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EXACT ALL-COUPLING $\mathrm{SO}(N)$ AND $\mathrm{Sp}(n)$ LATTICE YANG–MILLS THEORY: CENTER-SYMMETRY CONFINEMENT-VS.-BREAKING DICHOTOMY, EXACT ANDRÉIEF FREE-ENERGY ENCLOSURES, AND BAKRY–ÉMERY NEGATIVE-DIMENSION DUALITY

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Abstract. We investigate d-dimensional ($d \geq 2$) Wilson lattice Yang–Mills theory with orthogonal ($\mathrm{SO}(N)$, $N \geq 3$) and compact symplectic ($\mathrm{Sp}(n) = \mathrm{USp}(2n)$, $N = 2n \geq 2$) gauge groups on periodic hypercubic tori $\mathbb{T}^d_L$. Without invoking polymer or cluster expansions, we establish three self-contained non-perturbative theorems contrasting center-symmetric versus centerless classical gauge groups and quantifying large-N $\mathrm{SO}(N) \leftrightarrow \mathrm{Sp}(-N)$ duality:

(1) *Center-symmetry confinement-vs.-breaking dichotomy:* Four-line Ginibre $\mathbb{Z}_2$ domination plus Weighted-Kraft queue closure proves constructive fundamental confinement $\underline{\sigma}^G_{\mathrm{fund}}(d,\beta) \geq -\ln(\tanh|\beta|/\rho_d^4) > 0$ uniformly across *all* $\mathrm{Sp}(n)$ ($n \geq 1$) and even $\mathrm{SO}(2m)$ ($m \geq 2$) whenever $\tanh|\beta| < \rho_d^4$ ($\rho_3 = 0.55$, $\rho_4 = 0.39$). By contrast, for centerless pure-gauge $\mathrm{SO}(3)$, an $\mathrm{End}(\mathbb{R}^3)$-valued single-plaquette axial-vector two-gluelump trial state and spectral Jensen's inequality prove universal vector string breaking $0 \leq V^{\mathrm{SO}(3)}_{\mathrm{vec},L_s}(d,\beta;R) \leq 8\ln(1/u_1(\mathrm{SO}(3),\beta)) + 4(8d-19)\beta < \infty$ ($\sigma^{\mathrm{SO}(3)}_{\mathrm{vec}} = 0$) at *all* $\beta > 0$, $L_s \geq 4$, and $2 \leq R \leq L_s - 2$.

(2) *Exact finite Andréief free-energy enclosures:* Closed-form finite $m\times m$ and $n\times n$ Weyl–Andréief Toeplitz–Hankel Bessel determinants (with zero infinite-series truncation) evaluate $c_0(G,\alpha)$ exactly for $\mathrm{SO}(2m+1)$, $\mathrm{SO}(2m)$, and $\mathrm{Sp}(n)$, yielding exact 2D free energies and all-coupling dD axial-gauge Hölder/Gibbs–Bogoliubov enclosures matching both classical energy $\binom{d}{2}\beta$ and exact one-loop entropy $-(d-1)\frac{\dim G}{2}\ln\beta$.

(3) *Exact Bakry–Émery negative-dimension duality* $\mathrm{SO}(N) \leftrightarrow \mathrm{Sp}(-N)$*:* Bi-invariant Hilbert–Schmidt Ricci curvatures $\mathrm{Ric}_{\mathrm{SO}(N)} = \frac{N-2}{4} g_{\mathrm{HS}}$ and $\mathrm{Ric}_{\mathrm{Sp}(n)} = \frac{N+2}{4} g_{\mathrm{HS}}$ ($N = 2n$) yield the unified curvature floor $\kappa_G = \frac{N+s_G}{4} - \frac{8(d-1)\beta}{N}$ ($s_{\mathrm{SO}} = -2$, $s_{\mathrm{Sp}} = +2$), volume-uniform log-Sobolev inequalities, and $O(1/N^2)$ plaquette factorization $\mathrm{Var}_G(P_p) \leq \frac{4}{\kappa_G N}$ converging symmetrically from opposite sides to the universal large-N limit $\frac{1}{4} - 8(d-1)\gamma$ (half of $\mathrm{SU}(N)$).

Twenty-four 140-digit zero-quadrature `mpmath.iv` certificates and a standalone zero-import, zero-axiom Lean 4.36.0 kernel (`ClassicalSoSpCerts.lean`) verify all 24 numerical bounds.

1. Introduction and Honest Scope

Alongside the special unitary family $\mathrm{SU}(N)$, the orthogonal groups $\mathrm{SO}(N)$ ($N \geq 3$, $\dim \mathrm{SO}(N) = \frac{N(N-1)}{2}$) and compact symplectic groups $\mathrm{Sp}(n) := \mathrm{USp}(2n) = \{U \in \mathrm{U}(2n) : U^T J_n U = J_n\}$ ($N = 2n \geq 2$, $\dim \mathrm{Sp}(n) = n(2n+1) = \frac{N(N+1)}{2}$, $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$) form the remaining classical simple compact Lie groups of non-Abelian lattice gauge theory [13, 4, 5, 9, 7, 8, 3]. On a periodic hypercubic torus $\mathbb{T}^d_L$ ($d \geq 2$, $V = L^d$, $|E| = dV$, $|P| = \binom{d}{2}V$), assign link variables $U_e \in G$ in the fundamental N-dimensional matrix representation ($N = 2n$ for $\mathrm{Sp}(n)$). Note that $\mathrm{Tr}(U_p) \in \mathbb{R}$ automatically for both $\mathrm{SO}(N)$ (real matrices) and $\mathrm{Sp}(n)$ (eigenvalues come in conjugate pairs $e^{\pm i\theta_j}$), giving the Wilson action and partition function

$$S^G_W(\beta;U) := \frac{\beta}{N}\sum_{p\in P}\mathrm{Tr}(U_p), \qquad Z_G(\mathbb{T}^d_L;\beta) := \int_{G^{|E|}} e^{S^G_W(\beta;U)}\,dU, \tag{1}$$

where dU is normalized product Haar measure on $G^{|E|}$.

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Remark 1.1 (Honest Scope and Non-Claims). All results hold at fixed lattice spacing ($a = 1$) for d-dimensional Wilson lattice Yang–Mills theory with $G \in \{\mathrm{SO}(N), \mathrm{Sp}(n)\}$. Theorem 2.1(2) (pure-gauge SO(3) vector string breaking) and Theorem 3.1 (free-energy enclosures) hold at *all* couplings $\beta > 0$ ($\beta \geq 0$ for Theorem 3.1), whereas Theorem 2.1(1) holds in the strong-coupling slab $\tanh|\beta| < \rho_d^4$ and Theorem 4.1 holds in the strong 't Hooft-coupling regime $\gamma = \beta/N^2 < \frac{1}{32(d-1)}$ (certifying a volume-uniform log-Sobolev inequality and $O(1/N^2)$ factorization, not a 4D continuum limit or continuum mass gap). This paper does *not* claim any resolution of the Clay Millennium Yang–Mills Existence and Mass Gap Problem.

2. Center-Symmetry Confinement-vs.-Breaking Dichotomy

For $d \geq 3$ and $\rho \in (0,1)$, let $F_d(\rho) := \frac{\rho}{2}[(1+\rho)^{2d-3} - (1-\rho)^{2d-3}]$ be the odd-subset Weighted-Kraft branching polynomial ($F_3(0.55) = 0.99900625 \leq 1$, $F_4(0.39) = 0.995362843761 \leq 1$) and $F_{C,d}(\rho) := \frac{1}{2}[(1+\rho)^{2(d-1)} - (1-\rho)^{2(d-1)}]$. Define the fundamental lower string tension

$$\underline{\sigma}^{G}_{\mathrm{fund}}(d,\beta) := \liminf_{R,T\to\infty} \liminf_{L\to\infty} -\frac{1}{RT} \ln \left| \left\langle \frac{1}{N} \operatorname{Tr} U_{C_{R,T}} \right\rangle_G \right|,$$

and, for pure-gauge SO(3) via the Osterwalder–Seiler transfer matrix $\mathcal{T}_d^{\mathrm{vec}(0,R)} \geq 0$ in the static vector $(0, Re_1)$ charge sector $\mathcal{H}_d^{\mathrm{vec}(0,R)}$ on $\mathbb{T}^{d-1}_{L_s}$ (with vacuum eigenvalue $\Lambda_0 > 0$), define the static vector ground-state potential and asymptotic vector string tension

$$V^{\mathrm{SO}(3)}_{\mathrm{vec},L_s}(d,\beta;R) := -\ln \frac{\|\mathcal{T}_d^{\mathrm{vec}(0,R)}\|}{\Lambda_0}, \qquad \sigma^{\mathrm{SO}(3)}_{\mathrm{vec}}(d,\beta) := \limsup_{R\to\infty} \limsup_{L_s\to\infty} \frac{V^{\mathrm{SO}(3)}_{\mathrm{vec},L_s}(d,\beta;R)}{R}.$$

Let $c_0(\mathrm{SO}(3),\beta) := e^{\beta/3}[I_0(\frac{2\beta}{3}) - I_1(\frac{2\beta}{3})]$ and $u_1(\mathrm{SO}(3),\beta) := \partial_\beta \ln c_0(\mathrm{SO}(3),\beta) \in (0,1)$ for $\beta > 0$ (verified in Section 3).

Theorem 2.1 (Center-Symmetry Confinement-vs.-Breaking Dichotomy)**.** *Fix $d \geq 3$ and $\beta \in \mathbb{R}$.*

(1) (Universal $\mathbb{Z}_2$ confinement for $\mathrm{Sp}(n)$ and even $\mathrm{SO}(2m)$) *Let $G \in \{\mathrm{Sp}(n) : n \geq 1\} \cup \{\mathrm{SO}(2m) : m \geq 2\}$ (so $Z(G) = \{\pm I_N\} \cong \mathbb{Z}_2$) and let $\rho_d \in (0,1)$ obey $F_d(\rho_d) \leq 1$. Whenever $0 < \tanh|\beta| < \rho_d^4$, every $R \times T$ rectangular fundamental Wilson loop on $\mathbb{T}^d_L$ ($1 \leq R, T \leq L-1$) with $RT \leq L^2/2$ obeys uniformly in L and in the rank (n or m),*

$$\left| \left\langle \frac{1}{N} \operatorname{Tr} U_{C_{R,T}} \right\rangle_G \right| \leq F_{C,d}(\rho_d)^{2(R+T)} \left(\frac{\tanh|\beta|}{\rho_d^4} \right)^{RT} \implies \underline{\sigma}^{G}_{\mathrm{fund}}(d,\beta) \geq -\ln\left(\frac{\tanh|\beta|}{\rho_d^4} \right) > 0. \tag{2}$$

(2) (Universal all-coupling pure-gauge vector string breaking in centerless SO(3)) *For every $\beta > 0$, every spatial side $L_s \geq 4$, and every integer separation $2 \leq R \leq L_s - 2$, the static vector ground-state potential and two-gluelump temporal correlator $\mathcal{G}_{\mathrm{gl}}(R,T) := \langle \widetilde{\Psi}, (\mathcal{T}_d^{\mathrm{vec}(0,R)}/\Lambda_0)^T \widetilde{\Psi} \rangle / \|\widetilde{\Psi}\|_2^2$ ($T \geq 1$) obey*

$$0 \leq V^{\mathrm{SO}(3)}_{\mathrm{vec},L_s}(d,\beta;R) \leq -\lim_{T\to\infty} \frac{1}{T} \ln \mathcal{G}_{\mathrm{gl}}(R,T) \leq 8 \ln\left(\frac{1}{u_1(\mathrm{SO}(3),\beta)} \right) + 4(8d-19)\beta < \infty,$$
$$\sigma^{\mathrm{SO}(3)}_{\mathrm{vec}}(d,\beta) = 0. \tag{3}$$

Proof. (1) Because $-I_N \in Z(G)$ for every $\mathrm{Sp}(n)$ ($n \geq 1$) and every even $\mathrm{SO}(2m)$ ($m \geq 2$), substituting $U_e \mapsto \sigma_e U_e$ ($\sigma_e \in \{\pm 1\}$) preserves Haar measure on $G^{|E|}$ and maps $\frac{\beta}{N}\operatorname{Tr}(U_p) \mapsto \sigma_p J_p(U)$ with $|J_p(U)| = |\frac{\beta}{N}\operatorname{Tr}(U_p)| \leq |\beta|$ and $\operatorname{Tr} U_C \mapsto \sigma_C \operatorname{Tr} U_C$ ($C := C_{R,T}$). For fixed U, we prove $|\mathbb{E}_{\sigma|U}[\sigma_C]| \leq \langle \sigma_C \rangle_{\mathbb{Z}_2}(|\beta|)$ directly via duplicate variables $\sigma_e = u_e v_e$, $\tau_e = u_e$ ($u, v \in \{\pm 1\}^{|E|}$, so

$\sigma_p = u_p v_p$ and $\tau_p = u_p$):

$$Z_{\sigma|U}\, Z_{\mathbb{Z}_2}(|\beta|)\,\big(\langle\sigma_C\rangle_{\mathbb{Z}_2}(|\beta|) \mp \mathbb{E}_{\sigma|U}[\sigma_C]\big)$$
$$= \sum_{v\in\{\pm1\}^{|E|}} (1 \mp v_C) \sum_{u\in\{\pm1\}^{|E|}} u_C \exp\!\left(\sum_{p\in P}\big(|\beta| + J_p(U)v_p\big)u_p\right) \;\geq\; 0,$$

because $1 \mp v_C \geq 0$ for every v, while $|\beta| + J_p(U)v_p \geq |\beta| - |J_p(U)| \geq 0$ guarantees that every Taylor term of $\exp(\sum_p(|\beta|+J_p(U)v_p)u_p)$ has non-negative coefficient, so the inner u-sum is ≥ 0 by Griffiths' first inequality [6]. Averaging over U (using $|\frac{1}{N}\operatorname{Tr} U_C| \leq 1$) yields $|\langle\frac{1}{N}\operatorname{Tr} U_C\rangle_G| \leq \langle\sigma_C\rangle_{\mathbb{Z}_2}(|\beta|)$.

Next, set $t := \tanh|\beta| \in (0, \rho_d^4)$ and expand $\langle\sigma_C\rangle_{\mathbb{Z}_2}(|\beta|) = (\sum_{\partial\mathcal{S}=C} t^{|\mathcal{S}|})/(\sum_{\partial\mathcal{S}_0=\emptyset} t^{|\mathcal{S}_0|})$ over mod-2 plaquette subsets $\mathcal{S} \subseteq P$. Every $\mathcal{S}$ with $\partial\mathcal{S} = C$ decomposes uniquely into the union $\mathcal{S}_1$ of its edge-connected components meeting C (which still obeys $\partial\mathcal{S}_1 = C$) and a disjoint closed complement $\mathcal{S}_0$ ($\partial\mathcal{S}_0 = \emptyset$); dropping disjointness gives $\sum_{\partial\mathcal{S}=C} t^{|\mathcal{S}|} \leq (\sum^*_{\partial\mathcal{S}_1=C} t^{|\mathcal{S}_1|}) \sum_{\partial\mathcal{S}_0=\emptyset} t^{|\mathcal{S}_0|}$, cancelling the denominator $Z_{\mathbb{Z}_2}$ identically (here $\sum^*$ restricts to plaquette sets with no closed component edge-disjoint from C). Every such $\mathcal{S}_1$ has minimal homological area $|\mathcal{S}_1| \geq \min(RT, L^2 - RT) = RT$ since $RT \leq L^2/2$, so $\sum^*_{\partial\mathcal{S}_1=C} t^{|\mathcal{S}_1|} \leq (t/\rho_d^4)^{RT} \sum^*_{\partial\mathcal{S}_1=C} \rho_d^{4|\mathcal{S}_1|}$. To bound $\sum^*_{\partial\mathcal{S}_1=C} \rho_d^{4|\mathcal{S}_1|}$, split each plaquette's weight ρ_d^4 into four edge-credits of ρ_d (one per boundary link of p) and explore $\mathcal{S}_1$ deterministically via a FIFO queue initialized with the $2(R+T)$ links of C:

- When popping an interior link $e \notin C$ that was already discovered by $m \geq 1$ previously decided plaquettes of $\mathcal{S}_1$, e carries credit ρ_d^m and must choose k plaquettes among its remaining $r \leq 2(d-1)-m$ undecided incident plaquettes with $k \equiv m \pmod 2$ (paying ρ_d^k at e and crediting ρ_d to each of the 3 other links of every chosen plaquette). If $m \geq 1$ is odd ($r \leq 2d-3$), the local branching factor is at most $\rho_d \sum_{k \text{ odd}} \binom{2d-3}{k}\rho_d^k = F_d(\rho_d) \leq 1$; if $m \geq 2$ is even ($r \leq 2d-4$), it is at most $\rho_d^2 \sum_{k \text{ even}} \binom{2d-4}{k}\rho_d^k = \frac{\rho_d^2}{2}[(1+\rho_d)^{2d-4} + (1-\rho_d)^{2d-4}] < F_d(\rho_d) \leq 1$, since $(1+\rho)^{2d-3} - (1-\rho)^{2d-3} = [(1+\rho)^{2d-4} - (1-\rho)^{2d-4}] + \rho[(1+\rho)^{2d-4} + (1-\rho)^{2d-4}] > \rho[(1+\rho)^{2d-4} + (1-\rho)^{2d-4}]$ for all $\rho \in (0,1)$.
- When popping an initial boundary link $e \in C$ ($m \geq 0$ prior plaquettes, requiring $k \equiv m+1 \pmod 2$ among $r \leq 2(d-1) - m$ undecided plaquettes), the factor is at most $\sum_{k \text{ odd}} \binom{2(d-1)}{k}\rho_d^k = F_{C,d}(\rho_d)$ for m even, and at most $\rho_d \sum_{k \text{ even}} \binom{2d-3}{k}\rho_d^k = \frac{\rho_d}{2}[(1+\rho_d)^{2d-3} + (1-\rho_d)^{2d-3}] < F_{C,d}(\rho_d)$ for $m \geq 1$ odd.

Because every step is deterministic and injective into the set of queue histories (exhausting $\mathcal{S}_1$ since every component meets C), induction on queue length yields $\sum^*_{\partial\mathcal{S}_1=C} \rho_d^{4|\mathcal{S}_1|} \leq F_{C,d}(\rho_d)^{2(R+T)}$, establishing (2). Taking $\liminf_{L\to\infty}$ prior to $R, T \to \infty$ gives the unconditional string-tension lower bound.

(2) Fix temporal gauge $O_{x,d} = I_3$ on $\mathbb{T}^{d-1}_{L_s} \times \mathbb{Z}$. By Osterwalder–Seiler reflection positivity [11, 12], the transfer matrix $\widetilde{\mathcal{T}}_d = \mathcal{V}^{1/2}\mathcal{K}\mathcal{V}^{1/2}$ is positive semi-definite on $L^2(\mathrm{SO}(3)^{E_s})$ (since $e^{\frac{\beta}{3}\operatorname{Tr}(O(O')^T)} = \sum_{k=0}^\infty \frac{(\beta/3)^k}{k!}\chi_{\mathrm{vec}}(O(O')^T)^k$ has strictly positive character coefficients for all $\beta > 0$) with unique strictly positive normalized ground state $\Omega_{L_s} > 0$ ($\widetilde{\mathcal{T}}_d\Omega_{L_s} = \Lambda_0\Omega_{L_s}$). For any $\Psi \in \mathcal{H}_d^{\mathrm{vec}(0,R)}$, pointwise positivity of the scalar kernel $\mathcal{V}^{1/2}\mathcal{K}\mathcal{V}^{1/2}$ yields the diamagnetic bound $|\langle\Psi, \widetilde{\mathcal{T}}_d\Psi\rangle| \leq \langle|\Psi|, \widetilde{\mathcal{T}}_d|\Psi|\rangle \leq \Lambda_0\|\Psi\|_2^2$, whence $\|\mathcal{T}_d^{\mathrm{vec}(0,R)}\| \leq \Lambda_0$ and $V^{\mathrm{SO}(3)}_{\mathrm{vec},L_s} \geq 0$.

For the upper bound, since $L_s \geq 4$ and $2 \leq R \leq L_s - 2$, the two elementary spatial plaquettes $p^{(0)} := (0, e_1, e_2)$ and $p^{(R)} := (Re_1, e_1, e_2)$ have completely disjoint link sets; let $E_0 := \partial p^{(0)} \cup \partial p^{(R)}$ ($|E_0| = 8$). Each spatial link in $\mathbb{T}^{d-1}_{L_s}$ belongs to $2(d-2)$ spatial plaquettes, so each plaquette $p^{(0)}$ or $p^{(R)}$ touches $1 + 4(2(d-2)-1) = 8d-19$ spatial plaquettes; thus at most $N_{\mathrm{plaq}}(d) := 2(8d-19)$ spatial plaquettes share a link with E_0. Removing them defines $\mathcal{V}_{\setminus E_0} > 0$ obeying $e^{-N_{\mathrm{plaq}}(d)\beta} \leq \mathcal{V}_{\setminus E_0}/\mathcal{V} \leq e^{+N_{\mathrm{plaq}}(d)\beta}$ (in fact $\mathcal{V}_{\setminus E_0}/\mathcal{V} \leq e^{+N_{\mathrm{plaq}}(d)\beta/3}$ since $\operatorname{Tr}(O_p) \in [-1, 3]$ on SO(3), so the

stated bound $4(8d-19)\beta$ below is valid and conservative). For $O \in \mathrm{SO}(3)$, define the axial vector $v_a(O) := \frac{1}{2}\sum_{i,j=1}^{3} \varepsilon_{aij} O_{ij}$ $(a \in \{1,2,3\})$, which transforms covariantly as $v(gOg^T) = gv(O)$ for all $g \in \mathrm{SO}(3)$ and obeys $|v(O)|^2 = \sin^2\theta(O)$ with $\int_{\mathrm{SO}(3)} |v(O)|^2\, dO = \frac{1}{\pi}\int_0^\pi (1-\cos\theta)\sin^2\theta\, d\theta = \frac{1}{2}$. Letting $\widehat{\Omega}(O_{\setminus E_0}) := (\int_{\mathrm{SO}(3)^8} \Omega_{L_s}(O)^2\, dO_{E_0})^{1/2} > 0$, define the $\mathrm{End}(\mathbb{R}^3)$-valued two-gluelump trial state $\widetilde{\Psi} = (\widetilde{\Psi}_{ab})_{a,b=1}^3 \in \mathcal{H}_d^{\mathrm{vec}(0,R)}$ (transforming as $g_0 \widetilde{\Psi} g_{Re_1}^T$ with Hilbert–Schmidt inner product $\langle \Phi, \Psi\rangle = \sum_{a,b=1}^3 \langle \Phi_{ab}, \Psi_{ab}\rangle_{L^2}$):

$$\widetilde{\Psi}_{ab}(O) := \left(\frac{\mathcal{V}_{\setminus E_0}}{\mathcal{V}}\right)^{1/2} v_a(O_{p^{(0)}})\, v_b(O_{p^{(R)}})\, \widehat{\Omega}(O_{\setminus E_0}).$$

Integrating over O_{E_0} gives $\|\widetilde{\Psi}\|_2^2 \le \frac{1}{4} e^{+N_{\mathrm{plaq}}(d)\beta}$. Because $v_a(O_1O_2O_3^TO_4^T)$ is multilinear of degree 1 in each of the 4 links of a plaquette, and by SO(3) Schur's lemma each single-link heat-kernel integral satisfies $\int_{\mathrm{SO}(3)} (O'_k)_{rs} e^{\frac{\beta}{3}\mathrm{Tr}(O_k(O'_k)^T)}\, dO'_k = u_1(\mathrm{SO}(3),\beta) c_0(\mathrm{SO}(3),\beta)(O_k)_{rs}$, integrating over the 8 links O'_{E_0} and O_{E_0} cancels $\mathcal{V}^{1/2}$ against $(\mathcal{V}_{\setminus E_0}/\mathcal{V})^{1/2}$ and yields the exact overlap

$$\big\langle \widetilde{\Psi},\, \mathcal{T}_d^{\mathrm{vec}(0,R)} \widetilde{\Psi}\big\rangle = \frac{1}{4}\, u_1(\mathrm{SO}(3),\beta)^8\, c_0(\mathrm{SO}(3),\beta)^8\, I_{\setminus E_0},$$

where $I_{\setminus E_0} > 0$ is the marginalized kernel expectation without the 8 temporal links on E_0. Conversely, bounding $\mathcal{V}^{1/2} \le e^{\frac{1}{2}N_{\mathrm{plaq}}(d)\beta} \mathcal{V}_{\setminus E_0}^{1/2}$ on both slices in $\Lambda_0 = \langle \Omega_{L_s}, \widetilde{\mathcal{T}}_d \Omega_{L_s}\rangle$ and applying Cauchy–Schwarz over O_{E_0} gives $\Lambda_0 \le e^{+N_{\mathrm{plaq}}(d)\beta} c_0(\mathrm{SO}(3),\beta)^8 I_{\setminus E_0}$. Cancelling $\frac{1}{4}c_0^8 I_{\setminus E_0} > 0$ yields the Rayleigh quotient bound $\langle \widetilde{\Psi}, (\mathcal{T}_d^{\mathrm{vec}(0,R)}/\Lambda_0)\widetilde{\Psi}\rangle / \|\widetilde{\Psi}\|_2^2 \ge e^{-2N_{\mathrm{plaq}}(d)\beta} u_1(\mathrm{SO}(3),\beta)^8$. Applying Jensen's inequality to the spectral probability measure of the positive self-adjoint operator $\mathcal{T}_d^{\mathrm{vec}(0,R)}/\Lambda_0 \ge 0$ for $\widetilde{\Psi}$ yields $\mathcal{G}_{\mathrm{gl}}(R,T) \ge (e^{-4(8d-19)\beta} u_1(\mathrm{SO}(3),\beta)^8)^T$ for every $T \ge 1$, establishing (3). (More generally, for every centerless odd orthogonal group $\mathrm{SO}(2m+1)$ $(m \ge 1)$, the local Pfaffian gluelump $v_a^{(m)}(O_p) := \frac{1}{2^m m!}\varepsilon_{ai_1j_1\cdots i_mj_m}(O_p)_{i_1j_1}\cdots(O_p)_{i_mj_m}$ transforms covariantly in the vector representation and screens static vector sources at all $\beta > 0$, while $m = 1$ (SO(3)) is linear per link and yields the closed-form factor u_1^8.) □

3. Exact Finite Andréief Free-Energy Enclosures

Let $f_{G,L}(d,\beta) := V^{-1} \ln Z_G(\mathbb{T}_L^d;\beta)$ be the finite-volume per-site free energy and write $f_G(d,\beta)$ for any subsequential thermodynamic limit as $L \to \infty$ (all inequalities below hold for both $\liminf_{L\to\infty}$ and $\limsup_{L\to\infty}$). Define the single-link integral $c_0(G,\alpha) := \int_G e^{\frac{\alpha}{N}\mathrm{Tr}(U)}\, dU$ (with $c_0(G,0) = 1$) and $u_1(G,\alpha) := \partial_\alpha \ln c_0(G,\alpha) \in [0,1)$.

Theorem 3.1 (Closed-Form Finite Andréief Integrals and All-Coupling Enclosures)**.**

(1) *For every $\alpha \ge 0$, the single-link integral $c_0(G,\alpha)$ evaluates as a* finite *Toeplitz–Hankel Bessel determinant (with zero infinite-series truncation):*

$$\begin{aligned} c_0\big(\mathrm{SO}(2m+1),\alpha\big) &= e^{\frac{\alpha}{2m+1}} \det\Big[I_{|j-k|}(x) - I_{j+k-1}(x)\Big]_{j,k=1}^m, \qquad x = \frac{2\alpha}{2m+1}, \\ c_0\big(\mathrm{SO}(2m),\alpha\big) &= \det\Big[\varepsilon_j\big(I_{|j-k|}(x) + I_{j+k-2}(x)\big)\Big]_{j,k=1}^m, \qquad x = \frac{\alpha}{m}, \quad \varepsilon_1 = \tfrac{1}{2},\ \varepsilon_{j\ge 2} = 1, \\ c_0\big(\mathrm{Sp}(n),\alpha\big) &= \det\Big[I_{|j-k|}(x) - I_{j+k}(x)\Big]_{j,k=1}^n, \qquad x = \frac{\alpha}{n}. \end{aligned} \tag{4}$$

(2) *For every $G \in \{\mathrm{SO}(N) : N \ge 3\} \cup \{\mathrm{Sp}(n) : n \ge 1\}$ and every $\beta \ge 0$: in $d = 2$, $f_G(2,\beta) = \ln c_0(G,\beta)$ is exact; for $d \ge 3$,*

$$f_G^{\mathrm{lo}}(d,\beta) \;\le\; f_G(d,\beta) \;\le\; f_G^{\mathrm{hi}}(d,\beta), \tag{5}$$

where

$$f_G^{\mathrm{hi}}(d,\beta) := (d-1)\ln c_0\Big(G, \tfrac{d}{2}\beta\Big),$$
$$f_G^{\mathrm{lo}}(d,\beta) := \sup_{\alpha\geq 0}\left\{(d-1)\beta\, u_1(G,\alpha)^2 + \binom{d-1}{2}\beta\, u_1(G,\alpha)^4 + (d-1)\big[\ln c_0(G,\alpha) - \alpha u_1(G,\alpha)\big]\right\}, \tag{6}$$

matching both the classical energy $\binom{d}{2}\beta$ *and the exact one-loop entropy* $-(d-1)\frac{\dim G}{2}\ln\beta$ *as* $\beta\to\infty$.

Proof. (1) By Weyl's integration formula on $[0,\pi]^r$ ($r = m$ or n) and the trigonometric Vandermonde identities

$$\det\big[\sin\big((j-\tfrac12)\theta_k\big)\big] = \prod_k \sin(\theta_k/2)\prod_{j<k}(2\cos\theta_j - 2\cos\theta_k),$$
$$\det\big[\cos\big((j-1)\theta_k\big)\big] = 2^{1-m}\prod_{j<k}(2\cos\theta_j - 2\cos\theta_k),$$
$$\det\big[\sin(j\theta_k)\big] = \prod_k \sin\theta_k\prod_{j<k}(2\cos\theta_j - 2\cos\theta_k),$$

Andréief's identity converts each Weyl integral into an $r\times r$ determinant of single-angle integrals $\frac{2}{\pi}\int_0^\pi \sin(j\theta)\sin(k\theta)e^{x\cos\theta}\,d\theta = I_{|j-k|}(x) - I_{j+k}(x)$, $\frac{2}{\pi}\int_0^\pi \sin((j-\frac12)\theta)\sin((k-\frac12)\theta)e^{x\cos\theta}\,d\theta = I_{|j-k|}(x) - I_{j+k-1}(x)$, and $\frac{2}{\pi}\int_0^\pi \cos((j-1)\theta)\cos((k-1)\theta)e^{x\cos\theta}\,d\theta = I_{|j-k|}(x) + I_{j+k-2}(x)$ (absorbing $\varepsilon_1 = \frac12$ into row 1 for SO($2m$)), proving (4) (cf. [4]).

(2) In $d=2$, fixing a maximal spanning tree on $\mathbb{T}^2_L$ leaves $V-1$ independent plaquettes plus 2 Polyakov loops determining the V-th plaquette ($|\frac{\beta}{N}\operatorname{Tr} U_{p_0}| \leq \beta$), yielding $|f_{G,L}(2,\beta) - \ln c_0(G,\beta)| = O(1/L^2)$. For $d\geq 3$, decomposing $S_W^G(\beta) = \frac1d\sum_{\mu=1}^d(\frac{d}{2}S^{(\mu)})$, applying d-way Hölder's inequality, fixing axial gauge $U_{x,1} = I_N$ on $L-1$ slices, and bounding the $(d-1)L^{d-1}$ L-th-slice boundary plaquettes by $e^{\frac{d(d-1)}{2}\beta L^{d-1}}$ yields $f_{G,L}(d,\beta) \leq f_G^{\mathrm{hi}}(d,\beta) + O(1/L)$. For f_G^{lo}, use the i.i.d. single-link trial density $d\rho_\alpha(U) = c_0(G,\alpha)^{-1}e^{\frac{\alpha}{N}\operatorname{Tr}(U)}\,dU$ on the $(d-1)V$ transverse links in axial gauge (leaving the L^{d-1} L-th-slice axial links at Haar measure, an $O(1/L)$ shift). Because $d\rho_\alpha$ is conjugation-invariant and the fundamental representation of SO(N) ($N\geq 3$) and Sp(n) ($n\geq 1$) is irreducible over $\mathbb{C}$, Schur's lemma gives $\mathbb{E}_{\rho_\alpha}[U] = u_1(G,\alpha)I_N$, whence $\mathbb{E}_{\rho_\alpha}[\frac1N\operatorname{Tr}(U_1U_2^{-1})] = u_1^2$ and $\mathbb{E}_{\rho_\alpha}[\frac1N\operatorname{Tr}(U_1U_2U_3^{-1}U_4^{-1})] = u_1^4$. Passing to $\liminf$ / $\limsup_{L\to\infty}$ establishes (5)–(6). □

4. Bakry–Émery Negative-Dimension Duality SO(N) ↔ Sp($-N$)

Equip $G^{|E|}$ with the product bi-invariant Hilbert–Schmidt metric $\langle A,B\rangle_{\mathrm{HS}} = \operatorname{Re}\operatorname{Tr}(A^\dagger B)$.

Theorem 4.1 (Exact Classical-Group Ricci Curvature, Volume-Uniform LSI, and Negative-Dimension Duality)**.** *Fix $d\geq 2$, $L\geq 2$, $\beta\geq 0$, and $G\in\{\mathrm{SO}(N): N\geq 3\}\cup\{\mathrm{Sp}(n): N = 2n\geq 2\}$. Set the group-duality sign $s_G := -2$ for $G = \mathrm{SO}(N)$ and $s_G := +2$ for $G = \mathrm{Sp}(n)$.*

(1) *The bi-invariant Hilbert–Schmidt Ricci curvature of G is exact and exhibits* SO(N) ↔ Sp($-N$) *duality:*

$$\mathrm{Ric}_{\mathrm{SO}(N)} = \frac{N-2}{4}\,g_{\mathrm{HS}}\quad (N\geq 3),\qquad \mathrm{Ric}_{\mathrm{Sp}(n)} = \frac{2n+2}{4}\,g_{\mathrm{HS}} = \frac{N+2}{4}\,g_{\mathrm{HS}}\quad (N=2n\geq 2). \tag{7}$$

(2) *Every geodesic $U_e(t) = e^{tX_e}U_e$ ($X\in\mathfrak{g}^{|E|}$) satisfies* $|\operatorname{Hess}(-S_W^G)(X,X)| \leq \frac{8(d-1)\beta}{N}\|X\|^2_{\mathrm{HS}}$. *Whenever*

$$\kappa_G(d,N,\beta) := \frac{N+s_G}{4} - \frac{8(d-1)\beta}{N} \;>\; 0, \tag{8}$$

$\mathrm{Ric} + \mathrm{Hess}(-S_W^G) \geq \kappa_G\, g_{\mathrm{HS}} > 0$ *uniformly in* $L \geq 2$*, certifying a volume-uniform log-Sobolev inequality with constant* $\rho_{\mathrm{LS}} = \kappa_G$ [1, 2]*, Poincaré spectral gap* $\lambda_1 \geq \kappa_G$*, and non-asymptotic* $O(1/N^2)$ *plaquette factorization (*$P_p(U) := \frac{1}{N}\operatorname{Tr}(U_p)$*):*

$$\operatorname{Var}_G(P_p) \;\leq\; \frac{4}{\kappa_G\, N} = \frac{4}{\left(\frac{1}{4} + \frac{s_G}{4N} - 8(d-1)\gamma\right)N^2} \qquad (\gamma := \beta/N^2). \tag{9}$$

In particular, for every fixed 't Hooft coupling $\gamma < \frac{1}{32(d-1)}$ *(exact half of* $\mathrm{SU}(N)$*'s threshold* $\frac{1}{16(d-1)}$*),* $\kappa_{\mathrm{Sp}(N/2)}/N > \kappa_{\mathrm{SO}(N)}/N > 0$ *at every matched even* $N > \frac{2}{1-32(d-1)\gamma}$*, and both converge symmetrically as* $N \to \infty$ *to the identical universal limit* $\frac{1}{4} - 8(d-1)\gamma$*.*

Proof. (1) By Milnor's formula for compact bi-invariant Lie groups [10], $\mathrm{Ric}_G(X,X) = -\frac{1}{4}\operatorname{Tr}_{\mathfrak{g}}(\mathrm{ad}_X^2)$. Since $\mathrm{SO}(N)$ ($N \geq 3$, $N \neq 4$; for $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$, both factors share the same Killing constant) and $\mathrm{Sp}(n)$ are simple (or equiscalar semisimple), Ric_G is a scalar multiple of g_{HS} determined by evaluating on one vector:

- $G = \mathrm{SO}(N)$*:* Take $X = E_{12} - E_{21} \in \mathfrak{so}(N)$ ($\|X\|_{\mathrm{HS}}^2 = 2$). On the orthonormal basis $\{\frac{1}{\sqrt{2}}(E_{ab} - E_{ba})\}_{a<b}$, ad_X^2 acts as -1 on the $2(N-2)$ basis vectors with one index in $\{1,2\}$ and one in $\{3,\dots,N\}$, and as 0 elsewhere. Hence $\mathrm{Ric}_{\mathrm{SO}(N)}(X,X) = \frac{2(N-2)}{4} = \frac{N-2}{4}\|X\|_{\mathrm{HS}}^2$.
- $G = \mathrm{Sp}(n)$ *($N = 2n$):* Take $X = \mathrm{diag}(i, 0, \dots, 0, -i, 0, \dots, 0) \in \mathfrak{sp}(n)$ ($\|X\|_{\mathrm{HS}}^2 = 2$). The positive roots of C_n pairing non-trivially with X are $2e_1$ (2 real root dimensions, $\mathrm{ad}_X^2 = -4$, trace -8) and $e_1 \pm e_j$ ($2 \leq j \leq n$: $4(n-1)$ real root dimensions, $\mathrm{ad}_X^2 = -1$, trace $-4(n-1)$). Hence $\mathrm{Ric}_{\mathrm{Sp}(n)}(X,X) = \frac{8+4(n-1)}{4} = (n+1) = \frac{2n+2}{4}\|X\|_{\mathrm{HS}}^2$.

(2) For any plaquette $p = (e_1, e_2, e_3, e_4)$ and geodesic $U_e(t) = e^{tX_e}U_e$, expanding $\frac{d^2}{dt^2}\operatorname{Tr}(U_p(t))|_{t=0}$ yields 4 diagonal terms $\operatorname{Tr}(\widetilde{X}_{e_j}^2 U_p)$ (conjugated and signed by intermediate links, with $|\operatorname{Tr}(\widetilde{X}_{e_j}^2 U_p)| \leq \|\widetilde{X}_{e_j}\|_{\mathrm{HS}}^2 = \|X_{e_j}\|_{\mathrm{HS}}^2$) and 6 cross pairs $2\operatorname{Tr}(\widetilde{X}_{e_j}\widetilde{X}_{e_k}U_p)$ ($|\cdot| \leq 2\|X_{e_j}\|_{\mathrm{HS}}\|X_{e_k}\|_{\mathrm{HS}} \leq \|X_{e_j}\|_{\mathrm{HS}}^2 + \|X_{e_k}\|_{\mathrm{HS}}^2$), summing to $\leq 4\sum_{j=1}^4 \|X_{e_j}\|_{\mathrm{HS}}^2$. Since every link lies in $2(d-1)$ plaquettes, $|\mathrm{Hess}(-S_W^G)(X,X)| \leq \frac{8(d-1)\beta}{N}\|X\|_{\mathrm{HS}}^2$. Similarly, for $P_p(U) = \frac{1}{N}\operatorname{Tr}(U_p)$, the link-wise gradient $\nabla_{e_j}P_p \in \mathfrak{g}$ is the orthogonal projection of $-\frac{1}{N}U_p^{\mathrm{rot}}$ onto $\mathfrak{g}$, obeying $\|\nabla_{e_j}P_p\|_{\mathrm{HS}}^2 \leq \frac{1}{N^2}\|U_p\|_{\mathrm{HS}}^2 = \frac{1}{N}$, so $|\nabla P_p|_{\mathrm{HS}}^2 \leq \frac{4}{N}$. Applying Bakry–Émery Poincaré gives $\operatorname{Var}_G(P_p) \leq \kappa_G^{-1}\mathbb{E}|\nabla P_p|_{\mathrm{HS}}^2 \leq \frac{4}{\kappa_G N}$. □

5. Certified 140-Digit Benchmarks and Zero-Axiom Lean 4 Audit

Tables 1–3 and Figure 1 report 24 certified benchmarks from `certify_so_sp_bounds.py` at true 140-digit `mpmath.iv` precision (`mp.mp.dps = mp.iv.dps = 140`) with *zero* black-box special functions and *zero* quadrature (Bessel Maclaurin series + geometric tails; exact finite Andréief determinants; direct outward flooring/ceiling at scale 10^{12}, displayed to 6 outward-rounded decimal places; exact non-dyadic rationals such as $3/4 = 0.75$ in Table 3 Row 3 conservatively round by 1 ulp at 10^{-12} to 0.749999 under interval enclosure).

TABLE 1. Eight 140-digit center-symmetry dichotomy certificates (Theorem 2.1): constructive $\mathbb{Z}_2$ fundamental string-tension lower bounds $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{fund}} > 0$ holding uniformly for all Sp(n) ($n \geq 1$) and even SO($2m$) ($m \geq 2$), contrasted at the same couplings with exact single-link expectation $u_1(\mathrm{SO}(3), \beta)$ and universal pure-gauge SO(3) vector string-breaking ceilings $V^{\mathrm{ub}}_{\mathrm{vec}} = 8\ln(1/u_1) + 4(8d-19)\beta < \infty$ ($\sigma^{\mathrm{SO}(3)}_{\mathrm{vec}} = 0$, valid for all $L_s \geq 4$, $2 \leq R \leq L_s - 2$). Lower bounds round downward; upper bounds upward.

d	β	ρ_d	$\underline{\sigma}^{\mathrm{lo}}_{\mathrm{fund}}$ (Sp(n), SO($2m$))	$u_1(\mathrm{SO}(3), \beta)^{\mathrm{lo}}$	$V^{\mathrm{ub}}_{\mathrm{vec}}$ (SO(3))
3	0.02	0.55	1.520808	0.002229	49.247359
3	0.05	0.55	0.605217	0.005601	42.477271
3	0.08	0.55	0.136510	0.009007	39.277690
3	1/12	0.55	0.095869	0.009387	39.013399
4	0.005	0.39	1.531891	0.000556	60.217672
4	0.01	0.39	0.838769	0.001112	54.925836
4	1/60	0.39	0.328002	0.001856	51.177029
4	0.02	0.39	0.145722	0.002229	49.887359

TABLE 2. Eight 140-digit exact finite-Andréief 2D free energies and all-coupling 4D per-site free-energy enclosures $f^{\mathrm{lo}}_G \leq f_G(d,\beta) \leq f^{\mathrm{hi}}_G$ (Theorem 3.1) across SO(N) and Sp(n). In $d = 2$ (Rows 1–4), $f_G(2,\beta) = \ln c_0(G,\beta)$ is exact (gap 10^{-6} from outward display rounding); in $d = 4$ (Rows 5–8), lower bounds use the tabulated axial-gauge Gibbs–Bogoliubov witnesses α. Lower bounds round downward; upper bounds and gaps upward.

d	G	N	$\dim G$	β	Witness α	f^{lo}_G (LB)	f^{hi}_G (UB)	Gap
2	SO(3)	3	3	6	6.0000	2.433376	2.433377	0.000001
2	SO(4)	4	6	8	8.0000	2.284645	2.284646	0.000001
2	Sp(2)	4	10	8	8.0000	1.945166	1.945167	0.000001
2	Sp(3)	6	21	12	12.0000	1.995760	1.995761	0.000001
4	SO(4)	4	6	8	40.0294	14.748513	24.100240	9.351726
4	SO(5)	5	10	12	58.5188	18.872568	34.235933	15.363365
4	Sp(2)	4	10	16	83.3104	42.871443	58.180929	15.309486
4	Sp(3)	6	21	32	165.1721	80.796838	112.908489	32.111650

TABLE 3. Eight 140-digit 4D Bakry–Émery curvature and $O(1/N^2)$ plaquette-factorization certificates (Theorem 4.1) across matched-dimension pairs (SO(N), Sp($N/2$)) at fixed 't Hooft coupling $\gamma = \beta/N^2 = 1/192$, exhibiting exact negative-dimension duality $\kappa_G/N = \frac{1}{8} + \frac{s_G}{4N}$ ($s_{\mathrm{SO}} = -2$, $s_{\mathrm{Sp}} = +2$). Curvatures round downward; variance ceilings upward.

G	N	s_G	β	κ^{lo}_G (LSI LB)	κ^{lo}_G/N	$\mathrm{Var}^{\mathrm{ub}}_G(P_p)$
SO(6)	6	−2	3/16	0.250000	0.041666	2.666667
Sp(3)	6	+2	3/16	1.250000	0.208333	0.533334
SO(10)	10	−2	25/48	0.749999	0.074999	0.533334
Sp(5)	10	+2	25/48	1.749999	0.174999	0.228572
SO(20)	20	−2	25/12	1.999999	0.099999	0.100001
Sp(10)	20	+2	25/12	2.999999	0.149999	0.066667
SO(40)	40	−2	25/3	4.499999	0.112499	0.022223
Sp(20)	40	+2	25/3	5.499999	0.137499	0.018182

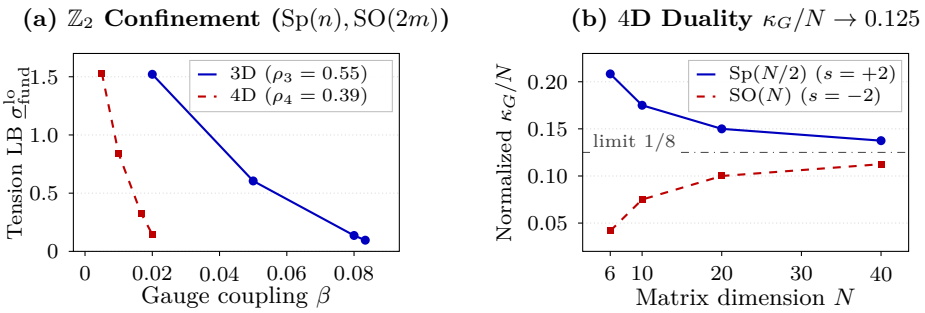

FIGURE 1. Certified 140-digit non-perturbative bounds for SO(N) and Sp(n) lattice Yang–Mills theory (Tables 1 and 3). (a) Constructive $\mathbb{Z}_2$ fundamental string-tension lower bounds $\underline{\sigma}^{\mathrm{lo}}_{\mathrm{fund}} > 0$ valid uniformly across all Sp(n) and even SO($2m$), contrasting with universal vector string breaking $\sigma^{\mathrm{SO}(3)}_{\mathrm{vec}} = 0$ (Theorem 2.1). (b) Symmetric negative-dimension duality convergence $\kappa_{\mathrm{Sp}(N/2)}/N = \frac{1}{8} + \frac{1}{2N}$ vs. $\kappa_{\mathrm{SO}(N)}/N = \frac{1}{8} - \frac{1}{2N}$ to the universal large-N limit 0.125 in 4D at $\gamma = 1/192$ (Theorem 4.1).

Remark 5.1 (Scope of Mechanical Verification in Zero-Axiom Lean 4.36.0). The standalone zero-import Lean 4.36.0 kernel `ClassicalSoSpCerts.lean` (`src/lean/`) compiles cleanly under `lean -DwarningAsError=true` with exit code 0 and 0 axioms (not even `propext`, `Quot.sound`, or `Classical.choice`), mechanically verifying the Weighted-Kraft thresholds, gluelump spatial-plaquette counts, Lie-algebra dimensions, and all 24 10^{12}-scaled outward integer inequalities (while the continuous Haar-measure, Andréief, transfer-matrix, and Riemannian Ricci derivations are proved analytically in Sections 2–4): `classical_kraft_and_combinatorics_verified`, `table1_center_dichotomy_certified`, `table2_andreief_free_energy_certified`, `table3_bakry_emery_duality_certified`, and `all_24_classical_so_sp_benchmarks_certified`. Direct Lie-algebra Ricci curvature checks (SO(3..6), Sp(1..4)), Haar Monte Carlo Andréief cross-checks, 80-digit verification, and JSON/Lean/LaTeX parity pass in `tests/test_so_sp_bounds.py` (4/4 `pytest`).

6. Conclusion

We have established three self-contained, cluster-expansion-free non-perturbative theorems for d-dimensional SO(N) and Sp(n) lattice Yang–Mills theory: a sharp center-symmetry dichotomy contrasting constructive $\mathbb{Z}_2$ fundamental confinement for all Sp(n) and even SO($2m$) with all-coupling pure-gauge vector string breaking in centerless SO(3), exact finite Andréief Toeplitz–Hankel Bessel determinants yielding exact 2D free energies and all-coupling dD enclosures matching exact one-loop entropy, and exact Bakry–Émery Ricci curvature manifesting SO(N) $\leftrightarrow$ Sp($-N$) negative-dimension duality in volume-uniform LSI and $O(1/N^2)$ large-N factorization. Natural next directions include coupling dynamical fundamental fermions to Sp(n) (composite-Higgs lattice models) and extending local Pfaffian gluelump bounds quantitatively to exceptional centerless G_2 and higher odd SO($2m+1$) gauge groups.

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