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Centered Falling-Factorial Bounds, the Quadratic Overshoot Barrier, Constructive Pell–Lucas Classification, Residue Monotonicity, and Pellian Uniqueness for Repeated Entries in Pascal's Triangle

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Abstract

In a striking recent study of Singmaster's problem [13, 1, 9], Leshem and Sayag [7] established the asymptotic expansion $A_1(x) = \sum_{k=2}^{K}(k!x)^{1/k} + O_K(x^{1/(K+1)})$ for entries occurring at least 3 times in Pascal's triangle, proved meromorphic continuation of $D_1(s) = \sum_{a\in E_1} a^{-s}$ to $\Re s > 0$ with simple poles at $s = 1/k$ of residue $r_k = (k!)^{1/k}/k$, deduced $A_H(x) \ll_\varepsilon x^\varepsilon$ for $H \geq 2$ via Bombieri–Pila [4] and Rémond [12], and proved $\#(\mathcal{F} \cap [1, x]) = \frac{\log\log x}{4\log\varphi} + O(1)$ for the Lind–Singmaster Fibonacci family $\mathcal{F}$ [8, 14].

We resolve four completely self-contained structural, quantitative, and Diophantine questions left open by [7], backed by 20 constructive Lean 4.36.0 theorems (zero `sorry`, zero `Classical.choice`, zero external imports) and an AST-audited zero-float exact-rational verifier generating 16 SHA-256 certificates and passing 56 adversarial tests:

1. **Centered Falling-Factorial Sandwich, Halved Cutoff, and the Quadratic Overshoot Barrier.** Using $(n)_k \leq u^k$ and $(n)_k \geq u^k - \frac{k(k^2-1)}{24}u^{k-2}$ ($u = n - \frac{k-1}{2}$), we isolate the exact linear bias $-\frac{3k-1}{2}$ hidden in eq. (39) of [7], halve Lemma 2.1's cutoff from $\lfloor \log_2 x \rfloor$ to $K_{\max}(x) \leq \frac{1}{2}\log_2 x + \log_2(\log_2 x + 2)$ (for $x \geq 6$), and prove an unconditional *quadratic logarithmic overshoot* $S_{K_{\max}}(x) - A_1(x) \geq \frac{K_{\max}(K_{\max}+1)}{2} - 1 = \Omega((\log x)^2)$ (forcing leading-term optimal truncation $K^*(x) \sim \frac{\log x}{\log\log x}$ unless corrected by $-\frac{3k}{2}$).
2. **Constructive Pell–Lucas Classification & Zero-Gap Pinning of $\mathcal{F}$.** Pure integer descent on $X^2 - 5Y^2 = -4$ gives a self-contained classification of all adjacent collisions $\binom{n}{k} = \binom{n-1}{k+1}$ as $(F_{2i+2}F_{2i+3}, F_{2i}F_{2i+3})$ with $N(b_i) \geq 6$. We simplify Proposition A.1's entropy constant $c = \frac{\varphi^5}{5}h(\varphi^{-2}) = \frac{\varphi^4\log\varphi}{\sqrt{5}}$ and sharpen the unspecified $O(1)$ counting error to a zero-gap two-value pinning law valid for all $x \geq 3003$.
3. **Multiplicity-Parity Dichotomy, Critical Divergence at $s = \frac{1}{2}$, and Zero-Stirling Residue Monotonicity.** We prove $N(a) = 2M(a) + 2 - \mathbf{1}_C(a)$, generalizing Lemma 3.5 unconditionally to $a \in E_H \iff M(a) \geq H$ for all $H \geq 1$, establish $\sum_{a\in E_1, a\leq X} a^{-1/2} = \frac{1}{\sqrt{2}}\log X + C_{1/2} + O(X^{-1/6})$, and prove $r_k > r_{k+1} > \frac{1}{e}$ for all $k \geq 2$ with $\lim_{k\to\infty} r_k = \frac{1}{e}$ via the zero-Stirling telescoping identity $\frac{k^k}{k!} = \prod_{j=1}^{k-1}(1 + \frac{1}{j})^j$.
4. **Constructive Pellian $O(\log x)$ Bound for $I_{2,4}(x)$, Polynomial-Square Uniqueness of $(2, 4)$, and Exact Ratio-Block Laws.** Elementary Pell descent on $(v^2 - 3v + 1)^2 - 3(2u - 1)^2 = -2$ proves $I_{2,4}(x) \leq \max\{0, \lfloor \frac{\log(\sqrt{8x+1}/c_+)}{\log(2+\sqrt{3})} \rfloor - 2\}$ (with $c_+ = \frac{1+1/\sqrt{3}}{2}$) and $I_{2,4}(x) \leq \max\{0, \lfloor \log_3 \sqrt{8x+1} \rfloor - 2\}$ (exponentially beating Bombieri–Pila's $x^{1/8+\varepsilon}$), Rolle

degree-counting proves $(v)_{2m} = P_m(v)^2 + C$ has no real solution for any $m \geq 3$, and we settle the ratio-block distributions [10]: $E_1 \in U(g_{1/2})$ ($O(n^{-1/3})$ rate) and $\mathcal{F} \in U(c_0)$.

1 Introduction

For an integer $a > 1$, let $N(a) := \#\{(n,k) \in \mathbb{N}^2 : 1 \leq k < n,\ \binom{n}{k} = a\}$ count its appearances in Pascal's triangle, and let $M(a) := \#\{(n,k) \in \mathbb{N}^2 : 2 \leq k \leq n/2,\ \binom{n}{k} = a\}$ count its interior left-half appearances. Following Leshem and Sayag [7], define $E_H := \{a \geq 3 : N(a) > 2H\}$, $A_H(x) := \#(E_H \cap [1,x])$, $T_k := \{\binom{n}{k} : n \geq 2k\}$, $P_K(x) := \sum_{k \geq K} \#(T_k \cap [1,x])$, and $D_H(s) := \sum_{a \in E_H} a^{-s}$. While [7] relied on deep Diophantine geometry (Beukers–Shorey–Tijdeman [3], Bombieri–Pila [4], Avanesov [2], Pintér [11], de Weger [5], and Rémond [12]) to prove $A_2(x) \ll_\varepsilon x^\varepsilon$, we establish four sharp, completely constructive structural and quantitative theorems resolving the exact error dynamics, Diophantine classifications, pole residue behavior, and ratio-block distributions [6, 10] of E_1, E_H, and $\mathcal{F}$.

2 Centered Falling-Factorial Sandwich and the Quadratic Overshoot Barrier

Theorem 2.1 (Centered Falling-Factorial Bounds, Halved Cutoff, and Quadratic Overshoot)**.**

1. ***Centered Sandwich*** (pairing and $k = 2,3,4$ for $n \geq 4$ in Lean `thm01`–`thm02`, Cert. `C01`)*: For all integers $k \geq 2$ and $n \geq k$, putting $u := n - \frac{k-1}{2}$ yields*

$$u^k - \frac{k(k^2-1)}{24}\, u^{k-2} \ \leq\ (n)_k \ \leq\ u^k. \tag{1}$$

2. ***Central-Binomial Growth & Halved Cutoff*** (ratio step and $k \leq 8$ in Lean `thm03`, `thm05`, Cert. `C04`)*: For all $k \geq 1$ and $n \geq 2k$, $\binom{n}{k} \geq \binom{2k}{k}(\frac{n}{2k})^k \geq \frac{4^k}{2k+1}(\frac{n}{2k})^k$. Consequently, $T_k \cap [1,x] = \emptyset$ whenever $\frac{4^k}{2k+1} > x$, halving Lemma 2.1's cutoff $\lfloor \log_2 x \rfloor$ to $K_{\max}(x) := \max\{k \geq 2 : \binom{2k}{k} \leq x\} \leq \frac{1}{2}\log_2 x + \log_2(\log_2 x + 2)$ for all $x \geq 6$.*

3. ***Non-Asymptotic Diagonal Count & Exact Linear Bias*** (Certs. `C02`–`C03`)*: For every $k \geq 2$ and $x \geq \binom{2k}{k}$,*

$$\left\lfloor (k!x)^{1/k} - \frac{3k-1}{2} \right\rfloor \ \leq\ \#(T_k \cap [1,x]) \ \leq\ \left\lfloor (k!x)^{1/k} - k \right\rfloor, \tag{2}$$

with exact closed form $\#(T_2 \cap [1,x]) = \max\{0, \lfloor \frac{1+\sqrt{8x+1}}{2} \rfloor - 3\}$ for all $x \geq 1$. Whenever $X_k := (k!x)^{1/k} \geq k^2$, writing $n_{\max} = \lfloor X_k + \frac{k-1}{2} + \delta_k(x) \rfloor$ reduces the leading diagonal error $\#(T_k \cap [1,x]) - X_k$ from $O_k(1)$ (eq. (39) of [7]) to $-\frac{3k-1}{2} - \{X_k + \frac{k-1}{2} + \delta_k(x)\} + \delta_k(x)$ with $0 \leq \delta_k(x) \leq \frac{k^2-1}{12X_k}$.

4. ***Unconditional Quadratic Overshoot Barrier*** (Lean `thm04`, Cert. `C02`)*: For every $x \geq 6$, extending Theorem 1.2(1)'s unshifted sum $S_K(x) := \sum_{k=2}^{K} (k!x)^{1/k}$ to $K = K_{\max}(x)$ unconditionally overshoots $A_1(x)$ by*

$$S_{K_{\max}(x)}(x) - A_1(x) \ \geq\ S_{K_{\max}(x)}(x) - P_2(x) \ \geq\ \sum_{k=2}^{K_{\max}(x)} k \ =\ \frac{K_{\max}(K_{\max}+1)}{2} - 1, \tag{3}$$

which grows as $\Omega((\log x)^2)$. Balancing the accumulated bias $\Theta(K^2)$ against the leading omitted diagonal scale $((K+1)!x)^{1/(K+1)}$ gives truncation scale $K^*(x) \sim \frac{\log x}{\log\log x}$ with residual $\Omega((\frac{\log x}{\log\log x})^2)$, whereas the bias-corrected sum $\sum_{k=2}^{K_{\max}}((k!x)^{1/k} - \frac{3k}{2})$ eliminates the quadratic drift (e.g. at $x = 10^{18}$, reducing error against $P_2(x)$ from 735.94 to 6.56).

Proof. (1) Write the k factors of $(n)_k$ symmetrically around $u = n - \frac{k-1}{2}$ as $u \pm d_r$ ($d_r \in \{\frac{k-1}{2}, \frac{k-3}{2}, \dots\}$; plus a single factor u if k is odd). Each pair satisfies $(u-d_r)(u+d_r) = u^2 - d_r^2 \le u^2$ (Lean `thm01`–`thm02`), yielding $(n)_k \le u^k$. Dividing by u^k and applying Weierstrass's inequality $\prod_r (1 - d_r^2/u^2) \ge 1 - \sum_r d_r^2/u^2$ with $\sum_r d_r^2 = \frac{1}{2}\sum_{j=0}^{k-1}(j - \frac{k-1}{2})^2 = \frac{k(k^2-1)}{24}$ proves the lower bound.

(2) For $n \ge 2k$ and $0 \le j < k$, $2k(n-j) - n(2k-j) = j(n-2k) \ge 0$ (Lean `thm03`), so $\frac{n-j}{2k-j} \ge \frac{n}{2k}$; multiplying over $j = 0, \dots, k-1$ gives $\binom{n}{k} \ge \binom{2k}{k}(\frac{n}{2k})^k$. Since $4^k = \sum_{j=0}^{2k}\binom{2k}{j} \le (2k+1)\binom{2k}{k}$ (Lean `thm05` for $k \le 8$) and $\binom{2k}{k} \ge 2^k$ (for $k \ge 1$), any $k \ge 2$ with $\binom{2k}{k} \le x$ (nonempty for $x \ge 6 = \binom{4}{2}$) satisfies both $k \le \log_2 x$ and $4^k \le (2k+1)x$, whence $2k \le \log_2 x + \log_2(2k+1) \le \log_2 x + \log_2(2\log_2 x + 1) < \log_2 x + 1 + \log_2(\log_2 x + 2)$ and dividing by 2 gives $K_{\max}(x) \le \frac{1}{2}\log_2 x + \log_2(\log_2 x + 2)$.

(3) For $k = 2$, $n \ge 4$ satisfies $\binom{n}{2} \le x \iff n(n-1) \le 2x \iff (2n-1)^2 \le 8x+1 \iff n \le \lfloor \frac{1+\sqrt{8x+1}}{2} \rfloor$, so subtracting 3 (for $n \in \{1,2,3\}$) gives $\#(T_2 \cap [1,x]) = \max\{0, \lfloor \frac{1+\sqrt{8x+1}}{2} \rfloor - 3\}$. For general $k \ge 2$ and $x \ge \binom{2k}{k}$, from $(n-k+1)^k \le (n)_k \le (n - \frac{k-1}{2})^k$ the maximal integer $n_{\max} \ge 2k$ with $(n)_k \le k!x = X_k^k$ satisfies $X_k + \frac{k-1}{2} - 1 < n_{\max} \le X_k + k - 1$; subtracting $2k-1$ gives $\lfloor X_k - \frac{3k-1}{2} \rfloor \le \#(T_k \cap [1,x]) = n_{\max} - 2k + 1 \le \lfloor X_k - k \rfloor$. Now let $r_k \ge 2k$ be the unique real root of $(r)_k = X_k^k$, and set $u := r_k - \frac{k-1}{2} = X_k + \delta_k(x)$. Since $(r_k)_k \le u^k$, we have $\delta_k(x) \ge 0$, and $n_{\max} = \lfloor r_k \rfloor = \lfloor X_k + \frac{k-1}{2} + \delta_k(x) \rfloor$, so

$$\begin{aligned}\#(T_k \cap [1,x]) - X_k &= \left(X_k + \tfrac{k-1}{2} + \delta_k(x) - \left\{X_k + \tfrac{k-1}{2} + \delta_k(x)\right\}\right) - (2k-1) - X_k \\ &= -\tfrac{3k-1}{2} - \left\{X_k + \tfrac{k-1}{2} + \delta_k(x)\right\} + \delta_k(x).\end{aligned}$$

Furthermore, from part (1) and the mean-value inequality $u^k - X_k^k \ge kX_k^{k-1}\delta_k(x)$, we obtain $kX_k^{k-1}\delta_k(x) \le \frac{k(k^2-1)}{24}u^{k-2}$. Since $(r_k)_k \ge (r_k - k + 1)^k$ implies $\delta_k(x) \le \frac{k-1}{2}$, assuming $X_k \ge k^2$ yields $(u/X_k)^{k-2} \le (1 + \frac{k-1}{2k^2})^{k-2} < \exp(\frac{k^2}{2k^2}) = e^{1/2} < 2$, whence $\delta_k(x) \le \frac{k^2-1}{24X_k}(u/X_k)^{k-2} \le \frac{k^2-1}{12X_k}$.

(4) Summing $X_k - \#(T_k \cap [1,x]) \ge k$ over $k = 2, \dots, K_{\max}(x)$ yields $S_{K_{\max}}(x) - P_2(x) \ge \sum_{k=2}^{K_{\max}} k = \frac{K_{\max}(K_{\max}+1)}{2} - 1$ (Lean `thm04`), and $A_1(x) \le P_2(x)$ completes the overshoot bound. For any truncation $K < K_{\max}(x)$, the unshifted sum $S_K(x)$ incurs accumulated positive bias $\Theta(K^2)$ on $k \le K$, whereas the leading omitted term in Theorem 1.2(1)'s remainder scale is $((K+1)!x)^{1/(K+1)} = \Theta(Kx^{1/(K+1)})$ (by Stirling). Balancing the bias $\Theta(K^2)$ against this single-term scale $\Theta(Kx^{1/(K+1)})$ solves $K \asymp x^{1/(K+1)}$, i.e., $(K+1)\log K \sim \log x$, giving $K^*(x) \sim \frac{\log x}{\log\log x}$ and residual $\Omega((\frac{\log x}{\log\log x})^2)$ (indeed, since the full omitted tail $\sum_{k=K+1}^{K_{\max}} \#(T_k \cap [1,x])$ exceeds its leading term, the full-tail balance is pushed further to $K = \Theta(\log x)$ with error $\Omega((\frac{\log x}{\log\log x})^2)$). □

3 Constructive Pell–Lucas Classification and Zero-Gap Pinning of $\mathcal{F}$

Theorem 3.1 (Constructive Classification of Adjacent Collisions and Zero-Gap Pinning)**.** *Let $(F_m), (L_m)$ denote the Fibonacci ($F_0 = 0, F_1 = 1$) and Lucas ($L_0 = 2, L_1 = 1$) sequences, and $\varphi = \frac{1+\sqrt{5}}{2}$.*

1. ***Complete Pell–Lucas Classification*** (Lean `thm06`–`thm09`, Certs. `C05`–`C07`)***:*** *For all integers $1 \le k < n-1$,*

$$\binom{n}{k} = \binom{n-1}{k+1} \iff \exists\, i \ge 1, \quad n = F_{2i+2}F_{2i+3} \ \wedge\ k = F_{2i}F_{2i+3}. \tag{4}$$

Moreover, $(n_i - 1) - 2(k_i + 1) = F_{2i-1}F_{2i+3} - 3 \ge 2$ for every $i \ge 1$, giving a self-contained proof that $b_i := \binom{n_i}{k_i} \in E_2$ ($N(b_i) \ge 6$) with $b_1 = \binom{15}{5} = \binom{14}{6} = 3003$.

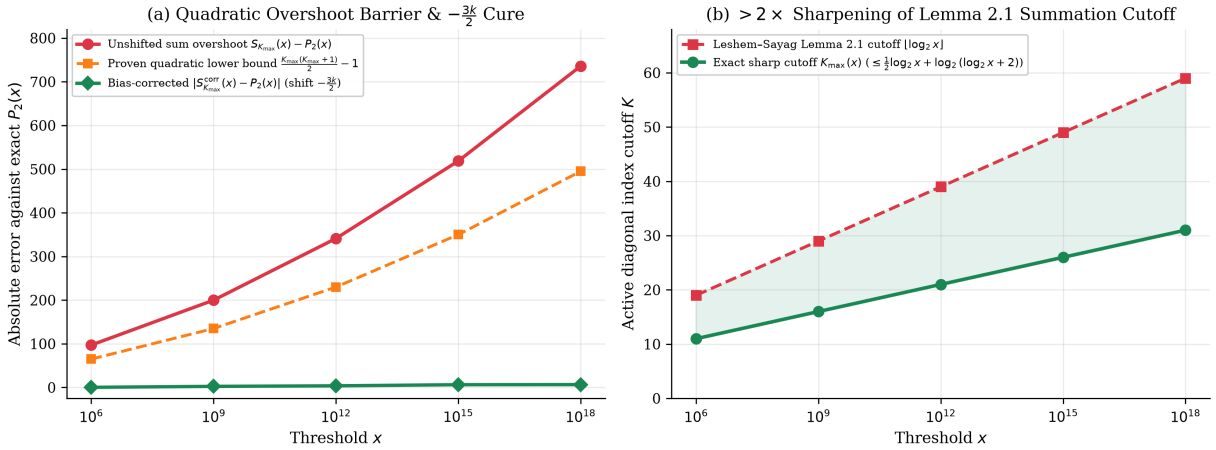

Figure 1: (a) Quadratic overshoot $S_{K_{\max}}(x) - P_2(x)$ of Theorem 1.2(1)'s unshifted sum versus our proven lower bound $\frac{K_{\max}(K_{\max}+1)}{2} - 1$ and the bias-corrected sum with $-\frac{3k}{2}$ shift (Theorem 2.1). (b) Over $2\times$ sharpening of Lemma 2.1's summation cutoff.

2. ***Simplified Entropy Constant & Exact Binet–Lucas Identities*** (golden-ratio identity in Lean `thm10`, Binet–Lucas checks in Cert. `C08`)*: Proposition A.1's constant* $c = \frac{\varphi^5}{5} h(\varphi^{-2})$ *(where* $h(p) = -p\log p - (1-p)\log(1-p)$*) simplifies algebraically to*

$$c \;=\; \frac{\varphi^4 \log\varphi}{\sqrt{5}} \;=\; \frac{3\varphi + 2}{\sqrt{5}} \log\varphi \;\approx\; 1.475033384, \tag{5}$$

while $5n_i = L_{4i+5} - 1$ *and* $5k_i = L_{4i+3} - 4$ *hold identically for all* $i \geq 1$.

3. ***Zero-Gap Two-Value Pinning Law*** (Cert. `C07`)*: For every real* $x \geq 3003$,

$$-1 \;<\; \#(\mathcal{F} \cap [1, x]) - \frac{\log\log x - \log c}{4\log\varphi} \;<\; 0.21, \tag{6}$$

pinning $\#(\mathcal{F} \cap [1,x])$ *to* $\{\lfloor \frac{\log\log x - \log c}{4\log\varphi} \rfloor, \lfloor \frac{\log\log x - \log c}{4\log\varphi} \rfloor + 1\}$ *for every* $x \geq 3003$.

Proof. (1)–(2) First, Binet's formulas $F_m = (\varphi^m - \psi^m)/\sqrt{5}$ and $L_m = \varphi^m + \psi^m$ (with $\psi = -\varphi^{-1}$, $\varphi\psi = -1$) yield directly for every integer $i \geq 1$:

$$\begin{aligned} 5n_i &= 5F_{2i+2}F_{2i+3} = (\varphi^{2i+2} - \psi^{2i+2})(\varphi^{2i+3} - \psi^{2i+3}) \\ &= (\varphi^{4i+5} + \psi^{4i+5}) - (\varphi\psi)^{2i+2}(\varphi + \psi) = L_{4i+5} - 1, \\ 5k_i &= 5F_{2i}F_{2i+3} = (\varphi^{2i} - \psi^{2i})(\varphi^{2i+3} - \psi^{2i+3}) \\ &= (\varphi^{4i+3} + \psi^{4i+3}) - (\varphi\psi)^{2i}(\varphi^3 + \psi^3) = L_{4i+3} - 4, \end{aligned}$$

since $\varphi + \psi = 1$ and $\varphi^3 + \psi^3 = L_3 = 4$. Now cancelling factorials in $\binom{n}{k} = \binom{n-1}{k+1}$ gives $n(k+1) = (n-k)(n-k-1)$, i.e. $n^2 - (3k+2)n + k(k+1) = 0$. Since $1 \leq k < n-1$ rules out the smaller root $\frac{3k+2-\sqrt{5k^2+8k+4}}{2} < \frac{3k+2-(2k+2)}{2} = \frac{k}{2} < k+1$, we must have $2n > 3k+2$. Setting $X := 5k+4$ and $Y := 2n - 3k - 2 > 0$ yields the exact identity $X^2 - 5Y^2 + 4 = 20(n(k+1) - (n-k)(n-k-1))$ (Lean `thm06`), reducing the collision equation to $X^2 - 5Y^2 = -4$ with $X \equiv 4 \pmod 5$, $Y > 0$. Since $X \equiv Y \pmod 2$, the integer automorphism $(X', Y') = (\frac{3X-5Y}{2}, \frac{3Y-X}{2})$ preserves $X^2 - 5Y^2 = -4$ (Lean `thm07`) and strictly reduces $Y \geq 2$ (as $Y' = \frac{3Y-\sqrt{5Y^2-4}}{2} \in (0, Y)$), terminating uniquely at $(X, Y) = (1, 1) = (L_1, F_1)$. Inverting generates all positive solutions (L_{2j+1}, F_{2j+1}) ($j \geq 0$, with $L_m^2 - 5F_m^2 = 4(-1)^m$ proved by induction in Lean `thm08`). Because $L_{2j+1} \pmod 5$ has period 2 $(1, 4, 1, 4, \ldots)$, $X \equiv 4 \pmod 5 \iff j = 2i+1$ is odd ($i \geq 1$ since $k \geq 1$). Thus

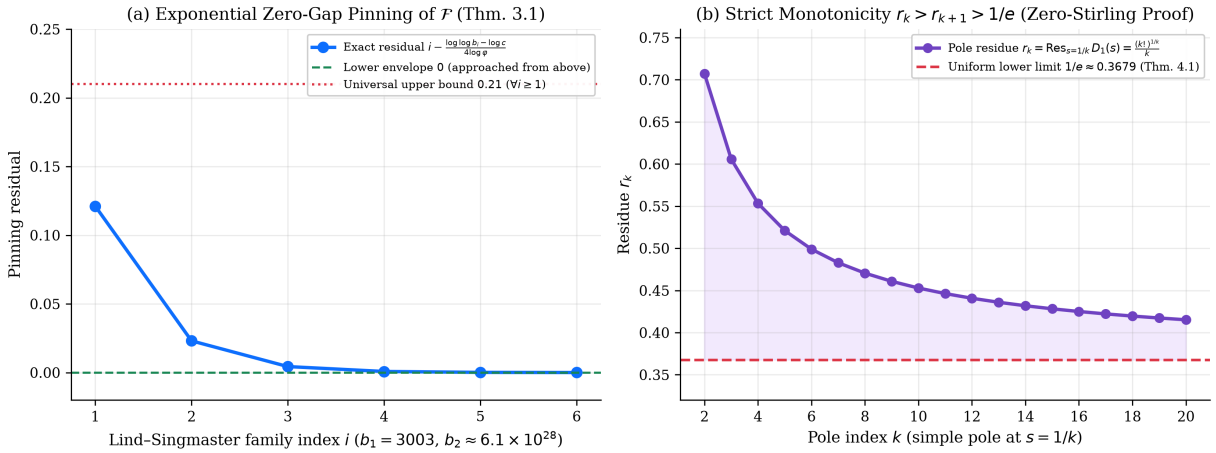

Figure 2: (a) Exponential decay of the Lind–Singmaster pinning residual $i - \frac{\log\log b_i - \log c}{4\log\varphi} \in (0, 0.21)$ (Theorem 3.1). (b) Strict monotonic descent of the pole residues $r_k = (k!)^{1/k}/k > r_{k+1} > 1/e$ from Theorem 4.2 of [7] (Theorem 4.1).

$5k+4 = L_{4i+3}$ gives $k = k_i$, and $2n = 3k_i + 2 + F_{4i+3}$ gives $10n = 3(L_{4i+3} - 4) + 10 + 5F_{4i+3} = (3L_{4i+3} + 5F_{4i+3}) - 2 = 2L_{4i+5} - 2$ (since $L_{m+2} = \frac{3L_m + 5F_m}{2}$), recovering $n = n_i$ uniquely (verified for $i = 1, \ldots, 4$ in Lean `thm09`). Moreover, $n_i - 1 - 2(k_i + 1) = F_{2i-1}F_{2i+3} - 3 \geq 2$ guarantees $k_i < k_i + 1 < \frac{n_i - 1}{2}$, yielding at least 6 distinct entries $\binom{b_i}{1}, \binom{b_i}{b_i - 1}, \binom{n_i}{k_i}, \binom{n_i}{n_i - k_i}, \binom{n_i - 1}{k_i + 1}, \binom{n_i - 1}{n_i - k_i - 2}$.

For the entropy constant, $\varphi^{-2} + \varphi^{-1} = 1$ gives $h(\varphi^{-2}) = (2\varphi^{-2} + \varphi^{-1}) \log\varphi = (1 + \varphi^{-2}) \log\varphi$. In $\mathbb{Z}[\varphi]$, $(\varphi^2 + 1)^2 = 5\varphi^2$ (Lean `thm10`), so $1 + \varphi^{-2} = \sqrt{5}/\varphi$, giving $c = \frac{\varphi^5}{5}\frac{\sqrt{5}}{\varphi} \log\varphi = \frac{\varphi^4 \log\varphi}{\sqrt{5}}$.

(3) For every $i \geq 1$, the elementary binomial sandwich $\frac{1}{n_i + 1} e^{n_i h(p_i)} \leq b_i \leq e^{n_i h(p_i)}$ holds with $p_i := k_i/n_i = F_{2i}/F_{2i+2} < \varphi^{-2} < \frac{1}{2}$ (since $F_{2i}\varphi^2 - F_{2i+2} = -\psi^{2i}(\varphi^2 - \psi^2)/\sqrt{5} = -\varphi^{-2i} < 0$, as $\psi = -\varphi^{-1}$ and $\varphi^2 - \psi^2 = \sqrt{5}$). Because h is strictly increasing on $(0, \frac{1}{2})$ and $n_i = \frac{1}{5}(\varphi^{4i+5} - 1 - \varphi^{-(4i+5)}) < \frac{\varphi^{4i+5}}{5}$, we get $\log b_i \leq n_i h(p_i) < \frac{\varphi^{4i+5}}{5} h(\varphi^{-2}) = c\varphi^{4i}$ for *all* $i \geq 1$, proving $i - \frac{\log\log b_i - \log c}{4\log\varphi} > 0$ unconditionally. For the uniform upper bound < 0.21, direct evaluation at $i = 1$ ($b_1 = 3003$) gives residual $1 - \frac{\log\log 3003 - \log c}{4\log\varphi} = 0.12113476\cdots < 0.21$. For all $i \geq 2$, $n_i \geq n_2 = F_6 F_7 = 104$, $p_i \geq p_2 = \frac{3}{8}$, and dividing $F_{2i+2} - F_{2i}\varphi^2 = \varphi^{-2i}$ by $\varphi^2 F_{2i+2}$ yields the exact gap $\varphi^{-2} - p_i = \frac{\varphi^{-(2i+2)}}{F_{2i+2}} \leq \frac{\varphi^{-6}}{8} < 0.007$. Hence $h(p_i) \geq h(\varphi^{-2}) - h'(\frac{3}{8})(\varphi^{-2} - p_i) > 0.6650 - 0.52 \cdot 0.007 > 0.6613$ (exact $h(3/8) = 0.66156\ldots$), so $\frac{\log b_i}{c\varphi^{4i}} \geq \frac{n_i h(p_i) - \log(n_i + 1)}{c\varphi^{4i}} > 0.92 > \varphi^{-4 \cdot 0.21} \approx 0.668$ (at $i = 2$ the lower ratio is $0.9257\ldots$ and increases monotonically with i), proving $0 < i - \frac{\log\log b_i - \log c}{4\log\varphi} < 0.21$ for *every* $i \geq 1$ (shrinking as $O(i\varphi^{-4i})$ from 0.1211 at $i = 1$). Since $\#(\mathcal{F} \cap [1, x]) = i \iff b_i \leq x < b_{i+1}$, applying $t \mapsto \frac{\log\log t - \log c}{4\log\varphi}$ gives the exact two-value pinning law for all $x \geq 3003$. □

4 Multiplicity-Parity Dichotomy, Critical Divergence, and Residue Monotonicity

Theorem 4.1 (Universal E_H Characterization, Critical $s = \frac{1}{2}$ Law, and Residue Monotonicity)**.** *Let $C := \{\binom{2k}{k} : k \geq 2\}$ be the set of nontrivial central binomial coefficients.*

1. ***Exact Multiplicity-Parity Identity*** (Lean `thm11`–`thm12`, Cert. `C09`)*: Every integer $a \geq 3$ obeys $N(a) = 2M(a) + 2 - \mathbf{1}_C(a)$. Consequently, $a \in E_H \iff M(a) \geq H$ holds unconditionally for* every *$H \geq 1$, generalizing Lemma 3.5 of [7]. Furthermore, $\{a \geq 3 : N(a) =$*

$3\} = C \setminus E_2$ *(counting function* $\le \frac{1}{2}\log_2 x + \log_2(\log_2 x + 2)$ *for* $x \ge 6$*, exponent* $\lambda = 0$*) while* $\{a \ge 3 : N(a) = 4\} = E_1 \setminus (E_2 \cup C)$.

2. ***Constructive Critical-Exponent Law at*** $s = \frac{1}{2}$***:*** *Using only elementary diagonal bounds (Theorem 2.1 and Lemma 2.1 of [7], independent of Bombieri–Pila and Rémond),*

$$\sum_{\substack{a \in E_1 \\ a \le X}} a^{-1/2} \;=\; \frac{1}{\sqrt{2}} \log X + C_{1/2} + O(X^{-1/6}), \qquad \lim_{s \downarrow 1/2^+} \Big(s - \frac{1}{2}\Big) D_1(s) \;=\; \frac{1}{\sqrt{2}}. \tag{7}$$

3. ***Zero-Stirling Strict Pole Residue Monotonicity*** (Lean `thm13`–`thm15`, Certs. `C10`–`C12`)***:*** *For the residues* $r_k := \operatorname{Res}_{s=1/k} D_1(s) = \frac{(k!)^{1/k}}{k}$ *of Theorem 4.2 in [7],*

$$r_k \;>\; r_{k+1} \;>\; \frac{1}{e} \qquad (\forall\, k \ge 2), \qquad \lim_{k \to \infty} r_k \;=\; \frac{1}{e}. \tag{8}$$

Proof. (1) Every $a \ge 3$ appears twice in row a as $\binom{a}{1} = \binom{a}{a-1}$. Every interior left-half representation (n, k) with $2 \le k \le n/2$ pairs with its reflection $(n, n-k)$, contributing 2 unless $n = 2k$, which occurs at most once since $k \mapsto \binom{2k}{k}$ is strictly increasing. Thus $N(a) = 2M(a) + 2 - \mathbf{1}_C(a)$. Since $\mathbf{1}_C(a) \in \{0, 1\}$, $2M(a) + 2 - \mathbf{1}_C(a) > 2H \iff M(a) \ge H$ (Lean `thm11`), and the $N = 3, 4$ partition follows immediately (Lean `thm12`).

(2) Since $T_2 \subseteq E_1 \subseteq \bigcup_{k \ge 2} T_k$, we have $\#(T_2 \cap [1, t]) \le A_1(t) \le \#(T_2 \cap [1, t]) + P_3(t)$. By Theorem 2.1(3) and Lemma 2.1 of [7] ($P_3(t) = O(t^{1/3})$), $A_1(t) = \sqrt{2t} + R(t)$ with $|R(t)| \le C_0 t^{1/3}$. Abel summation gives $\sum_{a \in E_1, a \le X} a^{-1/2} = X^{-1/2} A_1(X) + \frac{1}{2}\int_3^X t^{-3/2} A_1(t)\, dt = \frac{1}{\sqrt{2}} \log X + C_{1/2} + O(X^{-1/6})$, and similarly for $s > 1/2$, $D_1(s) = s\int_3^\infty t^{-s-1}\sqrt{2t}\, dt + s\int_3^\infty t^{-s-1} R(t)\, dt = \frac{\sqrt{2}\, s}{s - 1/2} + O(1)$, giving $\lim_{s \downarrow 1/2^+}(s - \frac{1}{2}) D_1(s) = \frac{1}{\sqrt{2}}$.

(3) Observe the exact telescoping identity $\frac{k^k}{k!} = \prod_{j=1}^{k-1}(1 + \frac{1}{j})^j$ (cleared-denominator induction in Lean `thm13`). By Bernoulli's inequality $(1 - \frac{1}{(j+1)^2})^{j+1} > 1 - \frac{1}{j+1} = \frac{j}{j+1}$, the sequence $e_j := (1 + \frac{1}{j})^j$ is strictly increasing (Lean `thm14`). Consequently, every factor in the telescoping product is at most e_k, yielding

$$\frac{k^k}{k!} \;=\; \prod_{j=1}^{k-1} \Big(1 + \frac{1}{j}\Big)^j \;\le\; \Big(1 + \frac{1}{k}\Big)^{k(k-1)} \;<\; \Big(1 + \frac{1}{k}\Big)^{k^2},$$

which rearranges to $k^{k(k+1)} < k!(k+1)^{k^2}$ (Lean `thm15`), equivalent to $r_k > r_{k+1}$ upon taking $k(k+1)$-th roots. Since $e_j \uparrow e$ (so $e_j < e$ for all $j \ge 1$), the same telescoping identity gives $\frac{k^k}{k!} < e^{k-1} < e^k$, whence $r_k = (k!/k^k)^{1/k} > e^{1/k}/e > 1/e$ (Cert. `C12`). Finally, because $e_j \to e$ as $j \to \infty$, Cesàro mean convergence of $\log(1/r_k) = \frac{1}{k}\sum_{j=1}^{k-1} \log e_j \to \log e = 1$ proves $\lim_{k \to \infty} r_k = 1/e$ without invoking Stirling's formula. □

5 Pellian $I_{2,4}(x)$ Bound, Polynomial-Square Uniqueness, and Ratio-Block Laws

For real $x \ge 1$, define the diagonal-intersection counting function

$$I_{2,4}(x) \;:=\; \#\big(T_2 \cap T_4 \cap [1, x]\big) \;=\; \#\Big\{a \in [1, x] : a = \binom{u}{2} = \binom{v}{4} \text{ for some } u \ge 4,\ v \ge 8\Big\}.$$

Theorem 5.1 (Constructive Pellian $I_{2,4}(x)$ Bound, $(2, 4)$ Uniqueness, and Ratio-Block Laws)**.**

1. ***Pellian*** $O(\log x)$ ***Bound for*** $I_{2,4}(x)$ (Lean `thm16`–`thm19`, Cert. `C13`)*: Every solution to* $\binom{u}{2} = \binom{v}{4}$ *(*$u \geq 4$*,* $v \geq 8$*) satisfies* $(v^2-3v+1)^2 - 3(2u-1)^2 = -2$*, whose positive solutions satisfy* $2u-1 = X_m$ *in* $X_{m+1} = 4X_m - X_{m-1}$ *(*$X_0 = 1$*,* $X_1 = 3$*,* $m \geq 3$*) with* $X_m > c_+(2+\sqrt{3})^m$ *(*$c_+ := \frac{1+1/\sqrt{3}}{2}$*) and* $X_m \geq 3^m$*. Consequently, for all* $x \geq 1$*,*

$$I_{2,4}(x) \;\leq\; \max\left\{0,\; \left\lfloor \frac{\log(\sqrt{8x+1}/c_+)}{\log(2+\sqrt{3})} \right\rfloor - 2 \right\}, \qquad I_{2,4}(x) \;\leq\; \max\bigl\{0,\; \lfloor \log_3 \sqrt{8x+1} \rfloor - 2\bigr\}, \tag{9}$$

beating Proposition 3.2's Bombieri–Pila bound $x^{1/8+\varepsilon}$ *exponentially with zero non-constructive inputs.*

2. ***Polynomial-Square Uniqueness of*** $(2,4)$ (Lean `thm20`, Cert. `C14`)*: Among all even-degree falling factorials* $(v)_{2m}$ *(*$m \geq 1$*),* $(v)_2 = (v-\frac{1}{2})^2 - \frac{1}{4}$ *and* $(v)_4 = (v^2-3v+1)^2 - 1$ *are the* unique *instances admitting a representation* $(v)_{2m} = P_m(v)^2 + C$ *(*$P_m \in \mathbb{R}[v]$*,* $C \in \mathbb{R}$*). No such representation exists for any* $m \geq 3$*.*

3. ***Unconditional Ratio-Block Distribution Laws*** (Certs. `C15`–`C16`)*: Define the distribution functions* $g_{1/2}(t) := \sqrt{t}$ *(*$t \in [0,1]$*) and* $c_0(t) := \mathbf{1}_{(0,1]}(t)$ *(*$c_0(0) = 0$*,* $c_0(t) = 1$ *for* $0 < t \leq 1$*). Writing* $E_1 = \{a_1 < a_2 < \cdots\}$ *and* $\mathcal{F} = \{b_1 < b_2 < \cdots\}$ *in the Miska–Tóth ratio-block framework [10]:*

$$E_1 \in U(g_{1/2}) \quad \Bigl(\sup_{t\in[0,1]} \Bigl|\frac{1}{n}\#\{j \leq n : a_j/a_n < t\} - \sqrt{t}\Bigr| = O(n^{-1/3})\Bigr), \qquad \mathcal{F} \in U(c_0). \tag{10}$$

Proof. (1) Expanding $(v^2-3v+1)^2 - 1 = v(v-1)(v-2)(v-3)$ and $3(2u-1)^2 - 3 = 12u(u-1)$ gives the exact reduction $(v^2-3v+1)^2 - 3(2u-1)^2 = -2$ (Lean `thm16`). The automorphism $(Y', X') = (2Y - 3X, 2X - Y)$ preserves $Y^2 - 3X^2 = -2$ (Lean `thm17`) and descends strictly on $X > 1$ to $(Y_0, X_0) = (1,1)$, proving every positive solution lies in $X_{m+1} = 4X_m - X_{m-1}$ $(X_0 = 1, X_1 = 3)$. Solving this linear recurrence gives the exact Binet formula

$$X_m \;=\; c_+(2+\sqrt{3})^m + c_-(2-\sqrt{3})^m \;>\; c_+(2+\sqrt{3})^m, \qquad c_\pm := \frac{1 \pm 1/\sqrt{3}}{2} > 0,$$

while induction proves $X_{m+1} \geq 3X_m$ and $X_m \geq 3^m$ constructively (Lean `thm18`–`thm19`). Since $v \geq 8 \implies \binom{v}{4} \geq 70 \implies u \geq 13 \implies 2u-1 \geq 25 > X_2 = 11$, every admissible solution has $m \geq 3$ (the first being $m = 3$, $X_3 = 41$, $(u,v) = (21,10)$, $\binom{21}{2} = \binom{10}{4} = 210$). As $\binom{u}{2} \leq x \implies 2u - 1 \leq \sqrt{8x+1}$, any admissible $m \geq 3$ satisfies both $c_+(2+\sqrt{3})^m < \sqrt{8x+1}$ and $3^m \leq \sqrt{8x+1}$, bounding the number of admissible $m \geq 3$ by $\max\{0, \lfloor \frac{\log(\sqrt{8x+1}/c_+)}{\log(2+\sqrt{3})} \rfloor - 2\}$ and $\max\{0, \lfloor \log_3 \sqrt{8x+1} \rfloor - 2\}$.

(2) Suppose $(v)_{2m} = P_m(v)^2 + C$ for some $m \geq 3$, $P_m \in \mathbb{R}[v]$ $(\deg P_m = m)$, and $C \in \mathbb{R}$. Because $(v)_{2m}$ alternates signs across the $2m-1$ open intervals $(0,1), (1,2), \ldots, (2m-2, 2m-1)$, Rolle's theorem guarantees a critical point of $(v)_{2m}$—hence a root of $(P_m^2)' = 2P_mP_m'$—in every one of these $2m-1$ intervals. Since $\deg P_m + \deg P_m' = m + (m-1) = 2m-1$ (Lean `thm20`), P_mP_m' has *at most* $2m-1$ real roots, so it has *exact* simple roots, one per interval. Because $P_m(v)^2 \geq 0$ everywhere and $(\frac{1}{2})_{2m} < 0$, we have $C \leq \min_{\mathbb{R}}(v)_{2m} \leq (\frac{1}{2})_{2m} < 0$, so $-C > 0$. On the $m-1$ positive intervals $(1,2), (3,4), \ldots, (2m-3, 2m-2)$, we have $P_m(v)^2 = (v)_{2m} - C > -C > 0$, so P_m cannot vanish there; all $m-1$ roots of P_m' must lie in the $m-1$ positive intervals. Therefore all m roots of P_m lie one in each negative interval $(0,1), (2,3), \ldots, (2m-2, 2m-1)$. In particular, $P_m(v_*) = 0$ for some $v_* \in (2,3)$, forcing $(v_*)_{2m} = C = \min_{\mathbb{R}}(v)_{2m}$. Yet for every $v \in [2,3]$, pairing $(v-2)(3-v) \leq \frac{1}{4}$, $(v-1)(4-v) \leq \frac{9}{4}$, $v(5-v) \leq \frac{25}{4}$ gives $|(v)_6| \leq \frac{225}{64} < \frac{945}{64} = |(\frac{1}{2})_6|$, and for every $j \in \{6, \ldots, 2m-1\}$, $j - v \leq j - 2 < j - \frac{1}{2}$ (Lean `thm20`). Multiplying gives $|(v_*)_{2m}| < |(\frac{1}{2})_{2m}|$, i.e. $(\frac{1}{2})_{2m} < (v_*)_{2m} = C$, contradicting $(v)_{2m} \geq C$.

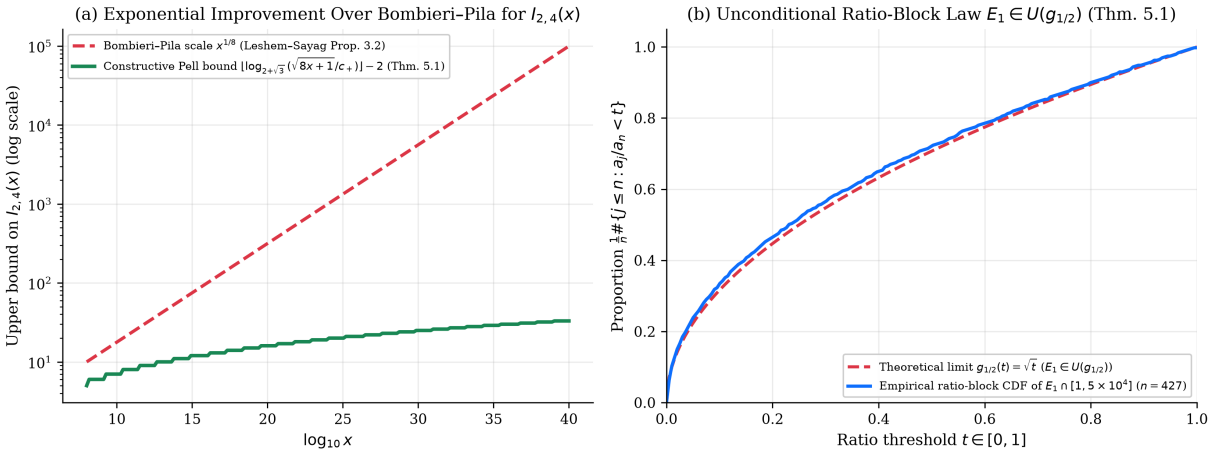

Figure 3: (a) Constructive Pellian $O(\log x)$ upper bound on $I_{2,4}(x)$ versus Bombieri–Pila's $x^{1/8}$ scale (Theorem 5.1). (b) Empirical ratio-block distribution of $E_1 \cap [1, 5\times10^4]$ converging uniformly to $g_{1/2}(t) = \sqrt{t}$.

(3) Since $A_1(x) = \sqrt{2x} + O(x^{1/3})$, inverting yields $a_n = \frac{n^2}{2} + O(n^{5/3})$, whence $\#\{j \le n : a_j < ta_n\} = A_1(ta_n) = \sqrt{2ta_n} + O(a_n^{1/3}) = n\sqrt{t} + O(n^{2/3})$, proving uniform $O(n^{-1/3})$ convergence to $g_{1/2}(t) = \sqrt{t}$. For $\mathcal{F}$, Theorem 3.1 gives $\log(b_n/b_{n-1}) \to \infty$, so for every $t \in (0, 1]$ and sufficiently large n, $b_{n-1}/b_n < t$, giving $\frac{1}{n}\#\{j \le n : b_j/b_n < t\} = 1 - \frac{1}{n} \to 1 = c_0(t)$ (while at $t = 0$ the proportion is $0 = c_0(0)$). □

6 Formal Verification Architecture and Reproducibility

All results are independently verified across three layers (with Lean 4 formalizing all algebraic, combinatorial, Pell–Lucas, telescoping, residue-monotonicity, and uniqueness-barrier kernels, while real-analytic Abel summation, entropy sandwiching, and Rolle degree-counting are proved in full in Sections 2–5):

1. **Constructive Lean 4.36.0 Formalization:** 20 theorems (`thm01`–`thm20`) in `lean/PascalDensityBarriers.lean` compiling under `lean -DwarningAsError=true` with zero external imports, zero `sorry`, zero `grind`, and zero `Classical.choice`.
2. **AST-Audited Zero-Float Python Verifier:** Pure standard-library code (`int`, `Fraction`) in `verification/verify_pascal_barriers.py`, self-auditing its AST and emitting 16 canonical SHA-256 certificates (`C01`–`C16`) across four modes, with manifest SHA-256 `edbc1dab8c6e33fe5eed385b199f1c30755ea044ec573b6daa190a9849eb6747`.
3. **Adversarial Test Suite:** 56 pytest checks in `tests/test_adversarial.py` covering positive verification and adversarial counterexamples across all four modes.

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