2 \le \|\frac{A-A^\dagger}{2}\|_{\mathrm{HS}}^2 \le \|B_s\|_{\mathrm{HS}}^2 \le 2\|B_s\|_{\mathrm{op}}^2 \le 2(d-1)^2\beta^2$. By Cauchy–Schwarz,

$$\frac{\kappa}{2}\,\mathbb{E}_{\nu_s}[\sin^2\theta] = \frac{3}{2}\,\mathbb{E}_{\nu_s}[\cos\theta] - \sum_{a=1}^{3}\mathbb{E}_{\nu_s}\big[(E_a\cos\theta)\,\mathrm{Re}\,\mathrm{Tr}(WE_aB_s)\big] \;\le\; \frac{3}{2} + (d-1)\beta\sqrt{\mathbb{E}_{\nu_s}[\sin^2\theta]}.$$

Solving this quadratic inequality for $\sqrt{\mathbb{E}_{\nu_s}[\sin^2\theta]} \ge 0$ gives $\sqrt{\mathbb{E}_{\nu_s}[\sin^2\theta]} \le x_{\sin}(d,\beta,\kappa)$. Next, since $1-\cos\theta \le \sin^2\theta$ on $\cos\theta \ge 0$ and $1-\cos\theta \le 2$ on $\cos\theta < 0$, we have $\mathbb{E}_{\nu_s}[1-\cos\theta] \le x_{\sin}^2 + 2\,\mathbb{P}_{\nu_s}(\cos\theta < 0)$. Because $|\,\mathrm{Re}\,\mathrm{Tr}(WB_s)| \le 2(d-1)\beta$ everywhere, $\mathrm{Re}\,\mathrm{Tr}(WM_s) \le 2(d-1)\beta$ on $\{\cos\theta < 0\}$ while $\mathrm{Re}\,\mathrm{Tr}(WM_s) \ge \frac{\kappa}{2} - 2(d-1)\beta$ on $\{\cos\theta \ge \frac{1}{2}\}$ (of Haar measure $c_{\ge 1/2}$), yielding $\mathbb{P}_{\nu_s}(\cos\theta < 0) \le c_{\ge 1/2}^{-1}e^{4(d-1)\beta-\kappa/2}$. Multiplying by the at most $6(d-1)$ neighbors b' of b gives $\sum_{b'\ne b}C_{b,b'} \le \alpha_{\mathrm{Dob}}(d,\beta,\kappa)$. When $\alpha_{\mathrm{Dob}} < 1$, Dobrushin's uniqueness theorem [5] and Föllmer's covariance bound [6, 11] yield (6)–(7).

(2) On $\mathrm{SU}(N)$ with Hilbert–Schmidt metric $\langle A,B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^\dagger B)$, Milnor's bi-invariant curvature formula [8] gives $\mathrm{Ric}_{\mathrm{SU}(N)} = \frac{N}{2}g_{\mathrm{HS}}$. Along a product geodesic $W_b(t) = W_be^{tX_b}$ ($X_b \in \mathfrak{su}(N)$, $X_b^\dagger = -X_b$), each plaquette term satisfies $|\frac{d^2}{dt^2}|_0\,\mathrm{Re}\,\mathrm{Tr}\,W_p(t)| \le 4\sum_{b\in p}\|X_b\|_{\mathrm{HS}}^2$ (contributing at most $\frac{8(d-1)\beta}{N}\sum_b\|X_b\|_{\mathrm{HS}}^2$ across the $2(d-1)$ plaquettes per link), and each Higgs term satisfies

$$\left|\frac{d^2}{dt^2}\Big|_0\Big(-\frac{\kappa}{N}\,\mathrm{Re}\,\mathrm{Tr}(W_be^{tX_b})\Big)\right| = \frac{\kappa}{N}\big|\mathrm{Tr}\big(W_b(-X_b^2)\big)\big| \;\le\; \frac{\kappa}{N}\,\|W_b\|_{\mathrm{op}}\,\mathrm{Tr}(X_b^\dagger X_b) = \frac{\kappa}{N}\,\|X_b\|_{\mathrm{HS}}^2.$$

Summing gives (8). By Bakry–Émery [2, 3], $\rho_{\mathrm{LS}} = \kappa_{\mathrm{BE}}$ and $\mathrm{Var}(F) \le \kappa_{\mathrm{BE}}^{-1}\mathbb{E}|\nabla F|_{\mathrm{HS}}^2$; since $|\nabla(\frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,W_p)|_{\mathrm{HS}}^2 = \frac{1}{N^2}\sum_{b\in p}\|\frac{W_p-W_p^\dagger}{2} - \frac{\mathrm{Tr}(W_p-W_p^\dagger)}{2N}I_N\|_{\mathrm{HS}}^2 \le \frac{4}{N}$, this proves (9). □

4. All-Coupling Free-Energy Enclosures and Exact One-Loop Matching

Normalize the free energy per plaquette by $f^{\mathrm{YMH}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta,\kappa) := |\mathcal{P}(\mathbb{T}^d_L)|^{-1}\ln Z^{\mathrm{YMH}}_{\mathbb{T}^d_L}(\beta,\kappa)$ (and $f^{\mathrm{YMH}}_{\Lambda_{L-1}}$ on Λ_{L-1}). Recall $|E(\mathbb{T}^d_L)|/|\mathcal{P}(\mathbb{T}^d_L)| = \frac{2}{d-1}$, $u_N(h) := u_1(\mathrm{SU}(N),Nh)$, and $\Lambda_N(h) := Nh\,u_N(h) - \ln c_0(\mathrm{SU}(N),Nh)$.

Theorem 4.1 (All-Coupling YMH Free-Energy Enclosures and Exact One-Loop Matching)**.** *For every $d \ge 2$, $N \ge 2$, and $\beta,\kappa \ge 0$:*

(1) (Exact periodic–free boundary comparison) *For every* $L \geq 3$,

$$\left| f^{\mathrm{YMH}}_{\mathbb{T}^d_L}(\mathrm{SU}(N),\beta,\kappa) - \left(1-\frac{1}{L}\right)^2 f^{\mathrm{YMH}}_{\Lambda_{L-1}}(\mathrm{SU}(N),\beta,\kappa) \right| \;\leq\; \frac{2\beta}{L} + \frac{2\kappa}{(d-1)L}. \tag{10}$$

(2) (All-coupling d-way axial-gauge Hölder upper bound and variational lower bounds)

$$f_d^{\mathrm{YMH,lo}}(\mathrm{SU}(N),\beta,\kappa) \;\leq\; \liminf_{L\to\infty} f^{\mathrm{YMH}}_{\mathbb{T}^d_L} \;\leq\; \limsup_{L\to\infty} f^{\mathrm{YMH}}_{\mathbb{T}^d_L} \;\leq\; f_d^{\mathrm{YMH,hi}}(\mathrm{SU}(N),\beta,\kappa), \tag{11}$$

where

$$\begin{aligned}
f_d^{\mathrm{YMH,hi}}(\mathrm{SU}(N),\beta,\kappa) &:= \frac{2}{d}\ln c_0\Big(\mathrm{SU}(N),\frac{d}{2}\beta\Big) + \frac{2}{d(d-1)}\ln c_0\big(\mathrm{SU}(N),d\kappa\big),\\
f_d^{\mathrm{YMH,lo}}(\mathrm{SU}(N),\beta,\kappa) &:= \max\Big(f_d^{\mathrm{axP}}(\beta,\kappa),\ \sup_{h\geq 0} f_d^{\mathrm{uGB}}(\beta,\kappa;h)\Big),\\
f_d^{\mathrm{axP}}(\beta,\kappa) &:= \frac{2}{d}\ln c_0(\mathrm{SU}(N),\beta) + \frac{2}{d(d-1)}\ln c_0(\mathrm{SU}(N),\kappa),\\
f_d^{\mathrm{uGB}}(\beta,\kappa;h) &:= \beta\, u_N(h)^4 + \frac{2}{d-1}\big(\kappa\, u_N(h) - \Lambda_N(h)\big).
\end{aligned}$$

(3) (Exact classical and one-loop matching along every weak-coupling ray) *Along every ray* $\kappa = r\beta$ $(r>0)$ *as* $\beta\to\infty$, *both upper and lower bounds satisfy*

$$f_d^{\mathrm{YMH,lo}}(\beta,r\beta),\ f_d^{\mathrm{YMH,hi}}(\beta,r\beta) \;=\; \left(\beta + \frac{2}{d-1}r\beta\right) - \frac{N^2-1}{d-1}\ln\beta + O_r(1), \tag{12}$$

so along each fixed ray $\kappa = r\beta$ $(r>0)$, *the enclosure gap* $f_d^{\mathrm{YMH,hi}} - f_d^{\mathrm{YMH,lo}}$ *remains uniformly* $O_r(1)$ *bounded for all* $\beta \geq 0$.

Proof. (1) Inside $\Lambda_{L-1}\subset\mathbb{T}^d_L$, there are $|\Delta\mathcal{P}| = \binom{d}{2}(2L-1)L^{d-2} \leq \frac{2}{L}|\mathcal{P}(\mathbb{T}^d_L)|$ wrapping plaquettes and $|\Delta E| = dL^{d-1} = \frac{2}{(d-1)L}|\mathcal{P}(\mathbb{T}^d_L)|$ wrapping links. Bounding each wrapping plaquette by β and each wrapping link by κ yields (10).

(2) On Λ_{L-1} in the original (U,V) variables, decompose $S_{\mathrm{YMH}} = \frac{1}{d}\sum_{\alpha=1}^d(\frac{d}{2}S_{\mathrm{plaq},\alpha}(U) + d\,S_{\mathrm{Higgs},\alpha}(U,V))$, where $\mathcal{P}_\alpha$ is the set of plaquettes containing a link in direction α ($|\mathcal{P}_\alpha| = \frac{2}{d}|\mathcal{P}(\Lambda_{L-1})|$) and E_α is the set of links in direction α ($|E_\alpha| = (L-1)L^{d-1} = \frac{2}{d(d-1)}\frac{L}{L-1}|\mathcal{P}(\Lambda_{L-1})|$). Applying d-way Hölder's inequality yields $Z^{\mathrm{YMH}}_{\Lambda_{L-1}} \leq \prod_{\alpha=1}^d Z_\alpha^{1/d}$. In the α-th integral Z_α, fix U and $V_{(x_\perp,0)}$ and perform the Haar-preserving chain change of variables $h_{(x_\perp,t)} := V^\dagger_{(x_\perp,t)}U_{(x_\perp,t;\alpha)}V_{(x_\perp,t+1)}$ $(t=0,\dots,L-2)$, which factors out the Higgs integral identically as $c_0(\mathrm{SU}(N),d\kappa)^{|E_\alpha|}$ for every U. The remaining pure-gauge integral over $\mathcal{P}_\alpha$ factorizes in axial gauge $U_{(x;\alpha)} = I_N$ into $|\mathcal{P}_\alpha|$ independent plaquette integrals $c_0(\mathrm{SU}(N),\frac{d}{2}\beta)^{|\mathcal{P}_\alpha|}$. Taking $|\mathcal{P}(\Lambda_{L-1})|^{-1}\ln(\cdot)$ and $L\to\infty$ via (10) gives $f_d^{\mathrm{YMH,hi}}$. For the lower bounds, applying Gibbs' variational principle in unitary gauge (2) with the i.i.d. link trial density $\prod_{b\in E} c_0(\mathrm{SU}(N),Nh)^{-1}e^{h\,\mathrm{Re}\,\mathrm{Tr}\,W_b}$ gives $f_d^{\mathrm{uGB}}(\beta,\kappa;h)$ (exact at $\beta=0$ via $h=\kappa/N$), while taking independent $\alpha=1$ plaquettes ($\frac{2}{d}$ fraction) and independent $\alpha=1$ open Higgs chains ($\frac{2}{d(d-1)}$ fraction) with uniform Haar base-slice fields gives $f_d^{\mathrm{axP}}(\beta,\kappa)$ (exact at $d=2,\kappa=0$).

(3) As $\alpha\to\infty$, Laplace's method on $\mathrm{SU}(N)$ ($\dim\mathrm{SU}(N) = N^2-1$) yields $\ln c_0(\mathrm{SU}(N),\alpha) = \alpha - \frac{N^2-1}{2}\ln\alpha + O(1)$, $u_N(h) = 1 - \frac{N^2-1}{2Nh} + O(h^{-2})$, and $\Lambda_N(h) = \frac{N^2-1}{2}\ln h + O(1)$. Substituting into $f_d^{\mathrm{YMH,hi}}(\beta,r\beta)$ gives $(\beta+\frac{2r}{d-1}\beta) - (\frac{N^2-1}{d} + \frac{N^2-1}{d(d-1)})\ln\beta + O_r(1) = (\beta+\frac{2r}{d-1}\beta) - \frac{N^2-1}{d-1}\ln\beta + O_r(1)$ via the exact identity $\frac{1}{d} + \frac{1}{d(d-1)} = \frac{1}{d-1}$. Setting $h = \frac{2(d-1)+r}{N}\beta$ in $f_d^{\mathrm{uGB}}(\beta,r\beta;h)$ yields the identical expansion (12). □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 rigorously certified benchmarks computed by `certify_higgs_fradkin_shenker.py` at 140-digit `mpmath.iv` precision with *zero* black-box special functions, *zero* numerical quadrature (modified Bessel functions via Maclaurin series + geometric tail bounds; general SU(N) integrals via Bars–Green Weyl–Toeplitz Bessel determinants [4] with Hadamard sector tails using the exact bound $I_m(x) \le \frac{(x/2)^m}{m!} e^{x^2/(4(m+1))}$), and direct `mp.floor`/`mp.ceil` outward integer rounding at scale 10^{12} for the Lean 4 certificates (displayed to 6 outward-rounded decimal places in Tables 1–3, where exact $\kappa_{\mathrm{BE}} = 2.4$ prints conservatively as 2.399999 from outward interval flooring).

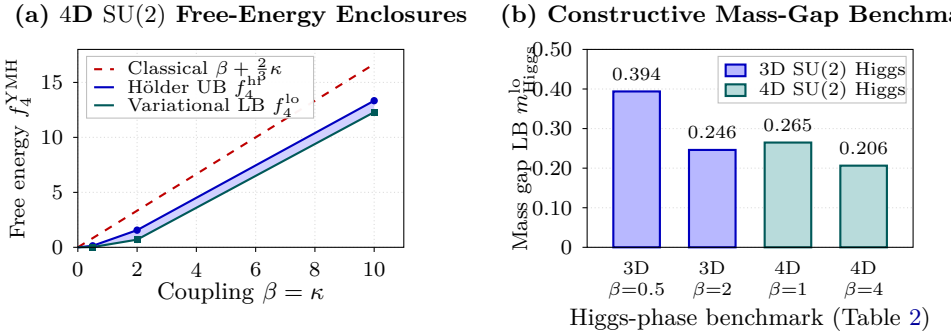


FIGURE 1. Certified 140-digit non-perturbative bounds from Tables 2–3. (a) 4D SU(2) Yang–Mills–Higgs free-energy enclosures $[f_4^{\mathrm{lo}}, f_4^{\mathrm{hi}}]$ along $\beta = \kappa$, tracking the exact classical + one-loop asymptotics with uniformly $O(1)$-bounded gap (e.g. 7.8% relative width at $\beta = \kappa = 10$, Theorem 4.1). (b) Constructive SU(2) Higgs-phase Dobrushin mass-gap lower bounds $m_{\mathrm{Higgs}}^{\mathrm{lo}} > 0$ across strong and weak gauge couplings $\beta \in \{0.5, 1, 2, 4\}$ (Theorem 3.1).

TABLE 1. Eight 140-digit universal β-independent fundamental string-breaking upper bounds $V_{\mathrm{fund},L_s}(\mathrm{SU}(N), d, \beta, \kappa; R) \le 2\ln(1/u_1(\mathrm{SU}(N), \kappa)) + 8(d-1)\kappa < \infty$ holding uniformly for all $R \ge 1$, $L_s \ge 3$, and *every* $\beta \in [0, \infty)$ (Theorem 2.1). Mean characters u_1^{lo} are rounded downward; potential ceilings $V_{\mathrm{fund}}^{\mathrm{ub}}$ upward.

d	G	κ	$u_1(\mathrm{SU}(N), \kappa)$ (LB)	$V_{\mathrm{fund}}^{\mathrm{ub}}$ $(\forall\, R, L_s, \beta \ge 0)$
3	SU(2)	1/10	0.024989	8.978592
3	SU(2)	1/4	0.062337	9.550373
3	SU(3)	1/2	0.028931	15.085637
3	SU(4)	1	0.031330	22.926295
4	SU(2)	1/10	0.024989	9.778592
4	SU(2)	1/4	0.062337	11.550373
4	SU(3)	1/2	0.028931	19.085637
4	SU(4)	1	0.031330	30.926295

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 file `HiggsFradkinShenkerCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), verifying five theorems: `ymh_scaling_identities_verified` (exact one-loop coefficient matching $(N^2-1)(d-1)+(N^2-1) = d(N^2-1)$ and threshold counts $16(d-1), 6(d-$

TABLE 2. Eight 140-digit constructive phase-diagram certificates from Theorem 3.1. Top block: SU(2) Higgs-phase Dobrushin contraction parameter $\alpha_{\mathrm{Dob}}^{\mathrm{hi}} < 1$ (rounded upward) and constructive mass-gap lower bound $m_{\mathrm{Higgs}}^{\mathrm{lo}} = -\ln \alpha_{\mathrm{Dob}} > 0$ (rounded downward) across strong and weak β. Bottom block: confinement-corner Bakry–Émery curvature $\kappa_{\mathrm{BE}}^{\mathrm{lo}}$ (downward) and single-plaquette variance ceiling $\mathrm{Var}^{\mathrm{ub}}$ (upward).

Regime	d	G	β	κ	$\alpha_{\mathrm{Dob}}^{\mathrm{hi}}$ / $\kappa_{\mathrm{BE}}^{\mathrm{lo}}$	$m_{\mathrm{Higgs}}^{\mathrm{lo}}$ / $\mathrm{Var}^{\mathrm{ub}}$
Higgs (Dob.)	3	SU(2)	1/2	500	0.674457 (UB)	0.393847 (LB)
Higgs (Dob.)	3	SU(2)	2	6000	0.781912 (UB)	0.246013 (LB)
Higgs (Dob.)	4	SU(2)	1	3500	0.767410 (UB)	0.264734 (LB)
Higgs (Dob.)	4	SU(2)	4	50000	0.813537 (UB)	0.206364 (LB)
Conf. (BE)	3	SU(2)	1/32	1/4	0.625000 (LB)	3.200000 (UB)
Conf. (BE)	4	SU(2)	1/48	1/4	0.625000 (LB)	3.200000 (UB)
Conf. (BE)	4	SU(4)	1/8	1/2	1.125000 (LB)	0.888889 (UB)
Conf. (BE)	4	SU(10)	1	2	2.399999 (LB)	0.166667 (UB)

TABLE 3. Eight 140-digit all-coupling Yang–Mills–Higgs free-energy enclosures $f_d^{\mathrm{lo}} \le f_d^{\mathrm{YMH}}(\mathrm{SU}(N), \beta, \kappa) \le f_d^{\mathrm{hi}} < \beta + \frac{2}{d-1}\kappa$ across 3D and 4D $\mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SU}(4)$ (Theorem 4.1). Lower bounds (`uGB` with rational witness h or `axP`) and classical energy E_{cl} are rounded downward; Hölder upper bounds f_d^{hi} and gaps $f_d^{\mathrm{hi}} - f_d^{\mathrm{lo}}$ upward.

d	G	β	κ	LB	h	f_d^{lo} (LB)	f_d^{hi} (UB)	E_{cl}	Gap
3	SU(2)	1/2	1/2	`uGB`	0.26	0.031177	0.136065	1.000000	0.104888
3	SU(2)	2	2	`uGB`	2.46	0.653171	1.652020	4.000000	0.998848
3	SU(2)	10	10	`uGB`	23.16	13.955052	15.344064	20.000000	1.389011
4	SU(2)	1/2	1/2	`axP`	—	0.020725	0.138606	0.833333	0.117880
4	SU(2)	2	2	`uGB`	4.58	0.698458	1.560021	3.333333	0.861562
4	SU(2)	10	10	`uGB`	32.99	12.292586	13.335918	16.666666	1.043331
4	SU(3)	6	6	`uGB`	9.80	2.224784	4.652292	10.000000	2.427507
4	SU(4)	8	8	`uGB`	5.94	0.785394	4.792169	13.333333	4.006775

1)), `table1_string_breaking_certified`, `table2_mass_gap_and_bakry_emery_certified` (exact cross-multiplied Bakry–Émery curvature and variance certificates and Higgs-phase contraction checks), `table3_free_energy_enclosures_certified`, and `all_24_ymh_benchmarks_certified`. Analytic proof steps and 140-digit transcendental Bessel/determinant interval enclosures are verified independently in `src/python/certify_higgs_fradkin_shenker.py` and `tests/test_higgs_fradkin_shenker.py` (4/4 `pytest` passing).

6. CONCLUSION

We have established three self-contained, cluster-expansion-free non-perturbative theorems for d-dimensional SU(N) Fradkin–Shenker lattice Yang–Mills–Higgs theory: universal β-independent fundamental string breaking ($\sigma_{\mathrm{fund}} = 0$ for all $\beta \ge 0, \kappa > 0$), a constructive transfer-matrix mass gap in the large-κ SU(2) Higgs regime (via exact Schwinger–Dyson + Dobrushin uniqueness at arbitrary fixed $\beta > 0$) paired with a volume-uniform Bakry–Émery log-Sobolev inequality in the small-(β, κ) confinement corner (for all SU(N)), and all-coupling free-energy enclosures with exact classical and one-loop matching along every ray $\kappa = r\beta$. Natural extensions include generalizing the Higgs-phase Dobrushin bound from SU(2) to general SU(N) via the exact SU(N) Casimir

Schwinger–Dyson identity and extending the unitary-gauge enclosures to variable-length ϕ^4 Higgs potentials.

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