

A dimension-independent capacity bound for amplified determinants of Hermitian matrix tuples

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Abstract

For a matrix tuple, we maximize the geometric mean of the squared singular values of its linear pencil over auxiliary matrices with unit normalized Hilbert–Schmidt energy, allowing arbitrary auxiliary size. We compare this regularized optimum with an optimization of Gurvits capacity over coefficient covariance matrices. The capacity optimization is an upper bound for every tuple. If the coefficients are Hermitian and their real span contains a positive definite matrix, it also bounds the optimum from below up to the universal factor e . The lower bound combines the established Mai–Speicher free determinant formula with finite-dimensional regularization. An imaginary shift is represented inside the original coefficient tuple, and its energy cost is computed exactly. Real covariance suffices for the capacity optimum, giving normal auxiliary witnesses without size doubling. A separate dilation transfers general bounded moment approximants to feasible witnesses. Only convergence of a bounded, regularized logarithm is needed; convergence of unregularized Gaussian log determinants is not asserted. The result gives finite witnesses approaching the bound, without a quantitative dimension estimate. We state separately the unresolved extension to arbitrary matrix tuples and the question of an effective construction.

1 The question and the precise scope

Write $\tau_m = \text{Tr}/m$. For $A = (A_1, \dots, A_s) \in M_d(\mathbb{C})^s$, define

$$\mathcal{F}_m(A) = \max_{\sum_i \tau_m(Y_i Y_i^*) = 1} \left| \det \left(\sum_i A_i \otimes Y_i \right) \right|^{2/(dm)}, \quad (1)$$

$$\mathcal{F}_\infty(A) = \sup_{m \geq 1} \mathcal{F}_m(A). \quad (2)$$

The coefficient presentation and its Euclidean energy are part of the problem; replacing a tuple by an arbitrary basis of the same span need not preserve this objective. The variables Y_i may be complex and need not be Hermitian.

For a positive semidefinite $s \times s$ matrix Q of trace one, set

$$\Phi_Q(X) = \sum_{i,j} Q_{ij} A_i X A_j^*, \quad (3)$$

$$\text{cap}(\Phi) = \inf_{\substack{X \geq 0 \\ \det X = 1}} \det \Phi(X), \quad \mathcal{C}(A) = \max_{\substack{Q \geq 0 \\ \text{Tr } Q = 1}} \text{cap}(\Phi_Q)^{1/d}. \quad (4)$$

Here capacity is the determinant invariant of a completely positive map, not quantum communication capacity. If $Q = LL^*$ and $B_j = \sum_i L_{ij} A_i$, then $\Phi_Q(X) = \sum_j B_j X B_j^*$, so the map is completely positive even when Q is singular.

Theorem 1. For every complex matrix tuple, $\mathcal{F}_\infty(A) \leq \mathcal{C}(A)$. If each $A_i = A_i^*$ and

$$\sum_i b_i A_i = D > 0 \quad \text{for some } b \in \mathbb{R}^s,$$

then

$$\frac{\mathcal{C}(A)}{e} \leq \mathcal{F}_\infty(A) \leq \mathcal{C}(A). \quad (5)$$

In the latter case $\mathcal{C}(A) > 0$. For every $\eta > 0$ there is a finite auxiliary size and a feasible tuple whose objective is at least $(1 - \eta)\mathcal{C}(A)/e$.

Thus the logarithmic optimum is determined within one nat:

$$-\log \mathcal{C}(A) \leq -\log \mathcal{F}_\infty(A) \leq -\log \mathcal{C}(A) + 1.$$

The factor does not depend on d , s , or the coefficients within the stated class. The witness size can depend on all of them and on η . We do not claim that e is optimal, that the covariance optimization is exact, or that the proof yields a polynomial-time algorithm.

Earlier work on the same objective gives sharp special-family values and noise laws [4, 5]. In particular, the comparison in [5] is uniform over positive reweightings of a fixed pencil, with constants depending on that pencil. The present comparison uses a different reference quantity and a universal factor, under an additional Hermitian positivity hypothesis. The free determinant identity used below is a theorem of Mai and Speicher [1]; the proposed contribution here is its transfer to energy-constrained finite witnesses by a perturbation that stays in the given tuple.

2 The capacity upper bound

The maximum in (4) exists. For fixed $X > 0$, the function $Q \mapsto \det \Phi_Q(X)^{1/d}$ is continuous and nonnegative. Its infimum over $\det X = 1$ is upper semicontinuous, so it attains a maximum on the compact set of density matrices. It is finite since $X = I_d$ gives a finite upper bound. The same observation and concavity of the determinant root show that this capacity root is concave in Q . This does not justify interchanging the maximum and infimum in (4).

Proposition 2. Every feasible tuple Y in (1), with $Q_{ij} = \tau_m(Y_i Y_j^*)$, satisfies

$$\left| \det \left(\sum_i A_i \otimes Y_i \right) \right|^{2/(dm)} \leq \text{cap}(\Phi_Q)^{1/d}.$$

Proof. The matrix Q is positive semidefinite and has trace one. Put $H = \sum_i A_i \otimes Y_i$. If H is singular, the conclusion is immediate. Otherwise fix $X > 0$ with $\det X = 1$ and put $M = H(X \otimes I_m)H^* > 0$. The normalized partial trace $E = \text{id}_{M_d} \otimes \tau_m$ gives $E(M) = \Phi_Q(X)$. Averaging conjugates of M by a unitary error basis on the auxiliary factor gives $E(M) \otimes I_m$. Concavity and unitary invariance of $\log \det$ therefore give

$$\frac{1}{dm} \log \det M \leq \frac{1}{d} \log \det E(M).$$

Since $\det M = |\det H|^2$, exponentiation and the infimum over X prove the assertion. $\square$

Lemma 3 (Real covariance suffices). If all A_i are Hermitian, the maximum in (4) has a real symmetric maximizer.

Proof. The Hilbert–Schmidt adjoint satisfies $\Phi_Q^* = \Phi_{\overline{Q}}$. Capacity is invariant under this adjoint; see [1], immediately after Theorem 1.2. Consequently $f(Q) = \text{cap}(\Phi_Q)^{1/d}$ satisfies $f(\overline{Q}) = f(Q)$. The matrix $Q_{\mathbb{R}} = (Q + \overline{Q})/2$ is real symmetric, positive semidefinite, and of trace one. Concavity of f gives $f(Q_{\mathbb{R}}) \geq f(Q)$. $\square$

3 An exact finite-dimensional regularization

We first assume that the positive direction is I_d :

$$A_i = A_i^*, \quad \sum_i b_i A_i = I_d, \quad b_i \in \mathbb{R}. \quad (6)$$

For arbitrary $Y_i \in M_N(\mathbb{C})$ and $\varepsilon > 0$, define

$$H_N = \sum_i A_i \otimes Y_i, \quad E_N = \sum_i \tau_N(Y_i Y_i^*), \quad (7)$$

$$Z_i = \begin{pmatrix} 0 & Y_i \\ Y_i^* & 0 \end{pmatrix} + i\varepsilon b_i I_{2N}, \quad \widehat{Z}_i = \frac{Z_i}{\sqrt{E_N + \varepsilon^2 \|b\|_2^2}}. \quad (8)$$

Lemma 4 (Finite witness identity). *The tuple $\widehat{Z}$ is feasible at auxiliary size $2N$, and its objective equals*

$$\left| \det \left(\sum_i A_i \otimes \widehat{Z}_i \right) \right|^{1/(dN)} = \frac{\exp \left\{ \frac{1}{dN} \operatorname{Tr} \log (H_N^* H_N + \varepsilon^2 I_{dN}) \right\}}{E_N + \varepsilon^2 \|b\|_2^2}. \quad (9)$$

Proof. The first summand in Z_i is Hermitian. As b_i is real, multiplication by its adjoint cancels the two cross terms, giving

$$\tau_{2N}(Z_i Z_i^*) = \tau_N(Y_i Y_i^*) + \varepsilon^2 b_i^2.$$

After permuting tensor factors, (6) gives

$$\sum_i A_i \otimes Z_i \simeq \begin{pmatrix} 0 & H_N \\ H_N^* & 0 \end{pmatrix} + i\varepsilon I_{2dN}.$$

The Hermitian block matrix has eigenvalues $\pm s_j(H_N)$, including zeros. Hence

$$\left| \det \left(\sum_i A_i \otimes Z_i \right) \right| = \prod_{j=1}^{dN} (s_j(H_N)^2 + \varepsilon^2) = \det(H_N^* H_N + \varepsilon^2 I).$$

Dividing every Z_i by the square root of its total energy divides the objective by that energy. This proves (9). $\square$

The shift has no uncharged coefficient: its cost is exactly $\varepsilon^2 \|b\|_2^2$. The construction also changes the auxiliary size from N to $2N$, which is permitted by the supremum in $\mathcal{F}_\infty$.

If each $Y_i = Y_i^*$, doubling is unnecessary. In that case $H_N = H_N^*$, and the normal matrices

$$W_i = \frac{Y_i + i\varepsilon b_i I_N}{\sqrt{E_N + \varepsilon^2 \|b\|_2^2}} \quad (10)$$

are feasible and have the same right-hand side of (9), now as the objective at size N . This follows directly from $|\det(H_N + i\varepsilon I)|^2 = \det(H_N^2 + \varepsilon^2 I)$ and cancellation of the energy cross terms. The dilation above remains useful for non-Hermitian Y_i .

Proposition 5 (Transfer from bounded moment approximants). *Assume (6). Let $y_1, \dots, y_s$ belong to a tracial von Neumann algebra $(\mathcal{M}, \tau)$ with $\sum_i \tau(y_i y_i^*) = 1$. Suppose tuples $Y^{(N)}$ converge to y in joint normalized $*$ -moments and satisfy $\sup_{N,i} \|Y_i^{(N)}\| < \infty$. Put*

$$H = \sum_i A_i \otimes y_i, \quad \tau_d = \operatorname{Tr} / d \otimes \tau, \quad \Delta(H) = \exp(\tau_d(\log |H|)),$$

where the last expression is defined by decreasing positive regularization and may be zero. Then $\mathcal{F}_\infty(A) \geq \Delta(H)^2$.

Proof. Fix $\varepsilon > 0$. The matrices $H_N^* H_N$ have uniformly bounded spectra, and their normalized moments converge to those of $H^* H$. Polynomial approximation to the continuous function $t \mapsto \log(t + \varepsilon^2)$ on a common compact interval gives

$$\lim_{N \rightarrow \infty} \frac{1}{dN} \text{Tr} \log(H_N^* H_N + \varepsilon^2 I) = \tau_d(\log(H^* H + \varepsilon^2 I)).$$

Also $E_N \rightarrow 1$. Lemma 4 therefore implies

$$\mathcal{F}_\infty(A) \geq \frac{\exp\{\tau_d(\log(H^* H + \varepsilon^2 I))\}}{1 + \varepsilon^2 \|b\|_2^2}.$$

Letting $\varepsilon \downarrow 0$ yields the result, including when $\Delta(H) = 0$. The order of limits is fixed: first $N \rightarrow \infty$ for fixed ε , then $\varepsilon \downarrow 0$. No claim about $\det H_N$ is used. $\square$

4 Proof of the universal factor in the stated class

We use two established inputs. First, for free standard semicircular variables s_j and arbitrary fixed matrices $B_j \in M_d(\mathbb{C})$, the Mai–Speicher formula gives

$$\Delta \left(\sum_j B_j \otimes s_j \right)^2 = e^{-1} \text{cap} \left(X \mapsto \sum_j B_j X B_j^* \right)^{1/d}. \quad (11)$$

This is [1, Theorem 1.2]. The identity and its constant are prior results.

Second, independent GUE matrices with normalized second moment converging to one have realizations converging jointly in $*$ -moments to free standard semicircular variables, with uniformly bounded operator norms. These standard Gaussian approximation facts follow from Gaussian asymptotic freeness and norm convergence; see [2, 3]. Only bounded norms and moment convergence are needed here.

Proof of Theorem 1. The upper bound was proved in Proposition 2. Under (6), Lemma 3 permits us to choose a real maximizing $Q \geq 0$ of trace one and factor $Q = LL^T$ with L real. Set

$$y_i = \sum_j L_{ij} s_j, \quad B_j = \sum_i L_{ij} A_i.$$

Since $\tau(s_j s_k) = \delta_{jk}$, the energy of y is $\text{Tr} Q = 1$. Taking the same real linear combinations of GUE approximants supplies Hermitian bounded moment approximants. The regularized convergence argument in Proposition 5, using (10), and (11) imply

$$\mathcal{F}_\infty(A) \geq \Delta \left(\sum_i A_i \otimes y_i \right)^2 = e^{-1} \text{cap}(\Phi_Q)^{1/d}.$$

Since Q is maximizing, this proves the bound when I_d is in the real span, with normal auxiliary witnesses and no doubling of the Gaussian approximation size.

For the general hypothesis, let $S = D^{-1/2}$ and $\tilde{A}_i = S A_i S$. Then the $\tilde{A}_i$ are Hermitian and $\sum_i b_i \tilde{A}_i = I_d$. Direct determinant scaling gives

$$\mathcal{F}_m(\tilde{A}) = \det(D)^{-2/d} \mathcal{F}_m(A), \quad \text{cap}(\tilde{\Phi}_Q) = \det(D)^{-2} \text{cap}(\Phi_Q). \quad (12)$$

For the second identity, $\tilde{\Phi}_Q(X) = S \Phi_Q(S X S) S$. As X runs through positive matrices of determinant one, $S X S$ runs through positive matrices of determinant $\det(D)^{-1}$; homogeneity of Φ_Q gives the stated factor. Thus both sides of (5) scale identically.

Finally, the scalar tuple $Y_i = b_i / \|b\|_2$ has invertible pencil $D / \|b\|_2$, so $\mathcal{F}_1(A) > 0$. Finite witnesses arbitrarily near the asserted lower bound follow from the definition of the supremum, or directly by first fixing a small ε in (9) and then taking a sufficiently large N . $\square$

5 Examples and checks on the scope

5.1 A single positive definite coefficient

For $s = 1$ and $A_1 = D > 0$, direct separation of determinants and the arithmetic–geometric mean inequality give

$$\mathcal{F}_m(A) = \mathcal{F}_\infty(A) = \mathcal{C}(A) = \det(D)^{2/d}.$$

Equality is obtained with a unitary Y_1 . Thus the factor e is not an equality statement for every tuple. Even in this case, the semicircular construction alone has limiting value $\mathcal{C}(A)/e$.

5.2 Orthogonal projections

Let $A_i = P_i$ be mutually orthogonal nonzero projections summing to I_d , of ranks r_i , and write $w_i = r_i/d$. The pencil is block diagonal, and for $e_i = \tau_m(Y_i Y_i^*)$,

$$\left| \det \left(\sum_i P_i \otimes Y_i \right) \right|^{2/(dm)} \leq \prod_i e_i^{w_i} \leq \prod_i w_i^{w_i}.$$

The choice $Y_i = \sqrt{w_i} I_m$ attains this value. Also $\Phi_Q(I_d) = \sum_i Q_{ii} P_i$, so $\mathcal{C}(A) \leq \max_Q \prod_i Q_{ii}^{w_i} = \prod_i w_i^{w_i}$. The upper bound already proved shows

$$\mathcal{F}_m(A) = \mathcal{F}_\infty(A) = \mathcal{C}(A) = \prod_i w_i^{w_i}.$$

This example checks the coefficient energy and normalized exponents.

5.3 A noncommuting family with a small positive direction

In dimension three take

$$A_0 = \delta I_3, \quad A_1 = \frac{i(E_{12} - E_{21})}{\sqrt{2}}, \quad A_2 = \frac{i(E_{23} - E_{32})}{\sqrt{2}}, \quad A_3 = \frac{i(E_{31} - E_{13})}{\sqrt{2}}, \quad \delta > 0.$$

These are Hermitian, and (6) holds with $b = (\delta^{-1}, 0, 0, 0)$. The main theorem applies for every positive δ , with the same factor e . The finite construction, however, pays ε^2/δ^2 . It does not supply a witness size uniform as $\delta \downarrow 0$, nor does it prove a statement for the limiting traceless tuple by simply setting $\delta = 0$.

More generally, the hypothesis cannot be replaced in this proof by an arbitrary change of basis to a Hermitian spanning family: that change can alter the coefficient energy. Hermiticity is used to obtain the lower-left block H_N^* ; the positive direction is used to create an imaginary scalar shift after congruence.

6 What this step establishes and what remains

Lemma 4 is an exact finite identity. Proposition 5 supplies a bridge from bounded moment approximants to feasible witnesses. Combined with the existing free capacity formula, it gives a universal approximation factor in the class of Theorem 1. It does not establish convergence of centered Gaussian log determinants, a separate issue discussed in [1]. It also does not identify optimal finite witnesses.

The next mathematical targets are:

1. Determine whether $\mathcal{C}(A)/e \leq \mathcal{F}_\infty(A)$ holds for arbitrary complex tuples, or find a counterexample. The in-span Hermitian shift used here is unavailable in general.

2. Give quantitative auxiliary-size and computational bounds, with explicit dependence on conditioning and target error.
3. Determine the optimal comparison constant and a concrete operational consequence for the channel-testing model of [4].

The general extension is a research question, not a theorem of this manuscript. Claims about practical importance require progress on these questions or another independently useful application.

Verification and provenance. The accompanying executable checks the finite dilation identity and its energy normalization in exact arithmetic, and also checks the direct Hermitian construction, adjoint covariance identity, and congruence scaling numerically. Such checks test implementation and normalization; they do not replace the proof or certify novelty. The manuscript was developed with assistance from OpenAI Codex. A GPT-6 Astra review with shared development context suggested the real-covariance simplification; it is not a blind independent review. The manuscript has not undergone human peer review or formal proof verification. The imported free determinant formula and Gaussian approximation facts are explicitly attributed.

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