

Exact values for a finite Ramsey-type problem from Green’s list (Problem 22):

$$N_0(1) = 9, N_0(2) = 54, N_0(3) = 774$$

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Abstract

For $r \geq 1$ let $N_0(r)$ be the least N such that every r -colouring of $\{1, \dots, N\}$ contains $x, y \geq 3$ (not necessarily distinct) with $x + y$ and xy in $\{1, \dots, N\}$ and of the same colour. This is the quantity in Problem 22 of Green’s list of open problems. Using SAT solving I find $N_0(1) = 9$, $N_0(2) = 54$ and $N_0(3) = 774$. The lower bounds $N_0(r) > N$ come from explicit colourings (given in the appendix) that are checked by independent programs, including Lean 4. The upper bounds come from unsatisfiability of the corresponding CNF formulas; for $r = 2, 3$ this is supported by four solvers, two encodings and DRUP proofs checked by a small checker written for this note. It is *not* a formally verified proof, and the note is a small computational datum, not progress on the asymptotic question. See the “Source disclosure” section.

1 The problem

Problem 22 in Ben Green’s *100 open problems* [1] reads: if $\{1, \dots, N\}$ is r -coloured then, for $N \geq N_0(r)$, there are integers $x, y \geq 3$ such that $x + y$ and xy have the same colour; find reasonable bounds for $N_0(r)$. Green comments that the existence of $N_0(r)$ follows from a theorem of Moreira [3], and that Green and Sawhney [2] have shown that $N_0(r) = \exp \exp(r^{50})$ is permissible.

The paper of Green and Sawhney, *Bounds for monochromatic solutions to $\{x + y, xy\}$* (arXiv:2511.09365, v2 of 19 Nov. 2025), states in its abstract and Theorem 1.1 that there is an absolute constant r_0 such that for $r \geq r_0$ and $N \geq \exp \exp(r^{50})$ every r -colouring of $[N]$ contains a monochromatic $\{x + y, xy\}$ with $x > y > 2$. In a remark after the theorem they describe a colouring of $[N]$, $N = (3^r + 7)/2$, obtained by colouring consecutive intervals $[a_i, a_{i+1})$, $a_i = (3^i + 9)/2$, with colour i (any colours on $1, 2, 3, 4$); it has no monochromatic $\{x + y, xy\}$ with $x > y > 2$, and they remark that the theorem is therefore at most one logarithm from optimal. For $r = 2, 3$ this gives only $N = 8$ and $N = 17$.

The Lean formalisation of the problem in the repository `google-deepmind/formal-conjectures` (file `GreensOpenProblems/22.lean`, read on 21 Oct. 2026) [4] calls a colouring of $\{1, \dots, N\}$ bad if there are $x, y \geq 3$ with $x + y$ and xy in range and of the same colour, and defines $N_0(r)$ as the least N such that every r -colouring of $\{1, \dots, N\}$ is bad. Here $x = y$ is allowed. I use exactly this definition:

$$N_0(r) = \min\{N : \text{every } c : \{1, \dots, N\} \rightarrow \{1, \dots, r\} \text{ admits } x, y \geq 3 \text{ with } xy \leq N \text{ and } c(x + y) = c(xy)\}.$$

Since $x, y \geq 3$ gives $x + y \leq xy$, the condition $x + y \leq N$ is implied by $xy \leq N$. The repository file also records the trivial bound $N_0(r) \geq 9$.

Monotonicity. If c is a colouring of $[1, N]$ with no monochromatic pair, its restriction to $[1, M]$, $M < N$, has none either, since every admissible pair in $[1, M]$ is admissible in $[1, N]$. Hence the set of N admitting a good colouring is an initial segment, and $N_0(r) = 1 + \max\{N : \text{a good } r\text{-colouring of } [1, N] \text{ exists}\}$.

2 Results

Theorem 1. $N_0(1) = 9$, $N_0(2) = 54$, $N_0(3) = 774$.

Equivalently: there are good 2-colourings of $[1, 53]$ and good 3-colourings of $[1, 773]$ (witnesses in the appendix), and none of $[1, 54]$, respectively $[1, 774]$. For $r = 1$ the statement is elementary: for $N \leq 8$ no $x, y \geq 3$ has $xy \leq N$, and for $N = 9$ we have $x = y = 3$ with $x + y = 6$, $xy = 9$.

Table 1 summarises the status of each claim.

r	Claim	Evidence	Strength
2	colouring of $[1, 53]$ exists	Python checker; Lean <code>native_decide</code> ; Lean kernel (<code>decide +kernel</code>)	rigorous
3	colouring of $[1, 773]$ exists	Python checker; Lean <code>native_decide</code>	rigorous (compiler trusted)
2	no colouring of $[1, 54]$	4 solvers; DRUP proof (1 lemma) checked; exhaustive Lean search by <code>native_decide</code>	computational
3	no colouring of $[1, 774]$	4 solvers; DRUP proof (2132 lemmas) checked by own RUP checker	computational

Table 1: Status of the claims.

For comparison with the Green–Sawhney remark: the interval construction yields good colourings only up to 8 and 17 for $r = 2, 3$, far below 53 and 773. I draw no conclusion about large r from these small values.

Under the stricter condition $x > y > 2$ used in [2] the values for $r = 2, 3$ are the same (53 colourable and 54 not; 773 colourable and 774 not; same encoding with $y \geq x + 1$). They differ for $r = 1$ (the strict variant gives 12, from $x = 4$, $y = 3$).

3 Method and verification

Encoding. Let $P_N = \{(x + y, xy) : 3 \leq x \leq y, xy \leq N\}$. For r colours use variables $v_{n,k}$ ($1 \leq n \leq N$, $0 \leq k < r$). The clauses are $\bigvee_k v_{n,k}$ for each n (at least one colour), and $\neg v_{a,k} \vee \neg v_{b,k}$ for each $(a, b) \in P_N$ and each k . “At most one colour” is not imposed (a colouring is read off by choosing any true colour), which is sound for both satisfiability and unsatisfiability. For $N = 54$, $r = 2$ the formula has 108 variables, 37 pairs and 128 clauses; for $N = 774$, $r = 3$ it has 2322 variables, 1489 pairs and 5241 clauses. When searching for the threshold the colour of 6 was fixed to 0 (valid, since $(3, 3)$ gives the pair $(6, 9)$ whenever $N \geq 9$, and colours may be permuted); the formulas used for the proofs have *no* symmetry breaking.

Solvers. Satisfiability was decided with CaDiCaL 1.5.3, Glucose 4 and MapleChrono through PySAT, and independently with OR-Tools CP-SAT using integer colour variables and no symmetry breaking (a second encoding). All four agree on the key instances ($N = 53, 54$ for $r = 2$; $N = 773, 774$ for $r = 3$), and bisection using monotonicity located the thresholds. A spurious debugging message (“found falsified original clause”) was printed by the PySAT CaDiCaL wrapper in some runs; the results were confirmed by the other solvers.

DRUP proofs and the checker. Glucose 4 was run with proof logging on the two unsatisfiable formulas (without symmetry breaking). The proofs have 1 lemma ($N = 54$) and 2132 lemmas ($N = 774$); deletion lines are ignored. I wrote a forward RUP checker (`rupcheck2.py`, about 70 lines, two-watched-literal unit propagation, independent of PySAT), and it accepts both proofs, ending in a conflict at decision level 0. Five deliberately false proofs are rejected: the empty clause given directly for $N = 774$; the empty clause given for the satisfiable $N = 53$ formula; the $N = 774$ proof with its first half removed; a proof with a false lemma “1 0” prepended; and the genuine $N = 774$ proof checked against the satisfiable $N = 773$ formula (rejected at lemma 1397). Three of these false proofs are in the supplement; the others are regenerated as described in its README. The checker is **not** drat-trim, not a verified checker such as `cake_lpr`, and not itself formally verified; the confidence it gives rests on the shortness of the code and on the negative tests.

Lean 4. The file `Green22.lean` (core Lean 4.35.0, no Mathlib) contains the two witnesses as strings, a Boolean function `good` that checks that no admissible pair is monochromatic, and proofs by `native_decide` that `good 53` and `good 773` hold for the witnesses. The 53-witness is also checked by `decide +kernel`, which does not trust the compiler. A short backtracking search (`exists_col`) is evaluated by `native_decide` and gives `exists_col 2 53 = true`, `exists_col 2 54 = false`, `exists_col 1 8 = true`, `exists_col 1 9 = false`. The file compiles (exit code 0, about two minutes). An attempt to check the 773-witness with `decide +kernel` (`G3k.lean`) was stopped by a 25-minute timeout (exit code 124) and gave no result.

What is not done. (i) There is no Lean kernel proof of unsatisfiability for $N = 774$. (ii) There is no LRAT or drat-trim certificate. (iii) `native_decide` trusts the Lean compiler, and there is no theorem linking `exists_col` to the statement in the formal-conjectures file, so the Lean search is not a formal proof that $N_0(2) = 54$; the formal definition of N_0 from that repository was not used. (iv) $N_0(4)$ has not been computed. (v) There has been no human review and no formal check of the whole pipeline.

4 Source disclosure

This is exploratory computational work carried out with AI assistance: the scripts, formulas, proof checker, Lean files and this text were produced by Claude Sonnet 5.5 agents under the direction of the author. There has been no human refereeing and no complete formal verification. The literature search for earlier values of $N_0(2)$ and $N_0(3)$ was limited (Green’s list, Green–Sawhney, Moreira, the formal-conjectures file and a few web searches); I found none, but absence of evidence is not evidence of priority, and I make no claim of priority. The numerical claims are reproducible with the supplement, and the reader should rerun them rather than rely on this text.

References

- [1] B. Green, *100 open problems*, manuscript (updated December 2025), Problem 22. <https://people.maths.ox.ac.uk/greenbj/papers/open-problems.pdf>
- [2] B. Green and M. Sawhney, *Bounds for monochromatic solutions to $\{x + y, xy\}$* , arXiv:2511.09365 (2025).
- [3] J. Moreira, *Monochromatic sums and products in $\mathbb{N}$* , Ann. of Math. **185** (2017), no. 3, 1069–1090.

- [4] Google DeepMind, *formal-conjectures*, `FormalConjectures/GreensOpenProblems/22.lean`, <https://github.com/google-deepmind/formal-conjectures>.

A Witness colourings

The colour of $n = 1, 2, \dots$ is written as a string of digits (the n th digit is the colour of n ; colours are 0, 1 for $r = 2$ and 0, 1, 2 for $r = 3$). No admissible pair $(x + y, xy)$ with $x, y \geq 3$ is monochromatic. The strings are also in `witness_r2_N53.txt` and `witness_r3_N773.txt`.

$r = 2, N = 53$.

```
0000001111000000000001001101101011011001101011001111110
```

$r = 3, N = 773$ (lines of 50 digits, last line 23).

```
00000021111112222222222222222220000000000000000000
0000000000000000000000000000000000000000000000000001111111111
11111111111111111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111111111111111
111111111120022220202202200220202222222202222022
20222202220200220222222202222022022022222200202
2202220022022220222220222220220022022222020022
0222222022222222202002222222020222220222202202202
2220220022022202200222222220022022220222222202
2002222202222022222202202222222222220220022220222
22202202222002202202022222022222220022220222202
02222202222002202202022220222222202222202220222
22022222020022222222002202222202222222022022022
022220220222222220022222202222002222022202202222
020222002202222022202220
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