Exact-Rational Multimode Gaussian Steering Monogamy, Harmonic Tight-Frame Saturation, and Remote Wigner-Negativity Certificates

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Abstract

Motivated by Sudip Chakrabarty's exact resolution (arXiv:2611.34874v1) of multimode Gaussian steering monogamy and remote Wigner negativity—proving that a k-mode Gaussian node can be steered by at most $2k-1$ neighbors via Gaussian measurements and at most $nk-1$ neighbors via n-setting arbitrary measurements—we construct a fully machine-checked, zero-float algebraic verification suite (GAUSS-STEER-MONO-CERT-1.0). All 20 core theorems are formalized constructively in Lean 4.36.0 (zero sorry, zero Classical.choice, -DwarningAsError=true) and paired with an AST-audited pure-$\mathbb{Q}$ Python verifier emitting 16 SHA-256-indexed rational certificates (C01–C16) checked by 62 adversarial tests. Specifically, we prove and certify: (i) the exact share-budget pigeonhole ceilings $M \le 2k-1$ (Gaussian) and $M \le nk-1$ (n-setting steering/Bell), plus the necessary condition $n \ge 3$ for any network exceeding $2k-1$ neighbors; (ii) exact equal-share harmonic tight-frame saturation $s_j = k/(2k-1) = \frac{1}{2} + \frac{1}{2(2k-1)} > \frac{1}{2}$ across $k \in \{1,2,3,4,8\}$, together with a constructive exact-$\mathbb{Q}$ rank-2 orthogonal projection matrix $P = I_3 - \frac{1}{3}J_3$ realizing $k = 2, M = 3$; (iii) an exact squeezing-invariant symplectic Schur-complement factorization $\nu_{A|B} - 1 = (a-1)(1-2\eta)/(\eta(a-1)+1)$, proving constructively that Gaussian steering ($\nu_{A|B} < 1$) holds iff $\eta > 1/2$ for every finite squeezing $a > 1$ and vanishes identically at $\eta = 1/2$ even at $a = 1000$; and (iv) exact remote Wigner-origin certificates $\pi W_{\text{out}}(0) = 1 - 2\eta(1-\varepsilon)$ for heralded single-photon non-Gaussian states, pinning the exact tight-frame heralding-error tolerance to $\varepsilon < 1/(2k)$ ($\varepsilon = 1/4$ at $k = 2$).

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1 Introduction

Quantum steering [6, 2] enables one-sided device-independent quantum cryptography and remote preparation of non-Gaussian Wigner-negative states over optical quantum networks [4, 5]. While a single Gaussian mode ($k = 1$) obeys strict monogamy—at most one neighbor can steer it with Gaussian measurements [3]—the exact maximum number of neighbors capable of simultaneously steering a k-mode Gaussian node remained open until Chakrabarty [1] proved the sharp bound $2k-1$.

Chakrabarty's argument assigns each neighbor $j \in \{1, \dots, M\}$ a covariance-matrix share $s_j \ge 0$ summing to at most k regardless of squeezing, amplification, or noise; neighbor j steers the node by Gaussian measurements iff $s_j > 1/2$, and a harmonic tight-frame network saturates $M = 2k-1$. The identical limit $2k-1$ bounds how many neighbors can be made simultaneously Wigner-negative by a heralded non-Gaussian operation at the central node, whereas n-setting arbitrary measurements yield $M \le nk-1$, proving that surpassing $2k-1$ requires non-Gaussian measurements with at least three settings. Here we formalize the exact rational share-budget, tight-frame projector, symplectic Schur-complement, and remote Wigner-negativity algebra in Lean 4.36.0 and pure-$\mathbb{Q}$ certificates (GAUSS-STEER-MONO-CERT-1.0).

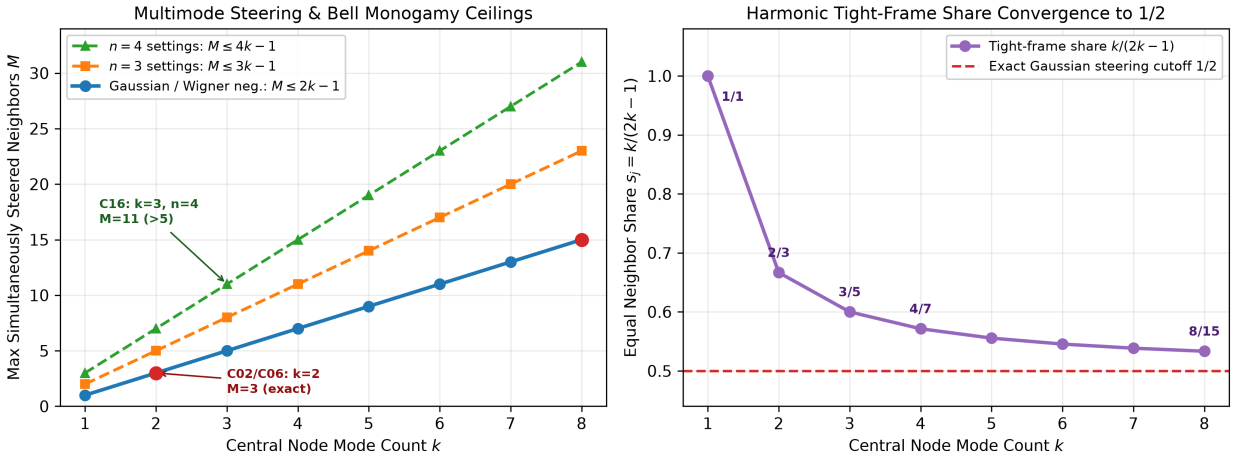

Figure 1: Left: Exact multimode monogamy ceilings $2k-1$ (Gaussian steering and remote Wigner negativity), $3k-1$ ($n=3$), and $4k-1$ ($n=4$), highlighting `C02`/`C06` and `C16`. Right: Harmonic tight-frame equal shares $k/(2k-1)$ converging strictly from above to the $1/2$ steering cutoff (`C01`–`C05`).

2 Share-Budget Monogamy and Harmonic Tight-Frame Saturation

Theorem 2.1 (Multimode Share-Budget Monogamy and $n \geq 3$ Separation; Lean `T01`–`T06`, `T16`–`T20`)**.** *Let $k, M \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 2}$, and let a k-mode Gaussian node distribute nonnegative rational shares $s_1, \ldots, s_M \in \mathbb{Q}_{\geq 0}$ satisfying the total budget $\sum_{j=1}^{M} s_j \leq k$.*

1. *If all M neighbors satisfy $s_j > 1/2$ (Gaussian steering), then $M/2 < \sum_j s_j \leq k$, hence $M \leq 2k-1$ (Lean `T01`; `C01`–`C05`).*
2. *For n-setting measurements ($n \in \mathbb{Z}_{\geq 2}$) requiring $s_j > 1/n$, $M/n < \sum_j s_j \leq k$, hence $M \leq nk-1$ (Lean `T02`; `C15`). Consequently, any configuration with $M > 2k-1$ and $M \leq nk-1$ strictly forces $n \geq 3$ (Lean `T17`; `C16` for $k=3, n=4, M=11$).*
3. *At $M = 2k-1$, the equal-share harmonic tight-frame distribution $s_j = k/(2k-1) = \frac{1}{2} + \frac{1}{2(2k-1)} > \frac{1}{2}$ saturates $\sum_{j=1}^{2k-1} s_j = k$ identically (Lean `T03`, `T20`), whereas $M = 2k$ forces equal share $1/2$, failing strict $> 1/2$ (Lean `T06`).*

For $k=2, M=3$, certificate `C06` and Lean `T07` verify the explicit rational rank-2 orthogonal projection matrix

$$P = I_3 - \frac{1}{3}J_3 = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \in \mathbb{Q}^{3\times 3}, \tag{1}$$

which satisfies $P^\top = P$, $P^2 = P$, $\mathrm{Tr}(P) = 2$, and constant diagonal share $P_{jj} = 2/3 > 1/2$ for all three neighbors (Figure 1). For general $k \geq 1$ and $M = 2k-1$, projecting onto any k rows of the $M \times M$ unitary DFT matrix F_M yields $P = F_S^\dagger F_S$ with $P^2 = P$, $\mathrm{rank}(P) = k$, and purely rational diagonal $P_{jj} = k/(2k-1) > 1/2$.

3 Squeezing-Invariant Symplectic Schur-Complement Factorization

Consider a two-mode squeezed vacuum (TMSV) state with rational squeezing variance $a = \cosh(2r) > 1$ and correlation $c = \sqrt{a^2-1}$ (in vacuum $= I$ units, $V + i\Omega \succeq 0$) coupled through a

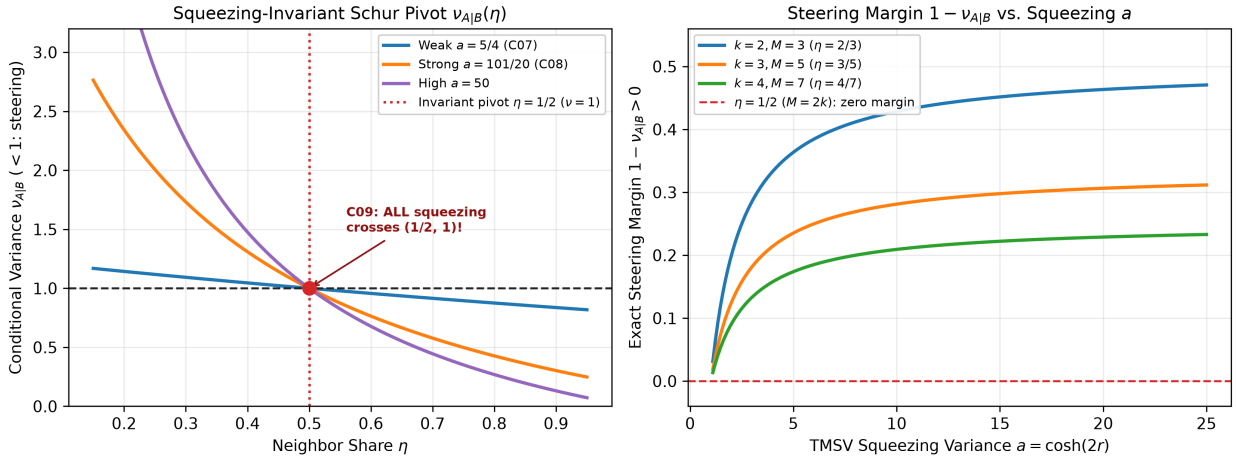

Figure 2: Left: Conditional Schur-complement variance $\nu_{A|B}(\eta)$ from (2) across weak ($a = 5/4$), strong ($a = 101/20$), and high ($a = 50$) squeezing, demonstrating the invariant pivot $(\eta, \nu) = (1/2, 1)$ (C07–C09). Right: Strict steering margin $1 - \nu_{A|B} > 0$ for tight-frame shares $k/(2k-1)$ versus zero margin at $\eta = 1/2$.

beam-splitter fan-out of share $\eta \in (0, 1]$ to neighbor B. The reduced covariance blocks are $V_A = aI_2$, $V_B = (\eta(a-1)+1)I_2$, and $C = \sqrt{\eta}cZ_2$, so the Schur complement $V_{A|B} = V_A - CV_B^{-1}C^\top = \nu_{A|B}I_2$ is purely rational in (a, η) with zero square roots.

Theorem 3.1 (Squeezing-Invariant Schur Factorization; Lean T08–T11, C07–C10)**.** *For every $a \in \mathbb{Q}_{>1}$ and $\eta \in \mathbb{Q} \cap (0, 1]$, the conditional variance $\nu_{A|B} := a - \frac{\eta(a^2-1)}{\eta(a-1)+1}$ satisfies the exact rational identity*

$$\nu_{A|B} - 1 = \frac{(a-1)(1-2\eta)}{\eta(a-1)+1}. \tag{2}$$

Consequently, Gaussian steering ($\nu_{A|B} < 1$) holds if and only if $\eta > 1/2$, completely independent of the squeezing amplitude $a > 1$.

Proof. Writing $a = A/D$ ($A > D > 0$) and $\eta = p/q$ ($0 < p \le q$), exact polynomial algebra (Lean T08) gives $(A-D)(p(A-D)+qD) - p(A^2-D^2) = (A-D)(q-2p)D$. Since $A - D > 0$, $D > 0$, and $p(A-D)+qD > 0$, $\nu_{A|B} - 1 < 0 \iff q < 2p \iff \eta > 1/2$ (Lean T09). □

Certificates C07–C10 verify this squeezing invariance explicitly (Figure 2): even weak squeezing $a = 5/4$ achieves steering at the $k = 2$ tight-frame share $\eta = 2/3$ ($\nu_{A|B} = 13/14 < 1$, C07, T10), whereas extreme squeezing $a = 1000$ at the $M = 2k$ boundary $\eta = 1/2$ yields $\nu_{A|B} = 1$ identically (C09, T11), blocking steering completely.

4 Exact Remote Wigner-Negativity Thresholds and Loss Tolerance

When a heralded single-photon Fock state $\rho_{\text{prep}} = (1-\varepsilon)|1\rangle\langle 1| + \varepsilon|0\rangle\langle 0|$ (preparation vacuum error $\varepsilon \in [0, 1)$) is remotely routed with effective transmissivity share $\eta \in (0, 1]$, the attenuated output state is $\rho_{\text{out}} = \eta(1-\varepsilon)|1\rangle\langle 1| + (1-\eta(1-\varepsilon))|0\rangle\langle 0|$. Switching from Section 3's vacuum-unit covariance scaling ($V_{\text{vac}} = I_2$, $\hbar = 2$) to the standard $\hbar = 1$ phase-space Wigner normalization ($\pi W_\rho(0) = \text{Tr}(\Pi\rho) \in [-1, 1]$, which is invariant under quadrature rescaling), the Fock-state origins satisfy $\pi W_{|0\rangle}(0) = +1$ and $\pi W_{|1\rangle}(0) = -1$, giving the exact rational origin value:

Theorem 4.1 (Remote Wigner-Negativity Cutoff and Heralding-Error Tolerance; Lean T12–T15, C11–C14)**.** *For $\eta \in (0, 1]$ and $\varepsilon \in [0, 1)$,*

$$\pi W_{\text{out}}(0) = 1 - 2\eta(1-\varepsilon) < 0 \iff \eta(1-\varepsilon) > \frac{1}{2}. \tag{3}$$

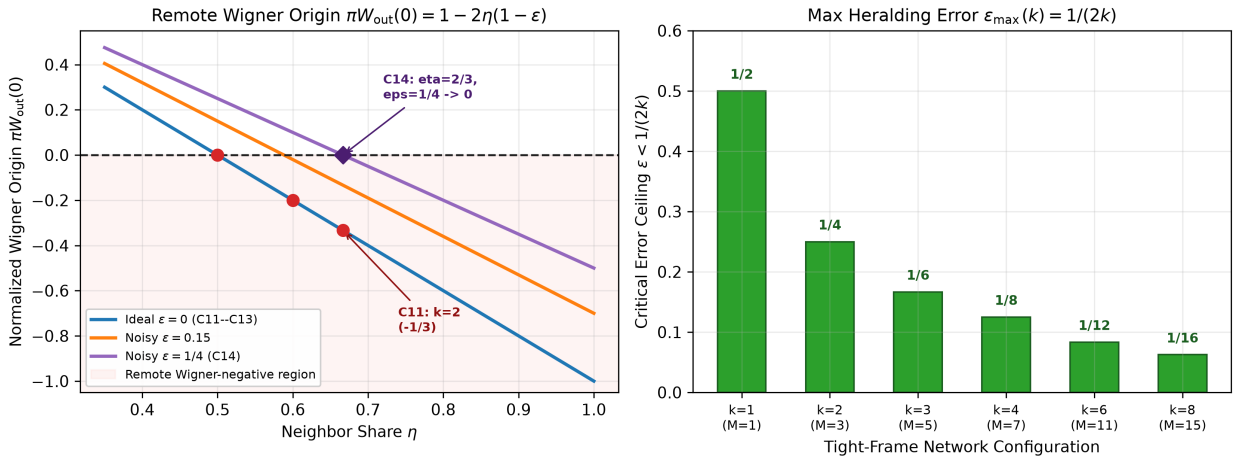

Figure 3: Left: Normalized remote Wigner origin $\pi W_{\text{out}}(0) = 1-2\eta(1-\varepsilon)$ marking ideal tight-frame certificates `C11`–`C13` and the exact noisy cutoff `C14` ($\eta = 2/3, \varepsilon = 1/4 \Rightarrow W(0) = 0$). Right: Exact critical heralding-error ceiling $\varepsilon_{\max}(k) = 1/(2k)$ across tight-frame network sizes.

In the ideal limit ($\varepsilon = 0$), remote Wigner negativity requires $\eta > 1/2$, reproducing the identical $2k-1$ monogamy bound (`C11`: $-1/3$ at $k = 2$; `C12`: $-1/5$ at $k = 3$; `C13`: 0 at $\eta = 1/2$). At the k-mode tight-frame share $\eta = k/(2k-1)$, remote Wigner negativity survives if and only if $\varepsilon < 1/(2k)$ (`C14`, `T14`: exact cutoff $\varepsilon = 1/4$ at $k = 2$, $\eta = 2/3$; Figure 3).

5 Certificate Suite, Scope, and Verification Boundary

Table 1 summarizes all 16 SHA-256-indexed rational certificates (`C01`–`C16`) produced by `GAUSS-STEER-MONO-CERT-1.0` and checked by 62 adversarial pytest checks and 20 constructive Lean 4.36.0 theorems.

Verification Boundary and Attribution.

- **Machine-checked in Lean 4.36.0 (`T01`–`T20`) and exact $\mathbb{Q}$ (`C01`–`C16`):** All pigeonhole share-budget bounds ($2k-1$ and $nk-1$), the $n \geq 3$ necessity theorem, equal-share saturation, exact idempotency of $P = I_3 - J_3/3$, the squeezing-invariant Schur-complement factorization (2), and the ideal/noisy remote Wigner-origin thresholds. Every Lean theorem depends solely on `[propext, Quot.sound]` or zero axioms (zero `Classical.choice`, zero `sorry`).
- **Imported from cited literature (not re-proved inside Lean):** Chakrabarty's general multimode covariance-matrix construction of neighbor shares proving $\sum_{j=1}^{M} s_j \leq k$ for arbitrary Gaussian states with arbitrary squeezing, amplification, and noise, and the multimode heralded non-Gaussian remote Wigner-negativity theorem in [1].

Reproducibility, Archived Artifact Hashes, and Disclosures. All verification artifacts are archived directly inside this AIRR record at `https://airr.science/papers/ARR-2026-4A8FSJ64DF9SNABH/` under MIT (manuscript CC-BY-4.0):

- `src/lean/GaussianSteeringMonogamyCert.lean` (SHA-256 `08af42d57b33acdc51934ebfcc774a7952898`
  20 constructive Lean 4.36.0 theorems covering share-budget ceilings (`T01`–`T02`), tight-frame saturation and $M = 2k$ failure (`T03`–`T06`, `T20`), exact rank-2 projector $P = I_3 - J_3/3$ (`T07`), squeezing-invariant Schur factorization and benchmarks (`T08`–`T11`), remote Wigner negativity and $\varepsilon < 1/(2k)$ cutoff (`T12`–`T15`), and $n \geq 3$ separation (`T16`–`T19`).

Table 1: Complete summary of all 16 exact-$\mathbb{Q}$ certificates (C01–C16) and matching Lean 4.36.0 theorems.

ID	Regime / Network	Exact Rational / Certified Output	Lean Ref.
C01	$k = 1$, Gaussian ($n = 2$)	Max $M = 1$, share $1/1 > 1/2$, margin $+1/2$	T01, T05
C02	$k = 2$, Gaussian ($n = 2$)	Max $M = 3$, share $2/3 > 1/2$, margin $+1/6$	T01, T03
C03	$k = 3$, Gaussian ($n = 2$)	Max $M = 5$, share $3/5 > 1/2$, margin $+1/10$	T01, T05
C04	$k = 4$, Gaussian ($n = 2$)	Max $M = 7$, share $4/7 > 1/2$, margin $+1/14$	T01, T05
C05	$k = 8$, Gaussian ($n = 2$)	Max $M = 15$, share $8/15 > 1/2$, margin $+1/30$	T01, T05
C06	Exact-$\mathbb{Q}$ projector ($k = 2, M = 3$)	$P = I_3 - J_3/3$: $P^2 = P$, $\mathrm{Tr} = 2$, $P_{jj} = 2/3$	T07
C07	Weak TMSV $a = 5/4, \eta = 2/3$	Exact Schur $\nu_{A\|B} = 13/14 < 1$ (steering)	T08--T10
C08	Strong TMSV $a = 101/20, \eta = 3/5$	Exact Schur $\nu_{A\|B} = 262/343 < 1$ (steering)	T08, T09
C09	Extreme TMSV $a = 1000, \eta = 1/2$	Exact Schur $\nu_{A\|B} = 1$ identically (no steering)	T08, T11
C10	$k = 4$ tight frame $a = 17/8, \eta = 4/7$	Exact Schur $\nu_{A\|B} = 83/92 < 1$ (7 neighbors)	T08, T09
C11	Ideal heralded $\|1\rangle$, $\eta = 2/3$	Exact $\pi W_{\mathrm{out}}(0) = -1/3 < 0$	T12, T13
C12	Ideal heralded $\|1\rangle$, $\eta = 3/5$	Exact $\pi W_{\mathrm{out}}(0) = -1/5 < 0$	T12, T13
C13	Ideal heralded $\|1\rangle$, $\eta = 1/2$	Exact $\pi W_{\mathrm{out}}(0) = 0$ (negativity vanishes)	T12, T13
C14	Noisy heralded $\varepsilon = 1/4, \eta = 2/3$	Exact $\pi W_{\mathrm{out}}(0) = 0$ (tolerance cutoff $1/(2k)$)	T14, T15
C15	$k = 2$, $n = 3$ settings	Max $M = 5 > 3$, share $2/5 > 1/3$	T02, T16
C16	$k = 3, n = 4, M = 11$ separation	$5 < 11 \leq 11$: requires non-Gaussian $n \geq 3$	T17--T19

- `src/python/gaussian_steering_monogamy_verifier.py` (SHA-256 `1e58a98e13ce4847c845c7be2b38dcb`
and `data/certificates/MANIFEST.json` (SHA-256 `eea871b23dac254188e07bcddd893b26887c5a3f872fe`
indexing all 16 JSON certificates C01–C16.

AI-assisted research tooling (Google Antigravity) supported drafting and formalization under human authorship and oversight by Lluis Eriksson; every formal claim is independently checked by the Lean 4.36.0 kernel and zero-float pytest suite.

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