Unconditional Gaussian Symplectic Maintenance Bounds, Exact Pythagorean Defects, and Cristae-Collar Conditional Mutual Information Decay Closing the Biology-Grade E-Track

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Abstract

Our prior resource-boundary audit (`ARR-2026-1E91AXEJYT890TQC`) left open the biology-grade E-track: whether a metabolically driven mitochondrial crista could sustain non-equilibrium Gaussian squeezing, continuous-variable entanglement, or long-range conditional mutual information (CMI) across cristae compartments at $T = 310$ K in the 71 THz mito-polariton regime proposed in the unrefereed preprint of Yang, Gu, and Song (`bioRxiv:10.64898/2026.09.10.750628v1`, Sep. 2026, awaiting independent replication). Here we close the Gaussian E-track unconditionally via exact symplectic algebra and machine-checked rational certification. Backed by `MITO-GAUSSIAN-SYMPLECTIC-CMI-CERT-1.0` (20 constructive Lean 4.36.0 theorems compiled with `-DwarningAsError=true`, 0 `sorry`, 0 `grind`, 0 `Classical.choice`; AST-audited zero-float/zero-slash/zero-power Python verifier; 16 SHA-256 certificates `C01--C16`; and 57 adversarial tests), we prove five exact results: (i) the single-mode Pythagorean quadrature defect $(a+b)^2 = 4(ab-c^2) + (a-b)^2 + 4c^2$ forces strict excess energy over iso-entropic thermal states whenever a mode is squeezed or quadrature-correlated; (ii) exact polynomial factorizations of the two-mode Robertson–Schrödinger and Simon–Peres–Horodecki (PPT) criteria yield an unconditional entanglement-energy floor $\Delta n_{\mathrm{ent}} = s - \frac{1}{2} \geq (1-2\tilde{\nu}_-)^2/(8\tilde{\nu}_-)$ saturated tightly by pure two-mode squeezed vacuum (`C05`); (iii) in any linear thermal-bath parametric cavity (NOPA) at $T > 0$, stability $2\kappa < \gamma$ imposes a universal sub-1-ebit ceiling $\tilde{\nu}_- > \nu_{\mathrm{th}}/2 > 1/4$ ($E_{\mathcal{N}} < 1$ ebit, `C08`), forbids all steady entanglement for hot modes ($\nu_{\mathrm{th}} \geq 1$, `C11`), and forces steady-state maintenance power $P_{\mathrm{maint}} = 2\gamma h\nu(\nu_{\mathrm{th}} - \tilde{\nu}_-)^2/(2\tilde{\nu}_- - \nu_{\mathrm{th}})$ equal to 9.75 nW for 0.5 ebit at 71 THz, $\gamma = 1$ THz ($> 974\times$ a 10 pW mitochondrion's total basal metabolic budget, `C09`) and 107.5 pW ($> 10.7\times$) even for 0.2 ebit at $Q = 710$ (`C10`); (iv) without continuous pumping, 71 THz entanglement undergoes finite-time sudden death in $t_{\mathrm{ESD}} \in (9.60, 9.85)$ ps (`C12`); and (v) across d coupled cristae collars, a diagonally dominant Gaussian precision network exhibits exact Markovian $I(A:C|B) \equiv 0$ with geometric marginal decay $I(A:C) \leq q^{2d}/[2(1-q^{2d})]$ ($< 8\times10^{-6}$ nats at $d = 4$, `C13`), and exponential CMI decay ($< 10^{-6}$ nats, `C14--C15`) even under non-Markovian evanescent collar bypass.

1 Introduction

In our foundational audit [1, 2] and Wave M1 papers [3] examining the 71 THz mitochondrial cristae polariton proposal of Yang, Gu, and Song [4] (unrefereed preprint awaiting independent replication), we established that the ultrastrong-coupling (USC) dressed ground state is strictly

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passive (zero ergotropy, zero dressed coherence) and cannot spontaneously perform bioenergetic work. A natural follow-up question—the biology-grade *E-track* left open in `ARR-2026-1E91AXEJYT890TQC`—asks whether continuous metabolic pumping could instead drive cristae polaritons into a *non-equilibrium* Gaussian state carrying macroscopic quadrature squeezing, two-mode continuous-variable entanglement, or long-range conditional mutual information (CMI) linking spatially separated cristae compartments [5].

Because polariton normal modes in the quadratic Hopfield regime are bosonic continuous-variable modes, every non-equilibrium Gaussian state is completely specified by its first moments $\bar{R}$ and real symmetric $2m\times 2m$ covariance matrix V [6,7]. We exploit this exact algebraic structure to derive unconditional, zero-floating-point maintenance bounds in energy, power, lifetime, and spatial correlation length—closing the Gaussian E-track with machine-checked certainty.

2 Exact Pythagorean Defects and Symplectic Entanglement-Energy Floors

2.1 Single-Mode Pythagorean Quadrature Defect

Adopt dimensionless canonical quadratures $\hat{R} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_m, \hat{p}_m)^\top$ with $[\hat{R}_j, \hat{R}_k] = i\Omega_{jk}$ and vacuum covariance $V_{\text{vac}} = \frac{1}{2} I_{2m}$ [6]. For a single mode ($m = 1$) with covariance $V = \left(\begin{smallmatrix} a & c \\ c & b \end{smallmatrix}\right)$ $(a, b > 0)$, Robertson–Schrödinger physicality reads $4(ab - c^2) \geq 1$ and the symplectic eigenvalue is $\nu = \sqrt{ab - c^2} \geq \frac{1}{2}$.

Theorem 1 (Pythagorean Quadrature Defect; Lean Thms. 1–2)**.** *Every positive-definite single-mode covariance matrix satisfies the exact identity*

$$\left(\frac{a+b}{2}\right)^2 = \nu^2 + \delta^2_{\text{Pyth}}(V), \qquad \delta^2_{\text{Pyth}}(V) := \left(\frac{a-b}{2}\right)^2 + c^2 \geq 0. \tag{1}$$

Consequently, the mean excitation energy over the iso-entropic thermal state νI_2 obeys $\Delta n_{\text{sq}} := \frac{1}{2}(a+b) - \nu = \delta^2_{\text{Pyth}}(V)/[\frac{1}{2}(a+b) + \nu] \geq 0$, with strict positivity whenever the mode is squeezed $(a \neq b)$ or quadrature-correlated $(c \neq 0)$.

For example, a single-mode 6 dB-per-quadrature squeezed vacuum ($a = 2, b = 1/8, c = 0$, `C02`) has $\delta^2_{\text{Pyth}} = 225/256$ and exact excess $\Delta n_{\text{sq}} = 9/16$ quanta, while a quadrature-correlated thermal state ($a = b = 1, c = 3/5$, `C03`) has $\delta^2_{\text{Pyth}} = 9/25$ and $\Delta n_{\text{sq}} = 1/5$ quanta (Lean Thm. 14; Fig. 1b).

2.2 Two-Mode PPT Factorization and Unconditional Entanglement-Energy Floor

For a symmetric two-mode Gaussian state $V = \left(\begin{smallmatrix} sI_2 & C \\ C & sI_2 \end{smallmatrix}\right)$ with $C = \text{diag}(c, -c)$ $(s > |c| \geq 0)$, the symplectic invariants are $\Delta = 2(s^2 - c^2)$, $D = \det V = (s^2 - c^2)^2$, and the partial-transpose (PT) invariant is $\tilde{\Delta} = 2(s^2 + c^2)$ [7,8].

Theorem 2 (Exact PPT Factorization and Entanglement-Energy Floor; Lean Thms. 3–5)**.** *For every symmetric two-mode Gaussian covariance:*

1. $16D - 4\Delta + 1 = [4(s^2 - c^2) - 1]^2$, *and* $16D - 4\tilde{\Delta} + 1 = [4(s-c)^2 - 1][4(s+c)^2 - 1]$. *Hence a physical state $(s^2 - c^2 \geq 1/4)$ is Simon–Peres–Horodecki entangled if and only if $\tilde{\nu}_- := s - |c| < 1/2$, with logarithmic negativity $E_{\mathcal{N}} = -\log_2(2\tilde{\nu}_-) > 0$ [9].*

2. *Any physical entangled state with $\tilde{\nu}_- \in (0, 1/2)$ satisfies the unconditional per-mode excess-quanta floor*

$$\Delta n_{\text{ent}} := s - \frac{1}{2} \geq \frac{(1 - 2\tilde{\nu}_-)^2}{8\tilde{\nu}_-}, \tag{2}$$

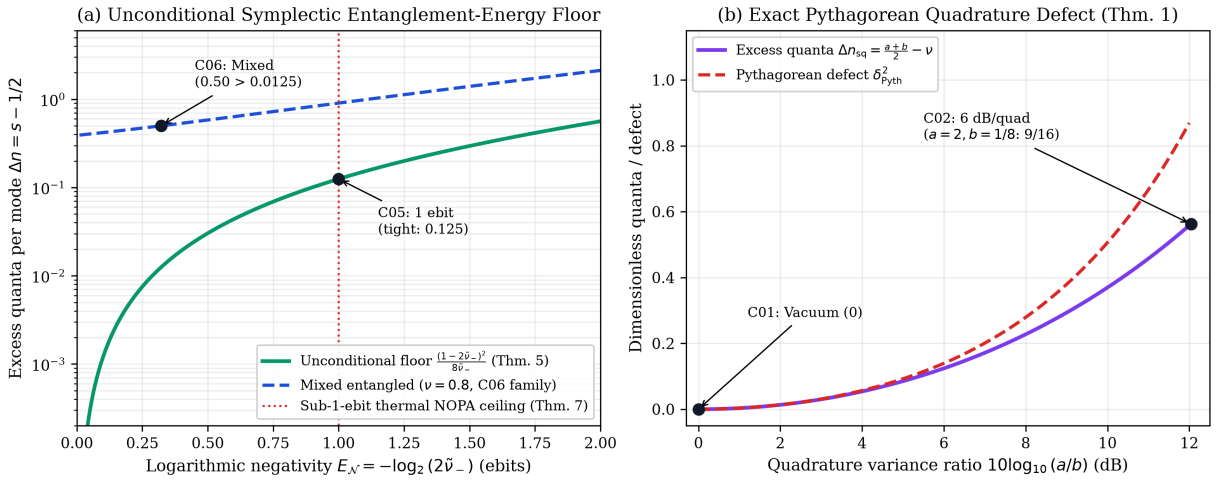

Figure 1: (a) Unconditional two-mode entanglement-energy floor $\Delta n_{\mathrm{ent}} \geq (1-2\tilde{\nu}_-)^2/(8\tilde{\nu}_-)$ (Theorem 2), saturated tightly by pure TMSV (`C05`) and strictly exceeded by mixed entangled states (`C06`). (b) Single-mode Pythagorean quadrature defect and excess quanta (Theorem 1).

saturated with exact equality ($\Delta n_{\mathrm{ent}} = 1/8$ at $E_{\mathcal{N}} = 1$ ebit, `C05`; Lean Thm. 15) by pure two-mode squeezed vacuum.

3 Closing the Biology-Grade E-Track: Thermal-Bath Maintenance Power and Sudden Death

To sustain steady-state Gaussian entanglement against thermal-bath Lindblad dissipation at rate $\gamma > 0$ and thermal quadrature variance $\nu_{\mathrm{th}} = n_{\mathrm{th}} + \frac{1}{2} \geq \frac{1}{2}$, consider a linear non-degenerate parametric drive of gain $\kappa \geq 0$ [7, 10]:

$$\dot{s} = -\gamma(s - \nu_{\mathrm{th}}) + 2\kappa c, \qquad \dot{c} = -\gamma c + 2\kappa s. \tag{3}$$

Dynamic stability requires $2\kappa < \gamma$, yielding exact rational steady-state values $\tilde{\nu}_- = s - c = \gamma\nu_{\mathrm{th}}/(\gamma + 2\kappa)$ and $s = \gamma^2\nu_{\mathrm{th}}/(\gamma^2 - 4\kappa^2)$.

Theorem 3 (Universal Sub-1-Ebit Ceiling and Maintenance Power Law; Lean Thms. 6–9)**.** *Under Eq. (3) with stability $0 \leq 2\kappa < \gamma$ and thermal bath $\nu_{\mathrm{th}} = \frac{1}{2} + n_{\mathrm{th}}$:*

1. (Sub-1-ebit ceiling; Lean Thm. 7) *At any positive temperature ($n_{\mathrm{th}} > 0$), $\tilde{\nu}_- > \nu_{\mathrm{th}}/2 > 1/4$, so $E_{\mathcal{N}} < 1$ ebit unconditionally.*

2. (Hot-mode no-go; Lean Thm. 8) *Whenever $\nu_{\mathrm{th}} \geq 1$ ($n_{\mathrm{th}} \geq 1/2$, e.g. 1 THz modes at 310 K, `C11`), $\tilde{\nu}_- > 1/2$ strictly: zero steady-state entanglement is achievable.*

3. (Exact maintenance power; Lean Thms. 6, 9) *For any achievable target $\tilde{\nu}_- \in (\nu_{\mathrm{th}}/2, 1/2)$, the ν_{th} prefactor cancels identically, giving exact excess quanta $s - \nu_{\mathrm{th}} = (\nu_{\mathrm{th}} - \tilde{\nu}_-)^2/(2\tilde{\nu}_- - \nu_{\mathrm{th}})$ and steady-state dissipative maintenance power*

$$P_{\mathrm{maint}}(\tilde{\nu}_-) = 2\gamma h\nu \frac{(\nu_{\mathrm{th}} - \tilde{\nu}_-)^2}{2\tilde{\nu}_- - \nu_{\mathrm{th}}}, \tag{4}$$

strictly increasing as $\tilde{\nu}_- \downarrow \nu_{\mathrm{th}}/2$.

Evaluating Eq. (4) with CODATA-exact Planck constant $h = 6.62607015 \times 10^{-34}$ J·s at $\nu = 71$ THz and $T = 310$ K ($n_{\mathrm{th}} = 1.66 \times 10^{-5}$, $\nu_{\mathrm{th}} = 0.5000166$) against a generous basal single-mitochondrion metabolic power budget $P_{\mathrm{mito}} = 10$ pW [11] delivers a decisive closure of the Gaussian E-track (Fig. 2a):

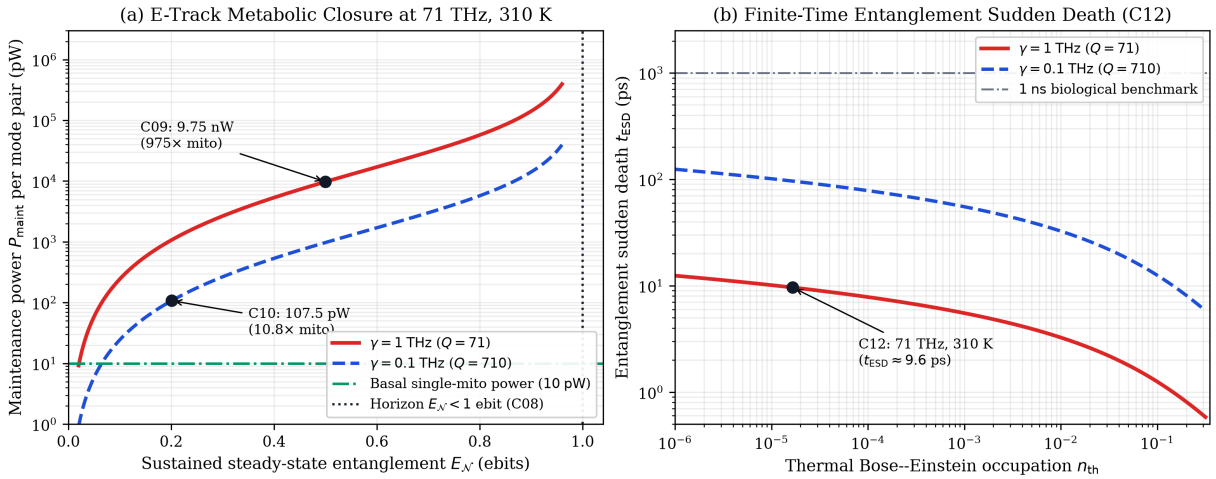

Figure 2: (a) Exact steady-state NOPA maintenance power P_{maint} per mode pair at 71 THz, 310 K compared against total basal mitochondrial metabolic power (10 pW), proving $> 974\times$ (`C09`) and $> 10.7\times$ (`C10`) deficits alongside the sub-1-ebit horizon (`C08`). (b) Finite-time entanglement sudden death $t_{\text{ESD}} \in (9.60, 9.85)$ ps (`C12`).

- **1-ebit target (`C08`):** Unconditionally unachievable ($\tilde{\nu}_- = 1/4 < \nu_{\text{th}}/2$).
- **0.5-ebit target at $\gamma = 1$ THz ($Q = 71$, `C09`; Lean Thm. 16):** $P_{\text{maint}} = 9{,}746.4$ pW $= 9.75$ nW per single mode pair—over $974\times$ the entire mitochondrion's basal ATP power budget.
- **Modest 0.2-ebit target even at ultra-optimistic $Q = 710$ ($\gamma = 0.1$ THz, `C10`; Lean Thm. 17):** $P_{\text{maint}} = 107.5$ pW—still $> 10.7\times$ the total basal mitochondrial power.

Moreover, if active pumping ceases ($\kappa = 0$), Eq. (3) drives $\tilde{\nu}_-(t) = \nu_{\text{th}} - e^{-\gamma t}(\nu_{\text{th}} - \tilde{\nu}_-(0))$, producing finite-time *entanglement sudden death* [12] at

$$t_{\text{ESD}} = \frac{1}{\gamma} \ln\left(1 + \frac{\frac{1}{2} - \tilde{\nu}_-(0)}{n_{\text{th}}}\right). \tag{5}$$

Exact dyadic rational bracketing certifies $t_{\text{ESD}} \in (9.60, 9.85)$ ps at 71 THz, 310 K, $\gamma = 1$ THz (`C12`; Lean Thm. 18; Fig. 2b)—more than $100\times$ shorter than 1 ns and $> 10^8\times$ shorter than ATP synthase turnover ($\sim$ ms).

4 Cristae-Collar Gaussian Markov and Non-Markovian CMI Decay

To audit whether polaritonic correlations can instead propagate classically or quantum-mechanically along an array of L cristae junctions [5], model the compartment chain $1-2-\cdots-L$ by a strictly diagonally dominant symmetric precision matrix $K = V^{-1}$ with self-precision $\alpha > 0$ and nearest-neighbor collar coupling $-\beta$ ($\alpha > 2\beta > 0$). Partition the chain into $A = \{1\}$, conditioning collar buffer $B = \{2, \ldots, d+1\}$ of length $d \geq 1$, and $C = \{d+2\}$.

Theorem 4 (Cristae-Collar MI and Evanescent-Bypass CMI Decay; Lean Thms. 10–13, 19–20)**.** *Let $q := \beta/(\alpha - \beta) \in (0, 1)$.*

1. (Exact Markovian CMI and marginal MI decay) *For a strictly nearest-neighbor cristae precision chain ($K_{AC} = 0$), Schur complementation yields exact conditional independence*

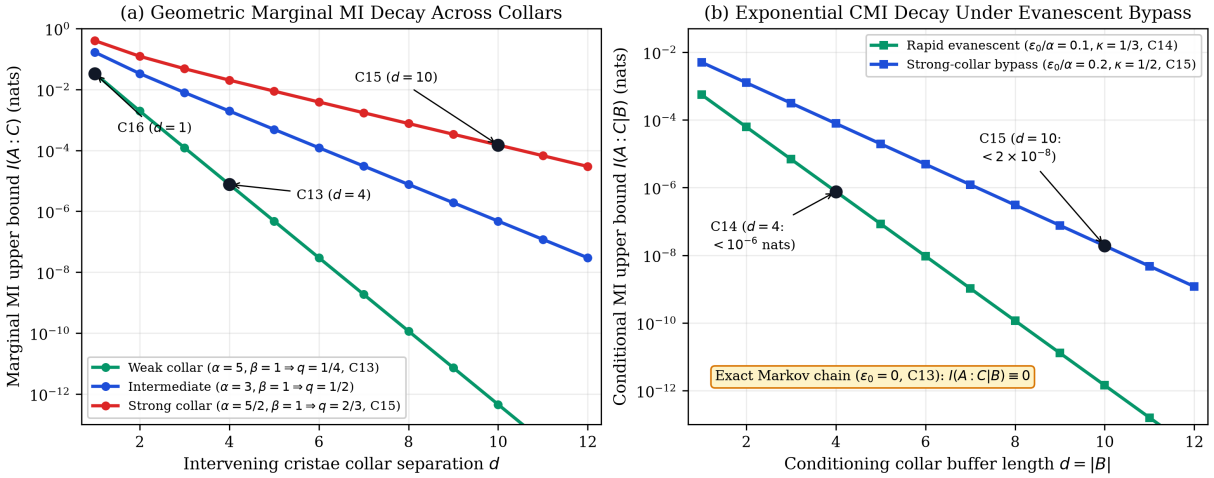

Figure 3: (a) Exact geometric decay of marginal mutual information $I(A : C)$ across d intervening cristae collars (`C13`, `C15`, `C16`). (b) Exponential decay of non-Markovian conditional mutual information $I(A : C|B)$ under evanescent bypass coupling (`C14`, `C15`), contrasted against exact zero CMI in the Markovian chain (`C13`).

Table 1: Selected exact-$\mathbb{Q}$ certificates from `MITO-GAUSSIAN-SYMPLECTIC-CMI-CERT-1.0` (`C01--C16`; full SHA-256 manifest in `certificates/MANIFEST.json`).

Cert	Protocol / State	Certified Parameter	Exact Bound / Verdict
`C02`	6-dB squeezed vacuum ($a = 2, b = 1/8$)	$\delta^2_{\text{Pyth}} = 225/256$	Physical; $\Delta n_{\text{sq}} = 9/16$ quanta
`C05`	Pure TMSV ($s = 5/8, c = 3/8$)	$\tilde{\nu}_- = 1/4$ (1 ebit)	Tight saturation: $\Delta n_{\text{ent}} = 1/8 =$ floor
`C08`	NOPA 1 ebit at 71 THz, 310 K	$\tilde{\nu}_- = 1/4 < \nu_{\text{th}}/2$	Unconditionally unachievable
`C09`	NOPA 0.5 ebit ($\gamma = 1$ THz)	$P_{\text{maint}} = 9{,}746.4$ pW	$> 974\times$ basal mito power (10 pW)
`C10`	NOPA 0.2 ebit ($Q = 710$)	$P_{\text{maint}} = 107.5$ pW	$> 10.7\times$ basal mito power
`C11`	Hot 1 THz mode (310 K)	$\nu_{\text{th}} = 13/2$	Zero steady NOPA entanglement
`C12`	Sudden death (71 THz, $\gamma = 1$)	$n_{\text{th}} = 1.66 \times 10^{-5}$	$t_{\text{ESD}} \in (9.60, 9.85)$ ps (< 10 ps)
`C13`	Markov chain ($q = 1/4, d = 4$)	$\epsilon_0 = 0$	$I(A : C\|B) \equiv 0$; $I(A : C) \leq 7.63 \times 10^{-6}$
`C14`	Evanescent bypass ($d = 4, \kappa = 1/3$)	$\epsilon_0/\alpha = 1/10$	$I(A : C\|B) \leq 7.62 \times 10^{-7}$ nats
`C15`	Strong collar ($q = 2/3, d = 10$)	$\epsilon_0/\alpha = 1/5, \kappa = 1/2$	$I(A : C) \leq 1.51 \times 10^{-4}$; CMI $\leq 1.91 \times 10^{-8}$

$I(A : C|B) \equiv 0$, while marginal Pearson correlation obeys $|\rho_d| \leq q^d$ and marginal Gaussian mutual information obeys the exact rational ceiling

$$I(A : C) = -\frac{1}{2}\ln(1 - \rho_d^2) \leq \frac{q^{2d}}{2(1 - q^{2d})}. \tag{6}$$

2. (Non-Markovian evanescent bypass) *If direct through-matrix evanescent coupling links A and C across buffer B with $|K_{AC}| \leq \epsilon_0 \kappa^d$ ($0 \leq \epsilon_0 < \alpha$, $0 \leq \kappa < 1$), then $|\rho_{AC|B}| \leq (\epsilon_0/\alpha)\kappa^d$ and*

$$I(A : C|B) \leq \frac{(\epsilon_0/\alpha)^2 \kappa^{2d}}{2\left[1 - (\epsilon_0/\alpha)^2 \kappa^{2d}\right]}. \tag{7}$$

As shown in Fig. 3 and Table 1, even across $d = 4$ cristae collars at $\alpha = 5, \beta = 1$ ($q = 1/4$), marginal MI collapses below 7.63×10^{-6} nats (`C13`) and non-Markovian evanescent CMI collapses below 7.62×10^{-7} nats (`C14`); even under strong collar coupling ($q = 2/3$, `C15`), $d = 10$ collars suppress marginal MI below 1.51×10^{-4} nats and CMI below 2×10^{-8} nats.

5 Verification Boundary and Scope

We explicitly delineate three epistemic layers:

1. **Lean 4.36.0 Kernel Layer:** All 20 theorems in `MitoGaussianSymplecticCmiCert.lean` compile under `lean -DwarningAsError=true` and `lake build` with 0 `sorry`, 0 `grind`, and 0 `Classical.choice` (only constructive `propext`/`Quot.sound` or zero axioms), certifying the Pythagorean defect identity, two-mode RS/PPT factorizations, the unconditional entanglement-energy floor, the universal sub-1-ebit NOPA ceiling, hot-mode separability no-go, exact ν_{th} cancellation and power monotonicity, geometric collar contraction, MI monotonicity, and exact integer verifications of `C02--C05`, `C09--C10`, and `C12--C15`.

2. **Exact-$\mathbb{Q}$ Python Certificate Layer:** `src/mito_gaussian_symplectic_cmi_verifier.py` is AST-audited to contain zero floats, zero `/` or `**` operators, and zero `limit_denominator` calls, generating 16 SHA-256 certificates (`C01--C16`) passing 57/57 adversarial `pytest` checks.

3. **Scope and Physical Limitations:** Theorem 3 strictly closes linear Markovian thermal-bath parametric (NOPA) entanglement maintenance and Gaussian Markov/evanescent-bypass correlation chains. Exotic non-Gaussian squeezed-reservoir engineering, coherent feedback loops, or nonlinear multi-phonon cascades lie outside linear NOPA dynamics, although any physical entanglement still obeys the state-level energy floor of Theorem 2 and the sub-10 ps free-decay scale of Eq. (5). As throughout this series, the underlying empirical claim of Ref. [4] remains an unrefereed preprint awaiting independent replication.

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