

Prescribed Tjurina Algebras and Complete Spectra in Osculating-Absorbing Gauss Fibres

Lluis Eriksson

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Abstract

We prove a simultaneous realization theorem for point-span osculating-absorbing Gauss fibres in every dimension. Given the smallest permitted support and any list of isolated tangent-section germs of order at least $s + 1$, a smooth hypersurface of every degree

$$D \geq 1 + \sum_i (\tau_i + 2)$$

can be chosen whose fibre has exactly the prescribed completed Tjurina algebras. Its length and dual multiplicity are respectively $\sum_i \tau_i$ and $\sum_i \mu_i$. For surfaces, the classical complete spectrum of ordinary plane multiple points then realizes every integer fibre length between

$$\binom{m+2}{2} \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor \quad \text{and} \quad \binom{m+2}{2} s^2,$$

while the dual multiplicity stays fixed.

We also give a short strong-Lefschetz construction of extremal three-variable Tjurina algebras. The explicit ordinary germs

$$x^{s+1} + y^{s+1} + z^{s+1} + (x + y + z)^{s+2}$$

attain the value $s(s+2)(2s-1)/3$, classically computed by Wahl as the minimum for positive-weight deformations of the Fermat cone. Our proof combines an Euler-reduced universal lower bound with an initial-ideal comparison and Stanley's theorem for $\mathbf{C}[x, y, z]/(x^s, y^s, z^s)$. This yields sharp absorbed threefold fibres at minimal reduced support. All statements are over $\mathbf{C}$; the degree thresholds are sufficient and not claimed minimal.

1 Introduction

For an isolated hypersurface germ $h \in \mathbf{C}[[x_1, \dots, x_d]]$, its Tjurina and Milnor numbers are

$$\tau(h) = \dim_{\mathbf{C}} \frac{\mathbf{C}[[x_1, \dots, x_d]]}{(h, \partial_1 h, \dots, \partial_d h)}, \quad \mu(h) = \dim_{\mathbf{C}} \frac{\mathbf{C}[[x_1, \dots, x_d]]}{(\partial_1 h, \dots, \partial_d h)}.$$

Fixing the order of h leaves a nontrivial minimization problem for τ . A recent finite-jet argument gives a universal family of lower bounds and determines the exact plane minimum [8]. In three variables its specialization is

$$\tau(h) \geq \frac{s(s+2)(2s-1)}{3},$$

which is sharp. For $s \geq 3$, the same value was already computed by Wahl as the minimum on positive-weight deformations of the degree- $s + 1$ Fermat cone [15, Example 4.7]; see also [1, Example 4.6]. We supply a short proof for the single uniform extremizer obtained by adding the $(s + 2)$ -nd power of a Lefschetz linear form. This is an explicit realization and a bridge to the global application, not a priority claim for the numerical minimum.

The main purpose is geometric. For a smooth hypersurface $X^d \subset \mathbf{P}^{d+1}$, the completed local algebra of a Gauss fibre at a point is the Tjurina algebra of the corresponding tangent-section germ [5]. Point-span order- s osculating absorption in the complete m -uple embedding forces both order at least $s + 1$ and at least $\binom{d+m}{d}$ reduced support points [7, 8]. Combining the universal floor with the explicit Lefschetz realization gives a sharp dimension-three fibre endpoint.

For surfaces, the endpoint was already known. We strengthen it in a different direction. Classical work on ordinary plane singularities says that every integral Tjurina number between the sharp minimum and the Milnor number occurs [2, 3, 4]. We prove that arbitrary finite lists of these germs can be installed simultaneously on an extremal absorbed Gauss fibre. Hence the entire integer interval of fibre lengths occurs while the multiplicity of the dual hypersurface is fixed.

No exhaustive priority assertion is intended. The local plane spectrum, Wahl's three-variable minimum, and the strong Lefschetz input are established results. The content here is the prescribed-list realization under the absorption constraint, its complete global spectrum consequence, and the direct Lefschetz extremizer used for the threefold endpoint.

2 The Euler-reduced lower bound in three variables

We use the convention $\binom{a}{b} = 0$ for $a < b$, including negative upper indices.

Lemma 2.1 (Euler reduction). *Let $R = \mathbf{C}[[x_1, \dots, x_d]]$ and $h \in \mathfrak{m}^{s+1}$. Put*

$$r = h - \frac{1}{s+1} \sum_{i=1}^d x_i \partial_i h.$$

Then $r \in \mathfrak{m}^{s+2}$ and

$$(h, \partial_1 h, \dots, \partial_d h) = (r, \partial_1 h, \dots, \partial_d h).$$

Proof. Euler's identity cancels the homogeneous component of degree $s + 1$. The equality of ideals follows directly from the displayed definition of r . Notice that the derivative generators are not replaced by the derivatives of r . $\square$

Theorem 2.2 (Exact three-variable floor). *Let $s \geq 1$, and let $h \in \mathbf{C}[[x, y, z]]$ have an isolated critical point and order at least $s + 1$. Then*

$$\tau(h) \geq T_s^{(3)} := \frac{s(s+2)(2s-1)}{3}. \quad (1)$$

This is the best possible bound for every s .

Remark 2.3 (Historical scope). For $s \geq 3$, Wahl computed the same sharp number in his study of positive-weight deformations of $x^{s+1} + y^{s+1} + z^{s+1}$ [15, Example 4.7]; Almirón records the formula explicitly in [1, Example 4.6]. The lower-bound proof below applies to every isolated germ of order at least $s + 1$. Proposition 3.2 then gives a particularly uniform one-term extremizer. We do not claim priority for the numerical minimum or for the abstract existence of extremal positive-weight deformations.

Proof of the lower bound. Work in $R/\mathfrak{m}^{2s}$, whose dimension is $\binom{2s+2}{3}$. By Lemma 2.1, the Tjurina ideal has three generators of order at least s and one generator of order at least $s + 2$. Their images are therefore spanned by at most

$$3 \binom{s+2}{3} + \binom{s}{3}$$

coefficient multiples. Rank is at most the dimension of the source, so

$$\tau(h) \geq \binom{2s+2}{3} - 3\binom{s+2}{3} - \binom{s}{3} = \frac{s(s+2)(2s-1)}{3}.$$

The Tjurina algebra surjects onto this truncated quotient. This proves the claim, including $s = 1, 2$ under the binomial convention. $\square$

3 A strong-Lefschetz extremizer

Put $\ell = x + y + z$ and

$$f_s = x^{s+1} + y^{s+1} + z^{s+1} + \ell^{s+2}. \quad (2)$$

Its initial form is Fermat and defines a smooth plane curve. Thus f_s is an ordinary isolated hypersurface singularity of multiplicity $s + 1$, and $\mu(f_s) = s^3$.

Lemma 3.1 (Lefschetz quotient). *For $s \geq 1$, let*

$$A_s = \mathbf{C}[x, y, z]/(x^s, y^s, z^s).$$

Then

$$\dim_{\mathbf{C}} A_s/(\ell^{s+2}) = \frac{s(s+2)(2s-1)}{3}.$$

Proof. For $s = 1$, the assertion is immediate from $A_1 = \mathbf{C}$, so assume $s \geq 2$. The linear form $\ell = x + y + z$ is a strong Lefschetz element of the monomial complete intersection A_s [14, 13]. If $H_j = \dim_{\mathbf{C}}(A_s)_j$, multiplication by ℓ^{s+2} from degree $j - s - 2$ to degree j has maximal rank. Therefore

$$\dim(A_s/(\ell^{s+2}))_j = \max\{H_j - H_{j-s-2}, 0\}.$$

The Hilbert function of A_s is symmetric and unimodal with socle degree $3(s-1)$. Its positive difference $H_j - H_{j-s-2}$ occurs precisely for $0 \leq j \leq 2s-1$, and the difference is nonpositive thereafter (eventually both terms vanish). Equivalently, the midpoint between j and $j - s - 2$ crosses the symmetry centre between $2s-1$ and $2s$. Summing the positive part through degree $2s-1$, using $\sum H_j t^j = (1-t^s)^3/(1-t)^3$, gives

$$\sum_{j=0}^{2s-1} (H_j - H_{j-s-2}) = \binom{2s+2}{3} - 3\binom{s+2}{3} - \binom{s}{3} = \frac{s(s+2)(2s-1)}{3}.$$

$\square$

Proposition 3.2 (Equality family). *For every $s \geq 1$, the germ f_s in (2) satisfies*

$$\mu(f_s) = s^3, \quad \tau(f_s) = \frac{s(s+2)(2s-1)}{3}.$$

Proof. Euler reduction applied to (2) gives

$$f_s - \frac{1}{s+1}(x\partial_x f_s + y\partial_y f_s + z\partial_z f_s) = -\frac{1}{s+1}\ell^{s+2}.$$

Hence ℓ^{s+2} belongs to the Tjurina ideal I_s . The lowest homogeneous forms of its three derivative generators are nonzero multiples of x^s, y^s, z^s . Thus the $\mathfrak{m}$ -adic initial ideal satisfies

$$\text{in}_{\mathfrak{m}}(I_s) \supseteq (x^s, y^s, z^s, \ell^{s+2}).$$

Here $\text{in}_{\mathfrak{m}}(I_s)$ denotes the ideal generated by the lowest nonzero homogeneous forms of all elements of I_s . Length is preserved on passage to the associated graded algebra, so Lemma 3.1 yields

$$\tau(f_s) \leq \dim_{\mathbf{C}} A_s/(\ell^{s+2}) = T_s^{(3)}.$$

The reverse inequality is Theorem 2.2. Finally, the ordinary leading form gives $\mu(f_s) = s^3$. $\square$

The containment of initial ideals, rather than an asserted equality, is important. It supplies only the upper bound needed to meet the universal lower bound.

4 Absorbed Gauss fibres

Let $X^d \subset \mathbf{P}_{\mathbf{C}}^{d+1}$ be a smooth non-linear hypersurface, let $\gamma_X : X \rightarrow X^\vee$ be its Gauss morphism, and fix a tangent hyperplane $\eta = [W]$. Write Γ_η for the scheme-theoretic fibre and $Z = (\Gamma_\eta)_{\text{red}}$. Let $H = \mathcal{O}_X(1)$ and

$$S_Z = \text{Im} \left(H^0(Z, H^m|_Z)^* \longrightarrow H^0(X, H^m)^* \right).$$

The support is *point-span order- s osculating-absorbing* if

$$\widehat{\text{Osc}}_p^s(H^m) \subseteq S_Z \quad (p \in Z).$$

We recall two inputs, with their scopes. First, if h_p is a local equation of $X \cap W$ at p , then

$$\widehat{\mathcal{O}}_{\Gamma_\eta, p} \cong \mathbf{C}[[z_1, \dots, z_d]] / (h_p, \partial h_p). \quad (3)$$

Second, absorption in the complete m -uple embedding, with $1 \leq s \leq m$, implies

$$h_p \in \mathfrak{m}_p^{s+1}, \quad |Z| \geq N_{d,m} := \binom{d+m}{d}. \quad (4)$$

These facts are proved in [5, 7, 8]; they are not reproved as new results here.

Corollary 4.1 (Sharp numerical floor in dimension three). *Suppose $d = 3$ and Z is point-span order- s osculating-absorbing in the complete H^m embedding, where $1 \leq s \leq m$. Then*

$$\text{length}(\Gamma_\eta) \geq \frac{s(s+2)(2s-1)}{3} |Z| \geq \frac{s(s+2)(2s-1)}{3} \binom{m+3}{3}.$$

Both factors can be attained simultaneously at sufficiently large ambient degree, as proved in Theorem 6.2 below.

Proof. Sum Theorem 2.2 over (3), then apply (4). □

5 Simultaneous realization

The following theorem isolates the interpolation argument used for both geometric consequences.

General deformation-theoretic realization results for prescribed singularities have a substantial literature, including [10]. The theorem below uses an elementary high-degree separator construction because it must retain the common tangent hyperplane, unisolvent support, and absorption constraint. Its separator–Bertini architecture already appears for two ordinary local types in [5]; here we record the heterogeneous finite-determinacy version and do not claim a new general method for prescribed singularities.

Theorem 5.1 (Prescribed Tjurina algebras on an extremal fibre). *Let $d \geq 1$, $1 \leq s \leq m$, and $N = \binom{d+m}{d}$. Choose N isolated convergent hypersurface germs*

$$g_i \in \mathbf{C}\{z_1, \dots, z_d\}, \quad \text{ord}(g_i) \geq s+1,$$

and put

$$E = \sum_{i=1}^N (\tau(g_i) + 2).$$

If

$$D \geq E + 1, \tag{5}$$

then there exist a smooth integral hypersurface $X \subset \mathbf{P}^{d+1}$ of degree D , a hyperplane W , and a tangent hyperplane point $\eta = [W] \in X^\vee$ such that:

- (i) $Z = (\gamma_X^{-1}(\eta))_{\text{red}}$ consists of exactly N points and $\dim S_Z = N < h^0(X, H^m)$;
- (ii) Z is point-span order- s osculating-absorbing;
- (iii) the tangent-section germ at the i -th point is contact equivalent to g_i , hence

$$\widehat{\mathcal{O}}_{\Gamma_\eta, p_i} \cong \mathbf{C}[[z_1, \dots, z_d]]/(g_i, \partial g_i);$$

(iv)

$$\text{length}(\Gamma_\eta) = \sum_{i=1}^N \tau(g_i), \quad \text{mult}_\eta(X^\vee) = \sum_{i=1}^N \mu(g_i).$$

The degree bound (5) is sufficient and is not asserted to be minimal.

Proof. Choose N points $Z \subset W \cong \mathbf{P}^d$ for which degree- m evaluation is an isomorphism. One explicit choice in an affine chart is the simplex lattice

$$[1 : i_1 : \dots : i_d], \quad i_j \in \mathbf{Z}_{\geq 0}, \quad \sum_j i_j \leq m;$$

multivariate Newton interpolation proves unisolvence.

Put $a_i = \tau(g_i) + 1$. Isolated hypersurface germs are contact $(\tau + 1)$ -determined [9, Corollary 2.24]; prescribing the a_i -jet therefore fixes the contact class of g_i . The elementary separator product makes

$$H^0(\mathbf{P}^d, \mathcal{O}(D)) \longrightarrow \bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}^d, p_i} / \mathfrak{m}_{p_i}^{a_i+1}$$

surjective once

$$D \geq \sum_i (a_i + 1) - 1 = E - 1.$$

Indeed, an a_i -jet at p_i can be multiplied by $\prod_{j \neq i} \ell_{ij}^{a_j+1}$, where ℓ_{ij} vanishes at p_j and is a unit at p_i . The stronger bound (5) leaves one degree after the product

$$P_u = \prod_{i=1}^N \ell_{i,u}^{a_i+1}, \quad \deg P_u = E,$$

so the affine system with the prescribed jets generates first jets at every $u \in W \setminus Z$.

A member singular at a specified $u \in W \setminus Z$ imposes $d+1$ linear conditions, while $\dim W = d$. Hence a general degree- D equation f on W is smooth away from Z and has exactly the chosen contact types at Z .

Embed $W = V(y)$ in $\mathbf{P}^{d+1}$ and set

$$F_G = f + yG, \quad G \in H^0(\mathbf{P}^{d+1}, \mathcal{O}(D-1)).$$

Choose G general and nonzero at Z . Then $X = V(F_G)$ is smooth: at Z its normal derivative is $G(p) \neq 0$; on $W \setminus Z$ this follows from $df \neq 0$; and off W the singular-point incidence has fibre codimension $d+2 > \dim \mathbf{P}^{d+1} = d+1$. The smooth hypersurface is connected and therefore integral. Locally on X , $y = -f/G$, so the tangent-section germs are contact equivalent to the prescribed germs.

The singular locus of $X \cap W$ is exactly Z , hence the reduced Gauss fibre over $[W]$ is exactly Z . Each $\tau(g_i) \geq 1$, while $N = \binom{d+m}{d} \geq m+1$; hence $D \geq E+1 > m$, and degree- m ambient forms identify with $H^0(X, H^m)$. A form vanishing on Z restricts to zero on W by unisolvence, and is consequently divisible by y . Along X , $y = -f/G$ has order at least $s+1$ at every support point. The reverse-annihilator criterion gives order- s absorption, while evaluation gives $\dim S_Z = N$. Properness follows from $h^0(X, H^m) = \binom{d+m+1}{d+1} > N$.

Finally, (3) gives the length formula. The dual multiplicity formula is the sum of the local Milnor numbers; see [12] for the classical dual-multiplicity result and [5, 6] for its use in this fibre setting. $\square$

6 Complete surface spectra and sharp threefold fibres

Set

$$T_s^{(2)} = \left\lfloor \frac{3s^2 + 4s - 3}{4} \right\rfloor.$$

An ordinary plane germ of multiplicity $s+1$ has Milnor number s^2 , and its Tjurina number assumes every integer from $T_s^{(2)}$ through s^2 [4, Theorem 5.1]; that theorem in turn attributes the spectrum to the classical sources recorded there. We use this local result rather than claiming it.

Theorem 6.1 (Complete absorbed surface spectrum). *Let $1 \leq s \leq m$, put $N = \binom{m+2}{2}$, and assume*

$$D \geq (s^2 + 2)N + 1.$$

For every integer L satisfying

$$NT_s^{(2)} \leq L \leq Ns^2,$$

there exist a smooth integral surface $X \subset \mathbf{P}^3$ of degree D and a point-span order- s osculating-absorbing Gauss fibre with

$$|Z| = N, \quad \text{length}(\Gamma_\eta) = L, \quad \text{mult}_\eta(X^\vee) = Ns^2.$$

More strongly, Theorem 5.1 realizes any specified list of N ordinary plane germs of multiplicity $s+1$, including their completed Tjurina algebras.

Proof. Write $L - NT_s^{(2)} = r_1 + \cdots + r_N$ with $0 \leq r_i \leq s^2 - T_s^{(2)}$. Such a decomposition exists by a greedy choice. For each i , choose an ordinary plane germ with Tjurina number $T_s^{(2)} + r_i$, using the cited complete local spectrum, and apply Theorem 5.1 with $d = 2$. $\square$

Theorem 6.2 (Sharp absorbed threefold fibres). *Let $1 \leq s \leq m$, put*

$$N = \binom{m+3}{3}, \quad T_s^{(3)} = \frac{s(s+2)(2s-1)}{3},$$

and assume

$$D \geq (T_s^{(3)} + 2)N + 1.$$

Then there exist a smooth integral threefold $X \subset \mathbf{P}^4$ of degree D and a point-span order- s osculating-absorbing Gauss fibre such that

$$|Z| = N, \quad \text{length}(\Gamma_\eta) = T_s^{(3)}N, \quad \text{mult}_\eta(X^\vee) = s^3N.$$

Thus Corollary 4.1 is sharp simultaneously in local length and reduced support size.

Proof. Apply Theorem 5.1 with $d = 3$ and all prescribed germs equal to f_s . Proposition 3.2 supplies their Tjurina and Milnor numbers. $\square$

7 Scope, reproducibility, and open directions

The argument is over $\mathbf{C}$. Euler division by $s + 1$, contact determinacy, the stated Gauss-fibre inputs, and strong Lefschetz are used in that setting. No positive-characteristic extension is asserted.

The replay accompanying the paper has three deliberately separated layers. Exact binomial arithmetic checks the formula and maximizing truncation for $1 \leq s \leq 50$. Exact rational matrix ranks check the relevant Lefschetz maps for $1 \leq s \leq 8$. Sparse quotient computations over two large prime fields directly check the displayed germ for $2 \leq s \leq 7$. These finite fixtures test the implementation and examples; they do not prove the universal theorems.

The paper identifies the Euler-reduced floor with Wahl’s sharp value in three variables and realizes it by the displayed Lefschetz family; no analogous assertion is made in four or more variables. Classification of all equality germs is also open. For higher dimensions a recent, non-refereed preprint gives stronger asymptotic ratio estimates and Fermat-deformation families [11]; it is not used here. The global degree thresholds come from transparent separator products and are probably far from optimal. The surface spectrum fixes only numerical length and dual multiplicity unless the stronger prescribed-list statement is invoked; different Tjurina algebras of the same length need not be isomorphic.

8 Conclusion

The prescribed-list theorem turns local Tjurina data into exact absorbed Gauss fibres at minimal reduced support. In the plane case it upgrades endpoint sharpness to the full integer spectrum at fixed dual multiplicity. In three variables, Euler reduction and strong Lefschetz give a direct uniform representative of Wahl’s sharp value and hence an exact global endpoint for threefold hypersurfaces. Equality classification, higher-variable sharpness, and optimal ambient degrees remain distinct problems.

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