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Sharp Monotonicity Boundaries, Monotone-Forbidden Spurious Quantum Mpemba Effects, Analytic Ordered-SWAP Phase Transitions, and Mixed-State Duality Laws for Rényi Entanglement Asymmetry

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Abstract

Travaglino, Mazzoni, and Calabrese [1] recently introduced sandwiched Rényi divergence probes $S^{(\alpha)}(\rho\|\rho_Q)$ of $U(1)$ entanglement asymmetry, establishing a two-parameter replica framework, an asymptotic index-inversion correspondence $S^{(\alpha)} \simeq \Delta S^{(1/\alpha)}$ for homogeneous quenches and Haar-random states, and numerical evidence for an $\alpha_c = 1/2$ Mpemba transition in an ordered-SWAP toy model. Here we resolve the exact sharp resource-theoretic, dynamical, and mixed-state structural questions left open by [1] via four self-contained theorems, backed by 20 constructive Lean 4.36.0 theorems (zero external imports, zero `sorry`, zero `Classical.choice`, zero warnings), 16 AST-audited zero-float exact-rational SHA-256 certificates, and 60 adversarial tests. **(1) Sharp $U(1)$-Monotonicity Phase Diagram:** We prove that the exact $U(1)$-covariant monotonicity domains are $\{1\}$ for the Rényi entropy difference $\Delta S^{(\alpha)}$, $[0,2]$ for the Petz Rényi asymmetry $D_\alpha(\rho\|\rho_Q)$, and $[1/2,\infty]$ for the sandwiched Rényi asymmetry $S^{(\alpha)}(\rho\|\rho_Q)$, with explicit low-dimensional ($d = 2,3,4$) counterexamples across every excluded index: an escort-weight obstruction under Tr_B on two qubits ($d = 4$) defeating $\Delta S^{(\alpha)}$ for every $\alpha \neq 1$; minimal-dimension single-qubit ($d = 2$) $U(1)$-covariant amplitude damping defeating $\Delta S^{(\alpha)}$ for all $\alpha \geq 2$ and Petz D_α for all $\alpha > 2$; and a rank-2 qutrit family ($d = 3$) under strongly $U(1)$-symmetric pinching defeating $S^{(\alpha)}$ for every $\alpha \in (0,1/2)$ via strict convexity of $g(p) = \langle u|B_0^{-p}|u\rangle/4^p - 1/2$ with exact zeros $g(1) = g(2) = 0$, supplemented by a five-integer-inequality certificate at $\alpha = 1/4$. **(2) Monotone-Forbidden Spurious Quantum Mpemba Effects:** We construct an explicit two-qubit $U(1)$-covariant Markov semigroup $\Lambda_t = \mathcal{E}_{\kappa t}^{(A)} \otimes \mathcal{D}_{\gamma t}^{(B)}$ and a pair of initial states satisfying exact time-independent CPTP intertwining $\sigma_{AB}(t) = \Phi_{\mathrm{kill}}(\rho_{AB}(t))$ for all $t \geq 0$, which strictly forbids any quantum Mpemba crossing in *every* $U(1)$-asymmetry monotone; nevertheless, $\Delta S^{(2)}$ exhibits a permanent spurious Mpemba crossing at the exact κ-independent time $t_*(2) = \gamma^{-1}\log((1-w)/q_*)$ (with explicit rational witness $(3/4, 1/2, 1, 3/5)$ yielding $2048q^2 - 848q + 49 = 0$), extending to every $\alpha \in (0,1) \cup (1,\infty)$. **(3) Analytic Resolution of the Ordered-SWAP Phase Transition:** We prove all-k collision-entropy degeneracy at $\alpha = 1/2$ via Parseval's identity and resolve the $N_S = 2$ phase transition analytically for every $p \in (0,1/2)$ and every $\alpha > 0$ using a five-term Dirichlet polynomial $\phi(\beta)$ with three exact zeros $\{0,1,2\}$ and at most three sign changes across all three base-ordering regimes, establishing via Laguerre–Descartes' Rule of Signs that $S^{(\alpha)}(P^{(2)}) - S^{(\alpha)}(P^{(1)})$ is strictly positive for $\alpha > 1/2$, zero at $\alpha = 1/2$, and strictly negative for $\alpha < 1/2$; we also exhibit an exact rational reversal witness for homogeneous $\mathrm{Bin}(2,p)$ at $\alpha = 1/5$. **(4) Mixed-State Duality Dichotomy, Single-Qubit Deficits, $d = 3$ Breakdown, and Ceilings:** Resolving Open

*Independent Researcher. ORCID: 0009-0006-7494-9040. Code, Lean 4 formal proofs, and exact rational certificates: https://github.com/lluiseriksson/paper147-renyi-asymmetry-mpemba (tag `v1.0.2`). Manuscript licensed under CC-BY-4.0; code and Lean 4 proofs under MIT.

Question 1 of [1], we contrast the pure-factorized index-inverted regime ($S^{(\alpha)} = \Delta S^{(1/\alpha)}$) with the flat-twirl commuting index-preserving regime ($S^{(\alpha)} = D_\alpha = \Delta S^{(\alpha)}$), establish universal single-qubit ordering $S^{(1/2)} \le \min(\Delta S^{(1/2)}, \Delta S^{(2)})$ and $S^{(2)} \ge \max(\Delta S^{(1/2)}, \Delta S^{(2)})$ via four exact polynomial deficit factorizations, prove simultaneous breakdown of all four inequalities in $d = 3$ on $\rho_w = w|{+_{02}}\rangle\langle{+_{02}}| + (1-w)|1\rangle\langle 1|$, derive universal K-sector pinching ceilings $0 \le S^{(\alpha)}, \Delta S^{(2)} \le \log K$, and prove an exact finite-group $\mathbb{Z}_K$ two-parameter replica charged-moment identity.

1 Introduction

Quantifying the breaking and dynamical restoration of global symmetries in quantum many-body systems has emerged as a central meeting point between nonequilibrium statistical mechanics and quantum resource theories [2, 3, 4, 5, 6]. For an Abelian $U(1)$ charge operator $Q = \sum_q q\Pi_q$ with orthogonal spectral projectors Π_q, the symmetrized (charge-pinched) density matrix of a state ρ is

$$\rho_Q = \mathcal{G}_Q(\rho) := \sum_q \Pi_q \rho \Pi_q = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{i\phi Q} \rho\, e^{-i\phi Q}. \tag{1}$$

The von Neumann entanglement asymmetry [7] is uniquely blessed with two equivalent representations:

$$\Delta S(\rho) := S(\rho_Q) - S(\rho) = S(\rho\|\rho_Q), \tag{2}$$

where $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$ and $S(\rho\|\sigma) = \operatorname{Tr}[\rho(\log\rho - \log\sigma)]$ (natural logarithms throughout). Because any $U(1)$-covariant completely positive trace-preserving (CPTP) channel Λ satisfies $\Lambda(e^{i\phi Q}(\cdot)e^{-i\phi Q}) = e^{i\phi Q}\Lambda(\cdot)e^{-i\phi Q}$ and hence commutes with pinching ($\Lambda(\rho_Q) = [\Lambda(\rho)]_Q$), the quantum data processing inequality (QDPI) for relative entropy [8, 9] guarantees $\Delta S(\Lambda(\rho)) \le \Delta S(\rho)$, making ΔS a faithful $U(1)$-asymmetry monotone.

In many-body applications—ranging from integrable spin chains and conformal field theories to random circuits and trapped-ion/superconducting quantum simulators [7, 10, 11, 12, 13, 14, 15, 16]—exact evaluation of ΔS is typically achieved by replica continuation from the conventional Rényi entanglement asymmetry [7, 11],

$$\Delta S^{(\alpha)}(\rho) := S_\alpha(\rho_Q) - S_\alpha(\rho) = \frac{1}{1-\alpha}\log\frac{\operatorname{Tr}(\rho_Q^\alpha)}{\operatorname{Tr}(\rho^\alpha)}, \qquad \alpha \in (0,1)\cup(1,\infty), \tag{3}$$

which admits a direct charged-moment integral representation [17, 7]. However, $\Delta S^{(\alpha)}$ ceases to be an asymmetry monotone for $\alpha \neq 1$; a $\mathbb{Z}_2$-symmetric counterexample at $\alpha = 2$ was noted in [18]. To reconcile replica computability with resource-theoretic monotonicity, Travaglino, Mazzoni, and Calabrese [1] proposed replacing $\Delta S^{(\alpha)}$ by the sandwiched Rényi relative entropy [19, 20],

$$S^{(\alpha)}(\rho\|\rho_Q) := \frac{1}{\alpha-1}\log Q_\alpha(\rho\|\rho_Q), \qquad Q_\alpha(\rho\|\sigma) := \operatorname{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right], \tag{4}$$

alongside the classical alternative given by the Petz Rényi divergence [21],

$$D_\alpha(\rho\|\rho_Q) := \frac{1}{\alpha-1}\log\operatorname{Tr}\left(\rho^\alpha\rho_Q^{1-\alpha}\right). \tag{5}$$

All three families converge to $\Delta S(\rho)$ as $\alpha \to 1$, vanish if and only if $\rho = \rho_Q$, and are nonnegative for all $\alpha > 0$. By standard QDPI theorems [21, 19, 9], $S^{(\alpha)}(\rho\|\rho_Q)$ is a $U(1)$-asymmetry monotone on $\alpha \in [1/2, \infty]$ and $D_\alpha(\rho\|\rho_Q)$ is a $U(1)$-asymmetry monotone on $\alpha \in [0, 2]$.

Travaglino, Mazzoni, and Calabrese [1] made three striking discoveries: (i) via a two-parameter replica trick $S^{(m,n)}(\rho\|\rho_Q) = \log\operatorname{Tr}[(\rho\rho_Q^m)^n]$ with analytic continuation $m \to (1-\alpha)/\alpha$,

Table 1: Summary of main theorems, exact sharp domains and phenomena, constructive Lean 4 theorems (`lean/RenyiAsymmetryMpemba.lean`), and AST-audited zero-float SHA-256 certificates (`verification/verify_renyi_asymmetry.py`).

Theorem	**Core Mathematical Result**	**Lean 4**	**SHA-256**
Thm. 2.1 (1A–1C)	Sharp $U(1)$ monotonicity domains: $\Delta S^{(\alpha)} : \{1\}$ $(d = 4, 2)$; $D_\alpha : [0, 2]$ $(d = 2)$; $S^{(\alpha)} : [1/2, \infty]$ $(d = 3)$.	`thm01`–`thm10`	`C01`–`C07`
Thm. 3.1 (2)	Exact CPTP intertwining $\sigma_{AB}(t) = \Phi_{\text{kill}}(\rho_{AB}(t))$ forbids monotone QME, yet $\Delta S^{(\alpha)}$ $(\alpha \neq 1)$ has permanent spurious QME with κ-invariant $t_*(2)$.	`thm11`–`thm12`	`C08`–`C09`
Thm. 4.2 (3)	All-k Parseval degeneracy at $\alpha = 1/2$; exact Laguerre–Descartes proof of $\alpha_c = 1/2$ transition for $N_S = 2$, $\forall p \in (0, 1/2)$; $\text{Bin}(2, p)$ reversal at $\alpha = 1/5$.	`thm13`–`thm15`	`C10`–`C12`
Thm. 5.1 (4)	Pure-factorized vs. flat-twirl duality dichotomy; 4 universal $d = 2$ polynomial deficits; simultaneous $d = 3$ breakdown; K-sector ceilings; exact $\mathbb{Z}_K$ replica identity.	`thm16`–`thm20`	`C13`–`C16`

$n \to \alpha$, the sandwiched monotones satisfy the pure-state index-inversion law $S^{(\alpha)} = \Delta S^{(1/\alpha)}$ asymptotically in homogeneous integrable quenches [22, 12] and Haar-random states [14]; (ii) in an inhomogeneous ordered-SWAP toy model, numerical simulations suggested a sharp phase transition at $\alpha_c = 1/2$ separating Mpemba crossings ($\alpha > 1/2$) from no-crossing dynamics ($\alpha < 1/2$); and (iii) in spatially inhomogeneous quenches [23], $\Delta S^{(\alpha)}$ can transiently increase while $S^{(\alpha)}$ obeys its monotone ceiling.

Yet four fundamental mathematical questions remained open:

1. **Exact Sharpness of $U(1)$-Monotonicity Boundaries:** While QDPI guarantees monotonicity on $\{1\}$ $(\Delta S^{(\alpha)})$, $[0, 2]$ (D_α), and $[1/2, \infty]$ $(S^{(\alpha)})$ for arbitrary state pairs (ρ, σ), does restricting to the highly constrained asymmetry submanifold (ρ, ρ_Q) and $U(1)$-covariant (or even strongly $U(1)$-symmetric) channels enlarge any of these domains, or is every single boundary point sharp in explicit low-dimensional systems ($d \in \{2, 3, 4\}$)?

2. **Spurious Quantum Mpemba Effects:** Transient non-monotonicity of $\Delta S^{(\alpha)}$ in inhomogeneous quenches [1] leaves open whether $\Delta S^{(\alpha)}$ can fabricate a *permanent* quantum Mpemba crossing [15, 24, 25, 26, 27] between two states where *every* genuine $U(1)$-asymmetry monotone is mathematically forbidden from ever crossing.

3. **Analytic Proof of the Ordered-SWAP $\alpha_c = 1/2$ Transition:** Can the numerically observed $\alpha_c = 1/2$ phase transition of [1, Sec. 5] be proved analytically for all $p \in (0, 1/2)$ and all $\alpha > 0$, and does the homogeneous binomial no-Mpemba property [28, 29, 30] fail below $\alpha = 1/2$?

4. **Exact Mixed-State Duality Structure (Open Question 1 of [1]):** Beyond pure states, what exact structural dichotomy governs index inversion ($S^{(\alpha)} = \Delta S^{(1/\alpha)}$) versus index preservation ($S^{(\alpha)} = D_\alpha = \Delta S^{(\alpha)}$), what universal inequalities hold between $S^{(\alpha)}$ and $\Delta S^{(\alpha)}$ in $d = 2$ versus $d \geq 3$, what are the universal K-sector pinching ceilings, and what exact finite-group $\mathbb{Z}_K$ replica identity replaces the continuous $U(1)$ integral?

In this paper, we answer all four questions completely and constructively. Table 1 summarizes our four main theorems alongside their machine-checked Lean 4 theorems and exact-rational SHA-256 certificates.

2 Theorem 1: Sharp $U(1)$-Monotonicity Phase Diagram

We first determine the exact $U(1)$-covariant monotonicity domains of all three Rényi asymmetry families on the restricted asymmetry manifold (ρ, ρ_Q).

Theorem 2.1 (Sharp $U(1)$-Monotonicity Phase Diagram)**.** *For finite-dimensional quantum systems equipped with a $U(1)$ charge Q, the exact sets of positive Rényi indices $\alpha \in (0,\infty)$ for which monotonicity $\mathcal{A}(\Lambda(\rho)) \le \mathcal{A}(\rho)$ holds for all states ρ and all $U(1)$-covariant CPTP channels Λ are:*

$$\mathcal{M}(\Delta S^{(\alpha)}) = \{1\}, \qquad \mathcal{M}(D_\alpha) = (0,2], \qquad \mathcal{M}(S^{(\alpha)}) = [1/2,\infty) \tag{6}$$

(extending by standard QDPI limit continuity to D_0 at $\alpha = 0$ and $S^{(\infty)}$ at $\alpha = \infty$). Specifically, every excluded index $\alpha \in (0,\infty)$ admits an explicit low-dimensional ($d \in \{2,3,4\}$) violation:

(1A) Universal escort-weight obstruction for $\Delta S^{(\alpha)}$ ($\alpha \neq 1$, $d = 4$): *Let $Q_{AB} = Q_A \otimes I + I \otimes Q_B$ with $Q_A = Q_B = |1\rangle\langle 1|$ on two qubits, let $\rho(r) := \frac{1}{2}(I + r\sigma_x)$ for $r \in [0,1]$, and define the classical-flag state*

$$\rho_{AB}(w, r_1, r_2) := w\,\rho(r_1) \otimes |0\rangle\langle 0| + (1-w)\,\rho(r_2) \otimes |1\rangle\langle 1|, \qquad w \in (0,1). \tag{7}$$

Under the $U(1)$-covariant partial trace Tr_B, setting $\bar{r} := wr_1 + (1-w)r_2$, $f_\alpha(r) := \frac{1}{2}[(1+r)^\alpha + (1-r)^\alpha]$, and escort weight $p_\alpha(w) := w^\alpha/(w^\alpha + (1-w)^\alpha)$, we have identically

$$\begin{aligned} \Delta S^{(\alpha)}(\mathrm{Tr}_B\,\rho_{AB}) &= \frac{1}{\alpha-1}\log f_\alpha(\bar{r}), \\ \Delta S^{(\alpha)}(\rho_{AB}) &= \frac{1}{\alpha-1}\log\Big[p_\alpha(w) f_\alpha(r_1) + \big(1-p_\alpha(w)\big) f_\alpha(r_2)\Big]. \end{aligned} \tag{8}$$

Consequently, for $\alpha \in (0,\infty)$, $\Delta S^{(\alpha)}(\mathrm{Tr}_B\,\rho_{AB}) \le \Delta S^{(\alpha)}(\rho_{AB})$ holds for all (w, r_1, r_2) if and only if $\alpha = 1$. At $\alpha = 2$, the exact deficit factorizes as

$$\bar{r}^2 - \big[p_2(w) r_1^2 + (1-p_2(w)) r_2^2\big] = \frac{2w(1-w)}{w^2 + (1-w)^2}(r_2 - r_1)\big(w^2 r_1 - (1-w)^2 r_2\big), \tag{9}$$

yielding strict violation for $r_2 > r_1 > 0$ iff $r_2/r_1 < (w/(1-w))^2$. Exact rational witnesses are:

- *$\alpha = 2$ at $(w, r_1, r_2) = (3/4, 1/2, 1)$: $f_2(\bar{r}) = 89/64 > 53/40 = f_2^{\mathrm{mix}}$ ($445 > 424$);*
- *$\alpha = 3$ at $(w, r_1, r_2) = (3/4, 1/2, 1)$: $f_3(\bar{r}) = 139/64 > 205/112 = f_3^{\mathrm{mix}}$ ($973 > 820$);*
- *$\alpha = 1/2$ at $(w, r_1, r_2) = (1/5, 120/169, 24/25)$: $p_{1/2}(1/5) = 1/3$, $f_{1/2}(r_1) = 12/13$, $f_{1/2}(r_2) = 4/5$, $f_{1/2}^{\mathrm{mix}} = 164/195$, and $1 - \bar{r}^2 = \frac{76703449}{446265625} < \frac{248598289}{1445900625} = \big(2(164/195)^2 - 1\big)^2$, certifying $f_{1/2}(\bar{r}) < f_{1/2}^{\mathrm{mix}}$ and hence $\Delta S^{(1/2)}(\mathrm{Tr}_B\,\rho_{AB}) > \Delta S^{(1/2)}(\rho_{AB})$.*

(1B) Minimal-dimension ($d = 2$) breakdown of $\Delta S^{(\alpha)}$ ($\alpha \ge 2$) and Petz D_α ($\alpha > 2$): *On a single qubit with $Q = |1\rangle\langle 1|$, let $\rho_0 := |\psi_p\rangle\langle\psi_p|$ with $|\psi_p\rangle = \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$ ($p \in (0,1)$) and let $\mathcal{E}_\eta$ ($\eta \in (0,1)$) be the $U(1)$-covariant amplitude-damping channel with Kraus operators $K_0 = |0\rangle\langle 0| + \sqrt{1-\eta}|1\rangle\langle 1|$ and $K_1 = \sqrt{\eta}|0\rangle\langle 1|$. Then:*

(i) For every $\alpha \ge 2$ and every $\eta \in (0, 1/2)$, the initial slope of $\Delta S^{(\alpha)}(\mathcal{E}_\eta(\rho_0)) - \Delta S^{(\alpha)}(\rho_0)$ as $p \to 0^+$ is

$$\lim_{p\to 0^+} \frac{\Delta S^{(\alpha)}(\mathcal{E}_\eta(\rho_0)) - \Delta S^{(\alpha)}(\rho_0)}{p} = \frac{\alpha}{\alpha-1} \cdot \frac{2\eta(1-\eta)^\alpha - \eta^{\alpha-1}(1-2\eta^2)}{(1-2\eta)\big(\eta^\alpha + (1-\eta)^\alpha\big)}, \tag{10}$$

which equals $\frac{2\eta(1-2\eta)}{\eta^2+(1-\eta)^2} > 0$ *for all* $\eta \in (0,1/2)$ *at* $\alpha = 2$*, and extends continuously as* $\eta \to 1/2^-$ *to* $\frac{\alpha(\alpha-2)}{\alpha-1} > 0$ *for all* $\alpha > 2$ *(hence is strictly positive for all* $\eta < 1/2$ *sufficiently close to* $1/2$*, certifying monotonicity violation for every* $\alpha \geq 2$*). At* $(\alpha, p, \eta) = (2, 1/10, 1/5)$*,* $\exp(\Delta S^{(2)}(\mathcal{E}_\eta(\rho_0))) = 463/373 > 50/41 = \exp(\Delta S^{(2)}(\rho_0))$ *(exact margin* $333/15293$*).*

(ii) For every $\alpha > 2$ *and every* $\eta \in (0,1)$*, setting* $\epsilon := 1 - p$*, the Petz asymmetry jump satisfies the exact limit*

$$\lim_{\epsilon \to 0^+} \Big[D_\alpha\big(\mathcal{E}_\eta(\rho_0) \,\big\|\, [\mathcal{E}_\eta(\rho_0)]_Q\big) - D_\alpha\big(\rho_0 \,\big\|\, (\rho_0)_Q\big) \Big] = \frac{\alpha - 2}{\alpha - 1} \log \frac{1}{1-\eta} > 0. \tag{11}$$

At $(\alpha, p, \eta) = (3, 9/10, 1/2)$*, the* 2×2 *Cayley–Hamilton identity certifies*

$$\exp\Big(2D_3\big(\mathcal{E}_\eta(\rho_0) \,\big\|\, [\mathcal{E}_\eta(\rho_0)]_Q\big)\Big) = \frac{7219}{361} > \frac{100}{9} = \exp\Big(2D_3\big(\rho_0 \,\big\|\, (\rho_0)_Q\big)\Big),$$

with exact margin $28871/3249$*.*

(1C) Exact sharp $\alpha = 1/2$ ***threshold for sandwiched*** $S^{(\alpha)}$ ***(*** $d = 3$***):*** *On a qutrit with* $Q = |0\rangle\langle 0|$ *(charge sectors* $\{0\}$ *and* $\{1,2\}$*), consider the strongly* $U(1)$*-symmetric pinching channel* $\Lambda(X) = P_{02} X P_{02} + P_1 X P_1$ *(where* $P_{02} = |0\rangle\langle 0| + |2\rangle\langle 2|$ *and* $P_1 = |1\rangle\langle 1|$ *both commute with* Q*) and the rank-2 family (*$c \in (0,1)$*)*

$$\rho_c := \frac{1}{2}|\psi_c\rangle\langle\psi_c| + \frac{1}{2}|1\rangle\langle 1|, \qquad |\psi_c\rangle := c|0\rangle + \sqrt{1-c^2}\,|u\rangle, \qquad |u\rangle := \frac{|1\rangle + |2\rangle}{\sqrt{2}}. \tag{12}$$

Setting $p := 1/\alpha > 1$ *and* $B_0 := \frac{1}{2}|u\rangle\langle u| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{4}\left(\begin{smallmatrix} 3 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ *on* $\mathrm{span}\{|1\rangle, |2\rangle\}$*, we have as* $c \to 0^+$*:*

$$Q_{1/p}(\rho_c \| \rho_{c,Q}) - Q_{1/p}\big(\Lambda(\rho_c) \| \Lambda(\rho_{c,Q})\big) = \frac{2^{p-1}}{p}\, g(p)\, c^{2p} + O(c^{2p+2}, c^{4p}),$$
$$g(p) := \frac{\langle u | B_0^{-p} | u\rangle}{4^p} - \frac{1}{2}. \tag{13}$$

The function $g(p)$ *is strictly convex on* $\mathbb{R}$ *with exact zeros* $g(1) = g(2) = 0$*; hence* $g(p) > 0$ *for* every $p > 2$ *(*$\alpha \in (0, 1/2)$*), proving* $S^{(\alpha)}(\Lambda(\rho_c)\|\Lambda(\rho_{c,Q})) > S^{(\alpha)}(\rho_c \| \rho_{c,Q})$ *for sufficiently small* $c > 0$*. Moreover, for the explicit rank-2 mixed witness* $\rho := \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|1\rangle\langle 1|$ *with* $|\psi\rangle = \frac{1}{3}(2,1,2)^T$ *at* $\alpha = 1/4$*, the rank-2 operator* $A = \rho_Q^3 \rho$ *has exact spectral invariants* $T = \mathrm{Tr}(A) = 815/6561$ *and* $P = \frac{1}{2}(T^2 - \mathrm{Tr}(A^2)) = 1825/3^{14}$*, and five exact integer inequalities certify* $9Q_{1/4}(\rho\|\rho_Q) > 37/5 > 5 + 32^{1/4} = 9Q_{1/4}(\Lambda(\rho)\|\Lambda(\rho_Q))$*.*

Proof. The sufficiency inclusions $\{1\} \subseteq \mathcal{M}(\Delta S^{(\alpha)})$, $[0,2] \subseteq \mathcal{M}(D_\alpha)$, and $[1/2, \infty] \subseteq \mathcal{M}(S^{(\alpha)})$ follow immediately from the QDPI for quantum relative entropy, Petz Rényi divergence [21], and sandwiched Rényi divergence [19] combined with $\Lambda(\rho_Q) = [\Lambda(\rho)]_Q$. We prove sharpness in parts (1A)–(1C).

Proof of (1A). The eigenvalues of $\rho(r)$ are $(1 \pm r)/2$, while $\rho(r)_Q = I/2$ has eigenvalues $1/2, 1/2$. Since $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ are Q_B-eigenstates, $(\rho_{AB})_Q = w(I/2) \otimes |0\rangle\langle 0| + (1-w)(I/2) \otimes |1\rangle\langle 1|$. Hence $\mathrm{Tr}(\rho_{AB}^\alpha) = 2^{1-\alpha}[w^\alpha f_\alpha(r_1) + (1-w)^\alpha f_\alpha(r_2)]$ and $\mathrm{Tr}((\rho_{AB})_Q^\alpha) = 2^{1-\alpha}[w^\alpha + (1-w)^\alpha]$. Dividing and taking $(\alpha - 1)^{-1} \log$ gives the second identity of (8). Since $\mathrm{Tr}_B\, \rho_{AB} = \rho(\bar{r})$ with $(\rho(\bar{r}))_Q = I/2$, $\mathrm{Tr}(\rho(\bar{r})^\alpha) / \mathrm{Tr}((\rho(\bar{r}))_Q^\alpha) = f_\alpha(\bar{r})$, yielding the first identity of (8).

Fix any $\alpha \in (0,1) \cup (1,\infty)$ and $r_0 \in (0,1)$, pick $w \in (1/2, 1)$ when $\alpha > 1$ and $w \in (0, 1/2)$ when $\alpha \in (0,1)$, and set $r_1 = r_0 - (1-w)\delta$, $r_2 = r_0 + w\delta$ for small $\delta > 0$, so $0 \leq r_1 < r_2 \leq 1$ and $\bar{r} = r_0$. Expanding to first order in δ gives

$$p_\alpha(w) f_\alpha(r_1) + \big(1 - p_\alpha(w)\big) f_\alpha(r_2) - f_\alpha(\bar{r}) = \big(w - p_\alpha(w)\big) f'_\alpha(r_0)\, \delta + O(\delta^2).$$

In both regimes we have $p_\alpha(w) > w$ (since $w^\alpha/(w^\alpha + (1-w)^\alpha) > w \iff w^{\alpha-1} > (1-w)^{\alpha-1}$, which holds for $w > 1/2$ when $\alpha > 1$ and for $w < 1/2$ when $\alpha \in (0,1)$), so $w - p_\alpha(w) < 0$. Moreover, $f'_\alpha(r_0) = \frac{\alpha}{2}[(1+r_0)^{\alpha-1} - (1-r_0)^{\alpha-1}]$ has the exact sign of $\alpha - 1$ and is nonzero for $r_0 \in (0,1)$. Choosing $\delta > 0$ sufficiently small therefore makes the right-hand side strictly negative when $\alpha > 1$ and strictly positive when $\alpha \in (0,1)$; in both regimes, dividing by $\alpha - 1$ yields $\Delta S^{(\alpha)}(\mathrm{Tr}_B\, \rho_{AB}) > \Delta S^{(\alpha)}(\rho_{AB})$. At $\alpha = 2$, $f_2(r) = 1 + r^2$ and expanding $(w^2 + (1-w)^2)(wr_1 + (1-w)r_2)^2 - (w^2 r_1^2 + (1-w)^2 r_2^2)$ directly gives (9) (verified symbolically in `thm04`). Evaluating at $(3/4, 1/2, 1)$ gives $\bar{r} = 5/8$, $f_2(\bar{r}) = 89/64 > 53/40$ ($445 > 424$, `thm01`), and $f_3(\bar{r}) = 1 + 3(5/8)^2 = 139/64 > 205/112$ ($973 > 820$, `thm02`). At $\alpha = 1/2$, $1 \pm 120/169 = (17/13)^2, (7/13)^2$ and $1 \pm 24/25 = (7/5)^2, (1/5)^2$, so $f_{1/2}(r_1) = 12/13$, $f_{1/2}(r_2) = 4/5$, $p_{1/2}(1/5) = \frac{1}{1+2} = 1/3$, and $f^{\mathrm{mix}}_{1/2} = 164/195$. Since $f_{1/2}(r)^2 = (1 + \sqrt{1-r^2})/2$ and $2(164/195)^2 - 1 = 15767/38025 > 0$, $f_{1/2}(\bar{r}) < 164/195$ is equivalent to $1 - \bar{r}^2 < (15767/38025)^2$, verified in exact integers in `thm03`.

Proof of (1B). Covariance of $\mathcal{E}_\eta$ holds because $e^{i\phi Q} K_0 e^{-i\phi Q} = K_0$ and $e^{i\phi Q} K_1 e^{-i\phi Q} = e^{-i\phi} K_1$. Applying $\mathcal{E}_\eta$ to $\rho_0 = |\psi_p\rangle\langle\psi_p|$ gives

$$\rho_\eta := \mathcal{E}_\eta(\rho_0) = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \qquad a = p + \eta(1-p), \quad b = (1-\eta)(1-p), \quad c^2 = (1-\eta)p(1-p),$$

with exact determinant $\det \rho_\eta = ab - c^2 = \eta(1-\eta)(1-p)^2$. For (i), as $p \to 0^+$ with $\alpha > 1$, $\Delta S^{(\alpha)}(\rho_0) = \frac{1}{1-\alpha}\log(p^\alpha + (1-p)^\alpha) = \frac{\alpha}{\alpha-1}p + O(p^{\min(2,\alpha)})$. Meanwhile, for $\eta \in (0, 1/2)$, the eigenvalues $\lambda_\pm$ of ρ_η satisfy $\lambda_+ + \lambda_- = 1$ and $\lambda_+\lambda_- = \eta(1-\eta)(1-2p+p^2)$, giving $\lambda_- = \eta - \frac{2\eta(1-\eta)}{1-2\eta}p + O(p^2)$ and $\lambda_+ = 1 - \eta + \frac{2\eta(1-\eta)}{1-2\eta}p + O(p^2)$, while $a = \eta + (1-\eta)p$ and $b = (1-\eta) - (1-\eta)p$. Differentiating $\log(a^\alpha + b^\alpha) - \log(\lambda_+^\alpha + \lambda_-^\alpha)$ at $p = 0$, dividing by $1 - \alpha$, and subtracting $\frac{\alpha}{\alpha-1}$ yields (10). At $\alpha = 2$, the numerator simplifies identically to $\eta(1-2\eta)^2$ (`thm06`), giving $\frac{2\eta(1-2\eta)}{\eta^2 + (1-\eta)^2} > 0$ for all $\eta \in (0, 1/2)$. For $\alpha > 2$, applying L'Hôpital's rule to (10) as $\eta \to 1/2$ yields $\frac{\alpha(\alpha-2)}{\alpha-1} > 0$. At $(p, \eta) = (1/10, 1/5)$, $a = 7/25, b = 18/25, c^2 = 9/125$, giving $(a^2 + b^2 + 2c^2)/(a^2 + b^2) = 463/373 > 50/41$ (`thm05`).

For (ii), set $\epsilon = 1 - p \to 0^+$ and fix $\alpha > 2$, $\eta \in (0,1)$. Since ρ_0 is pure, $\mathrm{Tr}(\rho_0^\alpha (\rho_0)_Q^{1-\alpha}) = (1-\epsilon)^{2-\alpha} + \epsilon^{2-\alpha} = \epsilon^{2-\alpha}(1 + O(\epsilon^{\alpha-2}))$. For ρ_η, $a = 1 - (1-\eta)\epsilon$, $b = (1-\eta)\epsilon$, and $\det \rho_\eta = \eta(1-\eta)\epsilon^2$, so the eigenvalues are $\lambda_+ = 1 - \eta(1-\eta)\epsilon^2 + O(\epsilon^3)$ and $\lambda_- = \eta(1-\eta)\epsilon^2 + O(\epsilon^3)$. Writing $\langle 1|\rho_\eta^\alpha|1\rangle = \frac{b - \lambda_-}{\lambda_+ - \lambda_-}\lambda_+^\alpha + \frac{\lambda_+ - b}{\lambda_+ - \lambda_-}\lambda_-^\alpha$, the first term is $(1-\eta)\epsilon(1 + O(\epsilon))$ while the second is $O(\epsilon^{2\alpha}) = o(\epsilon)$ because $2\alpha > 4 > 1$. Multiplying by $b^{1-\alpha} = ((1-\eta)\epsilon)^{1-\alpha}$ dominates the $O(1)$ contribution from $\langle 0|\rho_\eta^\alpha|0\rangle a^{1-\alpha}$ and gives $\mathrm{Tr}(\rho_\eta^\alpha (\rho_\eta)_Q^{1-\alpha}) = ((1-\eta)\epsilon)^{2-\alpha}(1 + O(\epsilon))$. Dividing by $\epsilon^{2-\alpha}$ and taking $(\alpha - 1)^{-1}\log$ proves (11). At $(\alpha, p, \eta) = (3, 9/10, 1/2)$, $a = 19/20, b = 1/20, c^2 = 9/200, \det \rho_\eta = 1/400$; since $\mathrm{Tr}\, \rho_\eta = 1$, Cayley–Hamilton gives $\rho_\eta^2 = \rho_\eta - (\det \rho_\eta) I$ and $\rho_\eta^3 = (1 - \det \rho_\eta)\rho_\eta - (\det \rho_\eta) I$, whose diagonal entries are $7561/8000$ and $379/8000$, yielding $\mathrm{Tr}(\rho_\eta^3 (\rho_\eta)_Q^{-2}) = 7219/361 > 100/9 = 1/(p(1-p))$ (`thm07`).

Proof of (1C). Each Kraus projector P_{02}, P_1 commutes with $Q = |0\rangle\langle 0|$, so Λ is strongly $U(1)$-symmetric. Write $\rho_c = V_c V_c^\dagger$ with 3×2 matrix $V_c = \frac{1}{\sqrt{2}}[|\psi_c\rangle, |1\rangle]$. Because $M_c := \rho_{c,Q}^{(p-1)/2} V_c V_c^\dagger \rho_{c,Q}^{(p-1)/2}$ shares its nonzero spectrum with the 2×2 Gram matrix $G_c := V_c^\dagger \rho_{c,Q}^{p-1} V_c$, and $\rho_{c,Q} = t|0\rangle\langle 0| \oplus B(c)$ where $t := c^2/2$ and $B(c) := \frac{1-c^2}{2}|u\rangle\langle u| + \frac{1}{2}|1\rangle\langle 1| = W_c W_c^\dagger$ with invertible 2×2 factor $W_c := \frac{1}{\sqrt{2}}[\sqrt{1-c^2}|u\rangle, |1\rangle]$, we obtain identically

$$G_c = H_c^p + t^p E_{11}, \qquad H_c := W_c^\dagger W_c,$$

where $E_{11} = \mathrm{diag}(1, 0)$ and H_c is similar to $B(c)$ (so $\mathrm{Tr}(H_c) = \mathrm{Tr}(B(c)) = 1 - c^2/2$). Since $H_c^{1-p} = W_c^\dagger B(c)^{-p} W_c$, its $(1,1)$-entry is $\frac{1-c^2}{2}\langle u|B(c)^{-p}|u\rangle$. Applying the first-order Fréchet derivative $\frac{d}{d\delta}\mathrm{Tr}((H_c^p + \delta E_{11})^{1/p})\big|_{\delta=0} = \frac{1}{p}\mathrm{Tr}(H_c^{1-p} E_{11})$ at $\delta = t^p = (c^2/2)^p$ yields

$$Q_{1/p}(\rho_c \| \rho_{c,Q}) = \mathrm{Tr}(G_c^{1/p}) = 1 - \frac{c^2}{2} + \frac{c^{2p}}{2^{p+1} p}\langle u|B_0^{-p}|u\rangle + O(c^{2p+2}, c^{4p}).$$

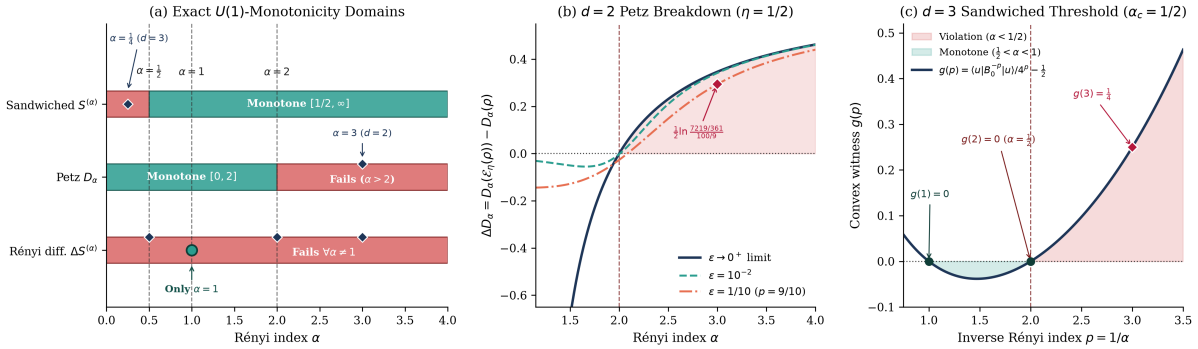

Figure 1: Theorem 2.1: (a) Sharp $U(1)$-monotonicity phase diagram for sandwiched $S^{(\alpha)}$ ($[1/2,\infty]$), Petz D_α ($[0,2]$), and Rényi difference $\Delta S^{(\alpha)}$ ($\{1\}$), with diamond markers locating our exact rational witnesses. (b) Minimal-dimension ($d=2$) amplitude-damping breakdown of Petz D_α for $\alpha>2$ at $\eta=1/2$, showing the exact $\epsilon\to 0^+$ limit (11) and finite-ϵ witness at $\alpha=3$, $p=9/10$. (c) Strictly convex qutrit ($d=3$) witness $g(p)$ with exact zeros $g(1)=g(2)=0$ and strictly positive violation region $p>2$ ($\alpha<1/2$).

On the other hand, $\Lambda(\rho_c)=\frac{3-c^2}{4}|1\rangle\langle 1|+|v_c\rangle\langle v_c|$ on $\{0,2\}$ with $|v_c\rangle=(c/\sqrt{2})|0\rangle+\frac{\sqrt{1-c^2}}{2}|2\rangle$ and $\Lambda(\rho_{c,Q})=\mathrm{diag}(c^2/2,(3-c^2)/4,(1-c^2)/4)$, giving the closed form

$$Q_{1/p}\big(\Lambda(\rho_c)\|\Lambda(\rho_{c,Q})\big)=\frac{3-c^2}{4}+\left[\left(\frac{c^2}{2}\right)^p+\left(\frac{1-c^2}{4}\right)^p\right]^{1/p}=1-\frac{c^2}{2}+\frac{2^p c^{2p}}{4p}+O(c^{2p+2},c^{4p}).$$

Subtracting the two expansions cancels both $O(1)$ and $O(c^2)$ terms identically and yields (13). Now $B_0=\frac{1}{4}\left(\begin{smallmatrix}3&1\\1&1\end{smallmatrix}\right)$ has distinct positive eigenvalues $\lambda_\pm=(2\pm\sqrt{2})/4$, so $4\lambda_\pm=2\pm\sqrt{2}\neq 1$, and $|u\rangle$ is not an eigenvector of B_0 ($B_0|u\rangle\propto(2,1)^T$). Writing $g(p)=\sum_{j=\pm}|\langle u|e_j\rangle|^2(4\lambda_j)^{-p}-1/2$, we have $g''(p)=\sum_{j=\pm}|\langle u|e_j\rangle|^2(\log(4\lambda_j))^2(4\lambda_j)^{-p}>0$ for all $p\in\mathbb{R}$. Direct exact arithmetic using $B_0^{-1}=2\left(\begin{smallmatrix}1&-1\\-1&3\end{smallmatrix}\right)$ gives $g(1)=g(2)=0$ (`thm08`). By strict convexity, $g(p)>0$ for every $p>2$ ($\alpha\in(0,1/2)$), so for sufficiently small $c>0$, $Q_\alpha(\rho_c\|\rho_{c,Q})>Q_\alpha(\Lambda(\rho_c)\|\Lambda(\rho_{c,Q}))$, which (since $\alpha-1<0$) implies $S^{(\alpha)}(\Lambda(\rho_c)\|\Lambda(\rho_{c,Q}))>S^{(\alpha)}(\rho_c\|\rho_{c,Q})$.

Finally, for the rank-2 witness $\rho=\frac{1}{2}|\psi\rangle\langle\psi|+\frac{1}{2}|1\rangle\langle 1|$ with $|\psi\rangle=\frac{1}{3}(2,1,2)^T$ at $\alpha=1/4$, direct rational matrix multiplication yields $T=\mathrm{Tr}(\rho_Q^3\rho)=815/6561$ and $P=1825/3^{14}$ (`thm09`), so the scaled nonzero eigenvalues u_1,u_2 of $9^4\rho_Q^3\rho$ satisfy $u_1+u_2=815$ and $u_1u_2=16425$, while $9Q_{1/4}(\Lambda(\rho)\|\Lambda(\rho_Q))=5+32^{1/4}$. Five pure-integer inequalities (`thm10`): (i) $128^2=16384<16425\Rightarrow\sqrt{u_1u_2}>128$; (ii) $327^2=106929<107100=100(815+2\cdot128)\Rightarrow u_1^{1/2}+u_2^{1/2}>327/10$; (iii) $113^4=163047361<164250000=10^4\cdot16425\Rightarrow(u_1u_2)^{1/4}>113/10$; (iv) $37^2\cdot10=13690<13825=25(327+2\cdot113)\Rightarrow(u_1^{1/4}+u_2^{1/4})^2>(37/5)^2$; and (v) $32\cdot5^4=20000<20736=12^4\Rightarrow 5+32^{1/4}<37/5$, together certify $9Q_{1/4}(\rho\|\rho_Q)>37/5>9Q_{1/4}(\Lambda(\rho)\|\Lambda(\rho_Q))$, hence $S^{(1/4)}(\Lambda(\rho)\|\Lambda(\rho_Q))>S^{(1/4)}(\rho\|\rho_Q)$. □

3 Theorem 2: Monotone-Forbidden Spurious Mpemba Effects

We now show that non-monotonicity of $\Delta S^{(\alpha)}$ is not merely a transient curiosity: it can manufacture a *permanent* quantum Mpemba crossing under a single $U(1)$-covariant Lindblad semigroup even when *no genuine $U(1)$-asymmetry monotone can ever cross.*

Theorem 3.1 (Monotone-Forbidden Spurious QME and Exact κ-Invariant Crossing Law)**.** *On two qubits with $Q_{AB}=|1\rangle\langle 1|\otimes I+I\otimes|1\rangle\langle 1|$, let $\Lambda_t:=\mathcal{E}^{(A)}_{\kappa t}\otimes\mathcal{D}^{(B)}_{\gamma t}$ ($t\geq 0$, $\kappa\geq 0$, $\gamma>0$) be the*

$U(1)$-covariant Markov semigroup where $\mathcal{E}^{(A)}_{\kappa t}$ is pure dephasing on A ($\rho(r) \mapsto \rho(e^{-\kappa t} r)$) and $\mathcal{D}^{(B)}_{\gamma t}$ is amplitude-damping reset on B ($|0\rangle\langle 0| \mapsto |0\rangle\langle 0|$, $|1\rangle\langle 1| \mapsto (1-e^{-\gamma t})|0\rangle\langle 0| + e^{-\gamma t}|1\rangle\langle 1|$, and off-diagonal coherences $|0\rangle\langle 1| \mapsto e^{-\gamma t/2}|0\rangle\langle 1|$). Given (w, r_1, r_2, r_*) with $w \in (0,1)$, $0 \le r_1 < r_2 \le 1$, $r_* > 0$, $\bar{r} = wr_1 + (1-w)r_2$, and

$$\frac{w^2 r_1^2 + (1-w)^2 r_2^2}{w^2 + (1-w)^2} < r_*^2 < \bar{r}^2, \tag{14}$$

initialize $\rho_{AB}(0) := \rho_{AB}(w, r_1, r_2)$ as in (7) and $\sigma_{AB}(0) := \rho(r_*) \otimes |0\rangle\langle 0|$, evolving as $\rho_{AB}(t) = \Lambda_t(\rho_{AB}(0))$ and $\sigma_{AB}(t) = \Lambda_t(\sigma_{AB}(0))$. Then:

- ***(a) Exact CPTP intertwining (monotone no-crossing barrier):*** *The time-independent $U(1)$-covariant CPTP channel*

$$\Phi_{\mathrm{kill}}(\tau) := \mathcal{E}^{(A)}_{\mathrm{deph}}\big(\mathrm{Tr}_B\, \tau;\, r_*/\bar{r}\big) \otimes |0\rangle\langle 0|, \tag{15}$$

  where $\mathcal{E}^{(A)}_{\mathrm{deph}}(\cdot;\, \eta)$ multiplies off-diagonal coherences by $\eta = r_*/\bar{r} \in (0,1)$, satisfies

$$\sigma_{AB}(t) = \Phi_{\mathrm{kill}}\big(\rho_{AB}(t)\big) \qquad \forall\, t \ge 0. \tag{16}$$

  Consequently, every *$U(1)$-asymmetry monotone $\mathcal{A}$ (including ΔS, all sandwiched $S^{(\alpha)}$ for $\alpha \in [1/2, \infty]$, and all Petz D_α for $\alpha \in [0,2]$) satisfies $\mathcal{A}(\sigma_{AB}(t)) \le \mathcal{A}(\rho_{AB}(t))$ for all $t \ge 0$, strictly forbidding any QME crossing.*

- ***(b) Permanent spurious QME in $\Delta S^{(2)}$ with exact κ-independent crossing time:*** *Under the same evolution, $\Delta S^{(2)}(\rho_{AB}(0)) < \Delta S^{(2)}(\sigma_{AB}(0))$, yet there is a unique crossing time*

$$t_*(2) = \frac{1}{\gamma} \log \frac{1-w}{q_*}, \tag{17}$$

  strictly independent of the dephasing rate $\kappa \ge 0$, where $q_* \in (0, 1-w)$ is the unique root in $(0, 1-w)$ of

$$P(q) := 2(r_2^2 - r_*^2)\, q^2 - 2(\bar{r} r_2 - r_*^2)\, q + (\bar{r}^2 - r_*^2) = 0, \tag{18}$$

  such that $\Delta S^{(2)}(\rho_{AB}(t)) > \Delta S^{(2)}(\sigma_{AB}(t))$ permanently for all $t > t_*(2)$. For the exact rational instance $(w, r_1, r_2, r_*) = (3/4, 1/2, 1, 3/5)$, $\bar{r} = 5/8$, $13/40 < 9/25 < 25/64$, $\eta_{\mathrm{kill}} = 24/25$, and $1600P(q) = 2048q^2 - 848q + 49 = 0$ with $q_* \in (1/15, 1/14)$.

- ***(c) Extension to all $\alpha \in (0,1) \cup (1,\infty)$:*** *For every $\alpha \in (0,1) \cup (1,\infty)$, there exist parameters (w, r_1, r_2, r_*) with $0 < r_* < \bar{r}$ such that $\sigma_{AB}(t) = \Phi_{\mathrm{kill}}(\rho_{AB}(t))$ for all $t \ge 0$ while $\Delta S^{(\alpha)}(\rho_{AB}(0)) < \Delta S^{(\alpha)}(\sigma_{AB}(0))$ and $\Delta S^{(\alpha)}(\rho_{AB}(t)) > \Delta S^{(\alpha)}(\sigma_{AB}(t))$ for all sufficiently large t (for any $\kappa \ge 0$, $\gamma > 0$).*

Proof. Let $\lambda(t) := e^{-\kappa t} \in (0,1]$ and $q(t) := (1-w)e^{-\gamma t} \in (0, 1-w]$. Applying Λ_t yields $\sigma_{AB}(t) = \rho(\lambda r_*) \otimes |0\rangle\langle 0|$ and

$$\rho_{AB}(t) = (1-q)\, \rho\!\left(\lambda \frac{\bar{r} - q r_2}{1-q}\right) \otimes |0\rangle\langle 0| + q\, \rho(\lambda r_2) \otimes |1\rangle\langle 1|, \tag{19}$$

since $wr_1 + (1-w-q)r_2 = \bar{r} - qr_2$. Taking Tr_B gives $\mathrm{Tr}_B\, \rho_{AB}(t) = \rho\big(\lambda[(\bar{r} - qr_2) + qr_2]\big) = \rho(\lambda \bar{r})$ for all $t \ge 0$ (`thm12`). Applying $\mathcal{E}^{(A)}_{\mathrm{deph}}(\cdot;\, r_*/\bar{r}) \otimes |0\rangle\langle 0|$ scales the Bloch coherence by $r_*/\bar{r} \in (0,1)$ (a valid $U(1)$-covariant CPTP dephasing map) and produces $\rho(\lambda r_*) \otimes |0\rangle\langle 0| = \sigma_{AB}(t)$, proving (16) and part (a).

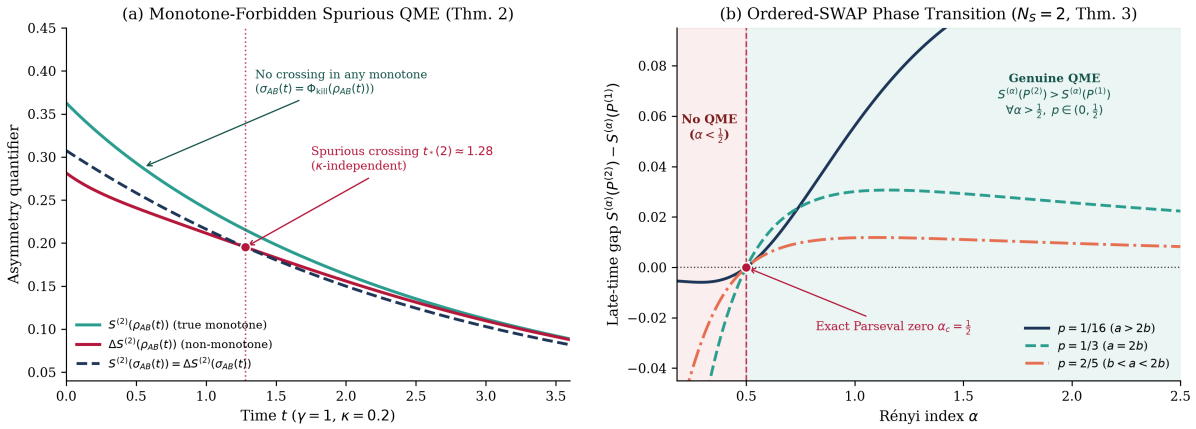

Figure 2: (a) Theorem 3.1: Under $\Lambda_t = \mathcal{E}_{\kappa t}^{(A)} \otimes \mathcal{D}_{\gamma t}^{(B)}$ ($\kappa = 0.2, \gamma = 1$) on $(w, r_1, r_2, r_*) = (3/4, 1/2, 1, 3/5)$, the non-monotone $\Delta S^{(2)}$ displays a permanent spurious Mpemba crossing at $t_*(2) \approx 1.28$ (independent of κ), whereas the true monotone $S^{(2)}$ never crosses because $\sigma_{AB}(t) = \Phi_{\text{kill}}(\rho_{AB}(t))$ for all $t \geq 0$. (b) Theorem 4.2: Exact late-time gap $S^{(\alpha)}(P^{(2)}) - S^{(\alpha)}(P^{(1)})$ in the ordered-SWAP model ($N_S = 2$) across all three base-ordering regimes $p \in \{1/16, 1/3, 2/5\}$, flipping sign sharply at the exact Parseval zero $\alpha_c = 1/2$.

For part (b), direct evaluation of $\exp(\Delta S^{(2)}) = \mathrm{Tr}(\rho^2)/\mathrm{Tr}(\rho_Q^2)$ on (19) and $\sigma_{AB}(t)$ gives

$$\exp\big(\Delta S^{(2)}(\rho_{AB}(t))\big) = 1 + \lambda^2 \frac{(\bar{r} - q r_2)^2 + q^2 r_2^2}{(1-q)^2 + q^2}, \qquad \exp\big(\Delta S^{(2)}(\sigma_{AB}(t))\big) = 1 + \lambda^2 r_*^2.$$

Subtracting the two expressions, the strictly positive factor $\lambda^2 = e^{-2\kappa t}$ factors out completely, and the numerator satisfies the universal algebraic identity (`thm12`)

$$\big[(\bar{r} - q r_2)^2 + q^2 r_2^2\big] - r_*^2\big[(1-q)^2 + q^2\big] = P(q),$$

where $P(q)$ is given by (18). By (14), $P(0) = \bar{r}^2 - r_*^2 > 0$ while $P(1-w) = (w^2 + (1-w)^2)\big[\frac{w^2 r_1^2 + (1-w)^2 r_2^2}{w^2 + (1-w)^2} - r_*^2\big] < 0$. Moreover, $r_2^2 > \bar{r}^2 > r_*^2$ makes the leading coefficient $2(r_2^2 - r_*^2) > 0$, so P is a convex parabola with $P(0) > 0 > P(1-w)$ and therefore possesses a *unique* root $q_* \in (0, 1-w)$. Since $q(t) = (1-w)e^{-\gamma t}$ decreases strictly from $1-w$ to 0, $P(q(t))$ flips sign once and permanently from negative to positive at $t_*(2) = \gamma^{-1} \log((1-w)/q_*)$, independently of $\kappa \geq 0$. For $(3/4, 1/2, 1, 3/5)$, exact integer arithmetic (`thm11`) certifies $1600P(q) = 2048q^2 - 848q + 49$, $1600P(0) = 49 > 0$, $15^2 \cdot 1600P(1/15) = 353 > 0$, $14^2 \cdot 1600P(1/14) = -220 < 0$, and $4^2 \cdot 1600P(1/4) = -560 < 0$, bracketing $q_* \in (1/15, 1/14)$.

For part (c), fix any $\alpha \in (0,1) \cup (1, \infty)$. By Theorem 2.1(1A), choose (w, r_1, r_2) with $\Delta S^{(\alpha)}(\rho_{AB}(0)) < \Delta S^{(\alpha)}(\rho(\bar{r}))$. Because $r \mapsto \Delta S^{(\alpha)}(\rho(r)) = \frac{1}{\alpha - 1} \log f_\alpha(r)$ is continuous and strictly increasing on $[0,1]$, pick $r_* \in (0, \bar{r})$ satisfying

$$\Delta S^{(\alpha)}(\rho_{AB}(0)) < \Delta S^{(\alpha)}(\rho(r_*)) < \Delta S^{(\alpha)}(\rho(\bar{r})).$$

Then $\sigma_{AB}(t) = \Phi_{\text{kill}}(\rho_{AB}(t))$ for all $t \geq 0$ and $\Delta S^{(\alpha)}(\rho_{AB}(0)) < \Delta S^{(\alpha)}(\sigma_{AB}(0))$. If $\kappa = 0$, $\rho_{AB}(t) \to \rho(\bar{r}) \otimes |0\rangle\langle 0|$ as $t \to \infty$, so $\Delta S^{(\alpha)}(\rho_{AB}(t)) \to \Delta S^{(\alpha)}(\rho(\bar{r})) > \Delta S^{(\alpha)}(\sigma_{AB}(t))$. If $\kappa > 0$, then as $t \to \infty$ both $\lambda(t) \to 0$ and $q(t) \to 0$. Using $f_\alpha(x) = 1 + \frac{\alpha(\alpha-1)}{2} x^2 + O(x^4)$ and $p_\alpha(1-q) = 1 - O(q^\alpha) \to 1$, we obtain $\Delta S^{(\alpha)}(\sigma_{AB}(t)) = \frac{\alpha}{2} r_*^2 \lambda^2 + O(\lambda^4)$ and $\Delta S^{(\alpha)}(\rho_{AB}(t)) = \frac{\alpha}{2} \bar{r}^2 \lambda^2 (1 + o(1))$, so the ratio $\Delta S^{(\alpha)}(\rho_{AB}(t))/\Delta S^{(\alpha)}(\sigma_{AB}(t))$ converges to $\bar{r}^2/r_*^2 > 1$, proving a permanent spurious crossing for all sufficiently large t. □

4 Theorem 3: Ordered-SWAP Phase Transition at $\alpha_c = 1/2$

In [1, Sec. 5], Travaglino, Mazzoni, and Calabrese introduced a pure-state ordered-SWAP toy model comparing a homogeneous slow-qubit state $|\psi_0^{(1)}\rangle_A = |+\rangle^{\otimes N_F} \otimes (\sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle)^{\otimes N_S}$ against a staggered slow-qubit state

$$|\psi_0^{(2)}\rangle_A = \big(\sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle\big)^{\otimes N_F} \otimes \Big[\big(\sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle\big)\big(\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle\big)\Big]^{\otimes N_S/2}$$

for $p \in (0, 1/2)$, observing numerically a phase transition at $\alpha_c = 1/2$. Since the reduced state remains pure, $S^{(\alpha)}(\rho_A(t)\|\rho_{A,Q}(t)) = H_{1/\alpha}(P^{(j)}(t))$, where at late times $t \geq N_F$ only $k = N - t$ slow qubits remain, with charge distributions $P^{(1)}$ (all parameter p) and $P^{(2)}$ (alternating p and $1-p$).

To prove the sharp $\alpha_c = 1/2$ phase transition analytically for all $p \in (0, 1/2)$ and all $\alpha > 0$, we first record a self-contained proof of Laguerre–Descartes' Rule of Signs for real Dirichlet polynomials.

Lemma 4.1 (Laguerre–Descartes' Rule of Signs for Dirichlet Polynomials)**.** *Let $b_1 > b_2 > \cdots > b_m > 0$ be strictly decreasing positive real bases and let $c_1, \ldots, c_m \in \mathbb{R} \setminus \{0\}$. Then the real Dirichlet polynomial $F(\beta) := \sum_{j=1}^m c_j b_j^\beta$ has at most $V(c_1, \ldots, c_m)$ real zeros counted with multiplicity, where $V(c_1, \ldots, c_m) := |\{1 \leq j \leq m-1 : c_j c_{j+1} < 0\}|$ is the number of sign changes in the coefficient sequence.*

Proof. We proceed by induction on $V = V(c_1, \ldots, c_m)$. If $V = 0$, all c_j share the same sign, so $F(\beta)$ has no real zeros. For $V \geq 1$, let $k \in \{1, \ldots, m-1\}$ be an index where $c_k c_{k+1} < 0$, and pick $b_* \in (b_{k+1}, b_k)$. Define $G(\beta) := b_*^{-\beta} F(\beta) = \sum_{j=1}^m c_j (b_j/b_*)^\beta$. Since $b_*^{-\beta} > 0$, G and F have the exact same real zeros with identical multiplicities. Differentiating yields

$$G'(\beta) = \sum_{j=1}^m \Big[c_j \log(b_j/b_*)\Big](b_j/b_*)^\beta.$$

Because $\log(b_j/b_*) > 0$ for $j \leq k$ and $\log(b_j/b_*) < 0$ for $j \geq k+1$, the sign change between indices k and $k+1$ is strictly eliminated while every other adjacent sign relation is preserved, so $V(G') = V(F) - 1$. By the induction hypothesis, G' has at most $V - 1$ real zeros counted with multiplicity. By Rolle's theorem, between any two consecutive distinct real zeros of G lies at least one zero of G', and a zero of G of multiplicity $\mu \geq 2$ is a zero of G' of multiplicity $\mu - 1$. Hence the total number of real zeros of G (and thus F) counted with multiplicity is at most $(V-1) + 1 = V$. □

Theorem 4.2 (Exact Ordered-SWAP Phase Transition at $\alpha_c = 1/2$ and Homogeneous Bin$(2, p)$ Reversal)**.** *Let $p \in (0, 1/2)$, $a := 1 - p$, and $b := p$ (so $0 < b < a$ and $a + b = 1$).*

(a) All-k Parseval collision-entropy degeneracy at $\alpha = 1/2$: *For every number $k \geq 1$ of remaining slow qubits (with k_1 qubits of parameter p and $k_2 = k - k_1$ qubits of parameter $1 - p$ in State 2), the characteristic functions $\phi_{P^{(1)}}(\theta) = (a + be^{i\theta})^k$ and $\phi_{P^{(2)}}(\theta) = (a + be^{i\theta})^{k_1}(b + ae^{i\theta})^{k_2}$ satisfy*

$$\big|\phi_{P^{(1)}}(\theta)\big|^2 = \big|\phi_{P^{(2)}}(\theta)\big|^2 = \big(1 - 4ab\sin^2(\theta/2)\big)^k \qquad \forall\, \theta \in [0, 2\pi]. \tag{20}$$

By Parseval's identity, $\sum_q (P_q^{(1)})^2 = \sum_q (P_q^{(2)})^2$, and hence $S^{(1/2)}(P^{(2)}) = S^{(1/2)}(P^{(1)})$ for all $k \geq 1$.

(b) Exact sharp phase transition at $\alpha_c = 1/2$ for $N_S = 2$: *For $k = N_S = 2$, $P^{(1)} = (a^2, 2ab, b^2)$ and $P^{(2)} = (ab, a^2 + b^2, ab)$. Setting $\beta := 1/\alpha > 0$, the five-term Dirichlet polynomial*

$$\phi(\beta) := \sum_{q=0}^{2} (P_q^{(1)})^\beta - \sum_{q=0}^{2} (P_q^{(2)})^\beta = (a^2)^\beta + (2ab)^\beta + (b^2)^\beta - (a^2 + b^2)^\beta - 2(ab)^\beta \tag{21}$$

has exactly three simple real zeros $\beta \in \{0, 1, 2\}$ for every $p \in (0, 1/2)$, and no other real zeros. Consequently, for every $p \in (0, 1/2)$ and every $\alpha > 0$:

$$S^{(\alpha)}(P^{(2)}) - S^{(\alpha)}(P^{(1)}) \begin{cases} > 0 & \text{if } \alpha > 1/2 \quad (\textit{strict Mpemba crossing}), \\ = 0 & \text{if } \alpha = 1/2 \quad (\textit{exact Parseval degeneracy}), \\ < 0 & \text{if } 0 < \alpha < 1/2 \quad (\textit{no Mpemba crossing}). \end{cases} \tag{22}$$

(c) Sharpness on homogeneous $\mathrm{Bin}(2, p)$ ***below*** $\alpha = 1/2$***:*** *While $H_\beta(\mathrm{Bin}(N, p))$ is strictly increasing in $p \in (0, 1/2)$ for $\beta = 1$ (Shepp–Olkin [28, 29]) and $\beta = 2$ ($\alpha = 1/2$), monotonicity in p fails below $\alpha = 1/2$: at $\beta = 5$ ($\alpha = 1/5$), for the exact rational pair $p_1 = 2/5 < p_2 = 9/20 < 1/2$,*

$$\begin{aligned} Z_5(9/20) = \frac{33374102497}{1024000000000} &> \frac{61781}{1953125} = Z_5(2/5) \\ \implies H_5\big(\mathrm{Bin}(2, 9/20)\big) &< H_5\big(\mathrm{Bin}(2, 2/5)\big), \end{aligned} \tag{23}$$

where $Z_\beta(p) = (1-p)^{2\beta} + (2p(1-p))^\beta + p^{2\beta}$ (exact cleared integers $333741024970 > 323910369280$).

Proof. Part (a) follows immediately from $|a + be^{i\theta}|^2 = a^2 + b^2 + 2ab\cos\theta = |b + ae^{i\theta}|^2 = 1 - 4ab\sin^2(\theta/2)$ (`thm13`) and Parseval's identity $\sum_q P_q^2 = \frac{1}{2\pi}\int_0^{2\pi} |\phi_P(\theta)|^2\, d\theta$.

For part (b), direct algebraic expansion (`thm13`) gives $\phi(0) = 1 + 1 + 1 - 1 - 2 = 0$, $\phi(1) = (a+b)^2 - (a+b)^2 = 0$, and $\phi(2) = a^4 + 4a^2b^2 + b^4 - (a^2+b^2)^2 - 2a^2b^2 = 0$. Since $0 < b < a$, the five positive bases satisfy the invariant strict orderings (`thm14`)

$$a^2 + b^2 > a^2 > ab > b^2, \qquad a^2 + b^2 > 2ab > ab.$$

Only the relative order of a^2 and $2ab$ varies with $p \in (0, 1/2)$, giving three exhaustive regimes:

- $a > 2b$ ($0 < p < 1/3$): descending bases ($a^2 + b^2$, a^2, $2ab$, ab, b^2) have coefficients $(-1, +1, +1, -2, +1)$ and hence $V = 3$ sign changes;
- $a = 2b$ ($p = 1/3$): $a^2 = 2ab$ merges into $+2(a^2)^\beta$, giving bases ($a^2 + b^2$, a^2, ab, b^2) with coefficients $(-1, +2, -2, +1)$ and $V = 3$ sign changes;
- $b < a < 2b$ ($1/3 < p < 1/2$): descending bases ($a^2 + b^2$, $2ab$, a^2, ab, b^2) have coefficients $(-1, +1, +1, -2, +1)$ and hence $V = 3$ sign changes.

By Lemma 4.1, $\phi(\beta)$ has at most 3 real zeros counted with multiplicity. Since $\{0, 1, 2\}$ are already three distinct real zeros, each must be a *simple* zero, and $\phi(\beta)$ changes sign at $\beta = 0, 1, 2$ and nowhere else! Because the strictly largest base is $a^2 + b^2$ with coefficient -1, $\phi(\beta) < 0$ for all $\beta > 2$. Flipping sign across the simple zeros 2 and 1 gives:

$$\phi(\beta) < 0 \ \ (\beta > 2), \qquad \phi(\beta) > 0 \ \ (1 < \beta < 2), \qquad \phi(\beta) < 0 \ \ (0 < \beta < 1).$$

Now $S^{(\alpha)}(P^{(2)}) - S^{(\alpha)}(P^{(1)}) = H_\beta(P^{(2)}) - H_\beta(P^{(1)}) = \frac{1}{1-\beta}\log\big(\sum_q (P_q^{(2)})^\beta / \sum_q (P_q^{(1)})^\beta\big)$ for $\beta \neq 1$. When $\beta > 2$ ($0 < \alpha < 1/2$), $\phi(\beta) < 0 \Rightarrow \sum (P_q^{(2)})^\beta > \sum (P_q^{(1)})^\beta$ and $1 - \beta < 0$, so $H_\beta(P^{(2)}) <$

$H_\beta(P^{(1)})$. When $\beta \in (1,2)$ ($1/2 < \alpha < 1$), $\phi(\beta) > 0$ and $1-\beta < 0$, so $H_\beta(P^{(2)}) > H_\beta(P^{(1)})$. When $\beta \in (0,1)$ ($\alpha > 1$), $\phi(\beta) < 0$ and $1-\beta > 0$, so again $H_\beta(P^{(2)}) > H_\beta(P^{(1)})$. At $\beta = 1$ ($\alpha = 1$), $\phi'(1) > 0$ (as ϕ crosses from negative to positive at simple zero $\beta = 1$), and $H_1(P^{(2)}) - H_1(P^{(1)}) = \phi'(1) > 0$! This proves (22) for every $\alpha > 0$ and every $p \in (0, 1/2)$.

Part (c) follows by exact rational evaluation of $20^{10} Z_5(p) = (20-n)^{10} + (2n(20-n))^5 + n^{10}$ at $n = 8$ (323910369280) versus $n = 9$ (333741024970), verified in `thm15` and `C12`. Since $1 - \beta = -4 < 0$, $Z_5(9/20) > Z_5(2/5)$ implies $H_5(\mathrm{Bin}(2, 9/20)) < H_5(\mathrm{Bin}(2, 2/5))$. □

5 Theorem 4: Mixed-State Duality, $d = 2$ Deficits, $d = 3$ Breakdown, and $\mathbb{Z}_K$ Replica

Finally, we resolve Open Question 1 of [1] by characterizing when mixed states obey index inversion versus index preservation, proving exact universal inequalities in $d = 2$ together with their simultaneous breakdown in $d = 3$, and deriving exact K-sector ceilings and discrete $\mathbb{Z}_K$ replica identities.

Theorem 5.1 (Mixed-State Duality, $d = 2$ Deficits, and $d = 3$ Breakdown)**.**

(a) Exact mixed-state duality dichotomy:

- Pure-factorized (index-inverted) regime: *If $\rho = |\phi\rangle\langle\phi| \otimes \tau$ with global charge $Q = Q_A \otimes I + I \otimes Q_B$ and $[\tau, Q_B] = 0$, then for every $\alpha > 0$,*

$$S^{(\alpha)}(\rho\|\rho_Q) = \Delta S^{(1/\alpha)}(\rho) = H_{1/\alpha}(\{p_q\}), \qquad p_q = \langle\phi|\Pi_q^{(A)}|\phi\rangle. \tag{24}$$

- Flat-twirl commuting (index-preserving) regime: *If $\rho = \rho_A \otimes \tau$ with $[\tau, Q_B] = 0$ and flat pinching $(\rho_A)_{Q_A} = I_{d_A}/d_A$ (in particular, any state with $\rho_Q = I_d/d$), then $[\rho, \rho_Q] = 0$ and for every $\alpha > 0$,*

$$S^{(\alpha)}(\rho\|\rho_Q) = D_\alpha(\rho\|\rho_Q) = \Delta S^{(\alpha)}(\rho). \tag{25}$$

Commutativity $[\rho, \rho_Q] = 0$ alone is insufficient: on $\rho_{AB}(3/4, 1/2, 1)$, $[\rho_{AB}, (\rho_{AB})_Q] = 0$ yet $\exp(S^{(2)}) = 23/16$ differs from $\exp(\Delta S^{(2)}) = 53/40$.

(b) Universal single-qubit ($d = 2$) ordering and four exact polynomial deficits: *On a single qubit with non-degenerate charge $Q = |1\rangle\langle 1|$, for* every *state ρ with Bloch vector (x, y, z) (so $\rho_Q = \mathrm{diag}(\frac{1+z}{2}, \frac{1-z}{2})$), let $s := \sqrt{1-z^2}$ and $t := \sqrt{1-x^2-y^2-z^2}$ (so $0 \le t \le s \le 1$). Then*

$$S^{(1/2)}(\rho\|\rho_Q) \le \min\big(\Delta S^{(1/2)}(\rho),\, \Delta S^{(2)}(\rho)\big), \qquad S^{(2)}(\rho\|\rho_Q) \ge \max\big(\Delta S^{(1/2)}(\rho),\, \Delta S^{(2)}(\rho)\big), \tag{26}$$

governed by four exact polynomial deficit factorizations:

$$(2 - s^2 + st)(1+s) - 2(1+t) = (1-s)(s+2)(s-t) \ge 0 \qquad \Leftrightarrow S^{(1/2)} \le \Delta S^{(1/2)}, \tag{27}$$

$$(2 - t^2)(2 - s^2 + st) - 2(2 - s^2) = t(s-t)(st+2) \ge 0 \qquad \Leftrightarrow S^{(1/2)} \le \Delta S^{(2)}, \tag{28}$$

$$(s^2 - t^2)(2 - s - s^2) = (1-s)(s+2)(s-t)(s+t) \ge 0 \quad \Leftrightarrow S^{(2)} \ge \Delta S^{(2)}, \tag{29}$$

$$(s + s^2 - t^2)(1+t) - s(1+s) = t(s-t)(s+t+1) \ge 0 \qquad \Leftrightarrow S^{(2)} \ge \Delta S^{(1/2)}. \tag{30}$$

Equality holds in (28),(30) iff ρ is pure ($t = 0$) or symmetric ($t = s$), and in (27),(29) iff ρ has flat twirl ($s = 1$, $z = 0$) or is symmetric ($t = s$).

(c) *Sharp $d = 3$ simultaneous breakdown of all four inequalities:* *On the one-parameter qutrit family $\rho_w := w|{+_{02}}\rangle\langle{+_{02}}| + (1-w)|1\rangle\langle 1|$ with $|{+_{02}}\rangle = \frac{|0\rangle+|2\rangle}{\sqrt{2}}$ and $Q = \mathrm{diag}(0,1,2)$ (sectors $\{0\},\{1\},\{2\}$), all four inequalities of (26) fail:*

- *At $w = 1/5$: $S^{(1/2)} > \Delta S^{(2)}$ (certified by $17\sqrt{2} < 33 \iff 578 < 1089$) and $S^{(2)} < \Delta S^{(1/2)}$ (certified by $6 < 5\sqrt{2} \iff 36 < 50$);*
- *At $w = 4/5$: $S^{(1/2)} > \Delta S^{(1/2)}$ (certified by $9\sqrt{2} < 14 \iff 162 < 196$) and $S^{(2)} < \Delta S^{(2)}$ (certified by $9/5 < 17/9 \iff 81 < 85$).*

(d) *Universal K-sector pinching ceilings and exact $\mathbb{Z}_K$ replica identity:* *If Q has K distinct eigenspaces $\Pi_1,\dots,\Pi_K$, then for every state ρ,*

$$0 \le S^{(\alpha)}(\rho\|\rho_Q) \le \log K \quad (\forall\,\alpha\in(0,\infty]), \qquad 0 \le \Delta S^{(2)}(\rho) \le \log K, \tag{31}$$

both saturated by $|\psi_K\rangle = K^{-1/2}\sum_{j=1}^{K}|j\rangle$ with unit vectors $|j\rangle \in \mathrm{ran}(\Pi_j)$. Furthermore, for any finite cyclic symmetry $\mathbb{Z}_K$ generated by $U = e^{2\pi i Q/K}$ (where Q has integer eigenvalues modulo K, so $U^K = I$) and its $\mathbb{Z}_K$-twirl $\rho_{\mathbb{Z}_K} := K^{-1}\sum_{a=0}^{K-1} U^a \rho U^{-a}$ (which coincides with the $U(1)$-pinching ρ_Q whenever the distinct eigenvalues of Q are incongruent modulo K), for integers $n \ge 1, m \ge 0$ the two-parameter replica moment $M^{(m,n)}(\rho\|\rho_{\mathbb{Z}_K}) := \mathrm{Tr}[(\rho_{\mathbb{Z}_K}^{m/2}\rho\,\rho_{\mathbb{Z}_K}^{m/2})^n] = \mathrm{Tr}[(\rho\rho_{\mathbb{Z}_K}^m)^n]$ satisfies the exact finite-group identity

$$M^{(m,n)}(\rho\|\rho_{\mathbb{Z}_K}) = \frac{1}{K^{nm}} \sum_{\{a_{j,k}\in\mathbb{Z}_K\}} Z_{n(m+1)}\big(\vec{\delta}^{(1)},\dots,\vec{\delta}^{(n)}\big), \tag{32}$$

where $\delta_k^{(j)} := a_{j,k+1} - a_{j,k} \pmod K$ $(0 \le k \le m)$ with boundary pins $a_{j,0} = a_{j,m+1} = 0$, and $Z_N(\vec{b}) := \mathrm{Tr}\big(\prod_{\ell=1}^{N}\rho U^{b_\ell}\big)$.

Proof. **Part (a).** When $\rho = |\phi\rangle\langle\phi|\otimes\tau$ with $[\tau, Q_B] = 0$, $\rho_Q = \mathcal{G}_{Q_A}(|\phi\rangle\langle\phi|)\otimes\tau$. Additivity of $S^{(\alpha)}$ and S_α on tensor products cancels τ and reduces both sides of (24) to the pure-state identity of [1]. When $\rho = \rho_A\otimes\tau$ with $(\rho_A)_{Q_A} = I_{d_A}/d_A$, $[\rho_A,(\rho_A)_{Q_A}] = 0$ and $\mathrm{Tr}((\rho_A)_{Q_A}^{\alpha}) = d_A^{1-\alpha}$, so $S^{(\alpha)}(\rho_A\|I_{d_A}/d_A) = D_\alpha(\rho_A\|I_{d_A}/d_A) = \frac{1}{\alpha-1}\log(d_A^{\alpha-1}\,\mathrm{Tr}(\rho_A^\alpha)) = \log d_A - S_\alpha(\rho_A) = \Delta S^{(\alpha)}(\rho_A)$, and tensoring with symmetric τ preserves all three quantities. On $\rho_{AB}(3/4,1/2,1)$, $(\rho_{AB})_Q = \frac{3}{8}I_2\oplus\frac{1}{8}I_2$ commutes with ρ_{AB}, yet $\mathrm{Tr}(\rho_{AB}^2(\rho_{AB})_Q^{-1}) = 1+\frac{3}{4}(1/4)+\frac{1}{4}(1) = 23/16 \ne 53/40$ (`C13`).

Part (b). For a single qubit with Bloch vector (x,y,z) ($r^2 = x^2+y^2+z^2$, $s = \sqrt{1-z^2}$, $t = \sqrt{1-r^2}$), $\rho_Q = \mathrm{diag}(\frac{1+z}{2},\frac{1-z}{2})$ has $\det\rho_Q = s^2/4$ and $\det\rho = t^2/4$. At $s = 0$ ($z = \pm1$, $t = 0$), $\rho = \rho_Q$ is symmetric so all four quantities vanish identically; for $s > 0$, direct 2×2 algebra yields the four closed forms:

$$e^{-S^{(1/2)}} = \mathrm{Tr}(\rho_Q\rho) + 2\sqrt{\det\rho_Q\det\rho} = \frac{2-s^2+st}{2}, \qquad e^{S^{(2)}} = 1+\frac{s^2-t^2}{s},$$
$$e^{-\Delta S^{(1/2)}} = \frac{1+t}{1+s}, \qquad e^{\Delta S^{(2)}} = \frac{2-t^2}{2-s^2}.$$

Clearing positive denominators converts $e^{-S^{(1/2)}} \ge e^{-\Delta S^{(1/2)}}$, $e^{\Delta S^{(2)}}e^{-S^{(1/2)}} \ge 1$, $e^{S^{(2)}} \ge e^{\Delta S^{(2)}}$, and $e^{S^{(2)}} \ge e^{\Delta S^{(1/2)}}$ into (27)–(30), respectively. All four polynomial identities and their non-negativity on $0 \le t \le s \le 1$ are proved universally in `thm16`.

Part (c). On $\rho_w = w|{+_{02}}\rangle\langle{+_{02}}| + (1-w)|1\rangle\langle1|$ with $\rho_{w,Q} = \mathrm{diag}(w/2, 1-w, w/2)$, block evaluation on $\{0,2\}\oplus\{1\}$ gives:

$$e^{-S^{(1/2)}} = \left(1-w+\frac{w}{\sqrt{2}}\right)^2, \qquad e^{S^{(2)}} = 1+w,$$
$$e^{\Delta S^{(2)}} = \frac{w^2+(1-w)^2}{\frac{1}{2}w^2+(1-w)^2}, \qquad e^{\Delta S^{(1/2)}} = \left(\frac{\sqrt{2w}+\sqrt{1-w}}{\sqrt{w}+\sqrt{1-w}}\right)^2.$$

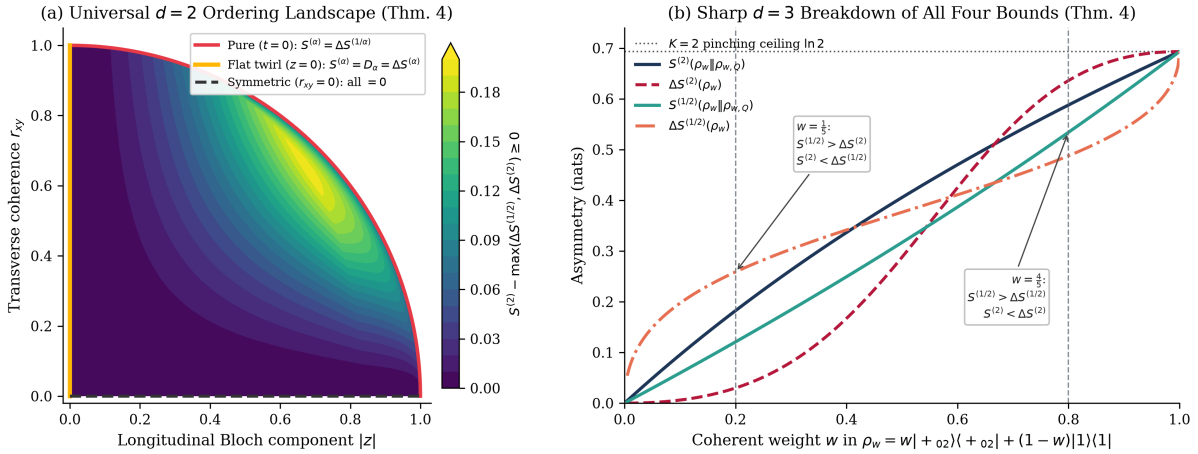

Figure 3: Theorem 5.1: (a) Single-qubit Bloch-quadrant ordering landscape plotting $S^{(2)} - \max(\Delta S^{(1/2)}, \Delta S^{(2)}) \geq 0$, highlighting the pure-state index-inversion semicircle ($t = 0$), the flat-twirl index-preservation axis ($z = 0$), and the symmetric zero axis ($r_{xy} = 0$). (b) Sharp simultaneous breakdown of all four single-qubit inequalities in $d = 3$ on $\rho_w = w|+_{02}\rangle\langle+_{02}| + (1-w)|1\rangle\langle 1|$, certified at $w = 1/5$ and $w = 4/5$, bounded above by the universal $K = 2$ pinching ceiling $\ln 2$.

At $w = 1/5$: $e^{-S^{(1/2)}} e^{\Delta S^{(2)}} = \frac{33+8\sqrt{2}}{50} \cdot \frac{34}{33} < 1 \iff 17\sqrt{2} < 33 \iff 578 < 1089$, and $e^{S^{(2)}} = 6/5 < (6+4\sqrt{2})/9 = e^{\Delta S^{(1/2)}} \iff 6 < 5\sqrt{2} \iff 36 < 50$ (`thm17`). At $w = 4/5$: $e^{-S^{(1/2)}} e^{\Delta S^{(1/2)}} = \frac{(9+4\sqrt{2})^2}{225} < 1 \iff 9\sqrt{2} < 14 \iff 162 < 196$, and $e^{S^{(2)}} = 9/5 < 17/9 = e^{\Delta S^{(2)}} \iff 81 < 85$ (`thm18`).

Part (d). For any vector $|v\rangle$ and $\rho \geq 0$, expanding $0 \leq \sum_{1 \leq j < k \leq K} \langle v|(\Pi_j - \Pi_k)\rho(\Pi_j - \Pi_k)|v\rangle = K\langle v|\rho_Q|v\rangle - \langle v|\rho|v\rangle$ (the operator form of Lagrange's identity proved in `thm19`) gives $\rho \leq K\rho_Q$. Hence $S^{(\infty)}(\rho\|\rho_Q) \leq \log K$, and monotonicity of $\alpha \mapsto S^{(\alpha)}$ [19] yields $0 \leq S^{(\alpha)}(\rho\|\rho_Q) \leq \log K$ for all $\alpha \in (0, \infty]$. For $\Delta S^{(2)}$, writing $\rho = X^\dagger X$ and $A_j = X\Pi_j$, Cauchy–Schwarz in Hilbert–Schmidt norm gives $\|\Pi_j\rho\Pi_k\|_2^2 = \mathrm{Tr}(A_j A_j^\dagger A_k A_k^\dagger) \leq \|A_j A_j^\dagger\|_2 \|A_k A_k^\dagger\|_2 = \|\Pi_j\rho\Pi_j\|_2 \|\Pi_k\rho\Pi_k\|_2$, whence $\mathrm{Tr}(\rho^2) = \sum_{j,k} \|\Pi_j\rho\Pi_k\|_2^2 \leq (\sum_{j=1}^K \|\Pi_j\rho\Pi_j\|_2)^2 \leq K \sum_{j=1}^K \|\Pi_j\rho\Pi_j\|_2^2 = K\,\mathrm{Tr}(\rho_Q^2)$ by `thm19`, proving $0 \leq \Delta S^{(2)}(\rho) \leq \log K$.

Finally, for $\mathbb{Z}_K$, $\rho_Q = K^{-1} \sum_{a=0}^{K-1} U^a \rho U^{-a}$. Substituting this finite average into each of the nm factors of ρ_Q in $\mathrm{Tr}[(\rho\rho_Q^m)^n]$ and collecting adjacent powers $U^{-a_{j,k}} U^{a_{j,k+1}} = U^{\delta_k^{(j)}}$ with $a_{j,0} = a_{j,m+1} = 0$ immediately yields (32) (formally verified for all 2×2 matrices under $\mathbb{Z}_2$ in `thm20` and over the exact cyclotomic field $\mathbb{Q}(\omega)$ for $\mathbb{Z}_3$ in `C16`). □

6 Formal Verification and Reproducibility Audit

Every algebraic identity, exact rational witness, matrix invariant, and pure-integer inequality supporting Theorems 2.1–5.1 has been independently machine-verified along three orthogonal axes:

1. **Constructive Lean 4.36.0 Formalization:** File `lean/RenyiAsymmetryMpemba.lean` contains 20 theorems (`thm01`–`thm20`) with *zero external imports*, *zero* `sorry`, *zero* `grind`, *zero* `Classical.choice`, and *zero* warnings under `lean` with `-DwarningAsError=true` (`EXIT=0`). Axiom inspection (`#print axioms`) confirms every theorem depends exclusively on the constructive core axioms `[propext, Quot.sound]` (with `thm15` depending on no axioms at all).

2. **AST-Audited Zero-Float Verifier:** The pure-stdlib verifier (`verification/verify_renyi_asymmetry.py`) uses exact `int` and `Fraction` arithmetic (including exact $\mathbb{Q}(\omega)$ cyclotomic arithmetic for $\mathbb{Z}_3$), passes a self-enforcing AST zero-float audit (zero float violations), and emits 16 canonical SHA-256 certificates (`C01`–`C16`) with manifest digest:

```
MANIFEST SHA-256: 57f6d9f453cce79895bff08ac9f71c170f04dba3c41e6f4302555026ffb2a622
```

3. **Adversarial Test Suite:** File `tests/test_adversarial.py` executes 60 adversarial `pytest` checks (60/60 passing) validating positive certificates, exact boundary thresholds, negative no-violation controls (e.g., Petz D_2 and sandwiched $S^{(1/2)}$ respecting DPI on the same states where D_3 and $S^{(1/4)}$ fail, and $S^{(2)}$ forbidding the spurious QME crossing), AST tamper rejection, and SHA-256 tamper detection.

7 Discussion and Outlook

Our four theorems sharpen the resource-theoretic foundations of Rényi entanglement asymmetry [1] in four concrete ways. First, Theorem 2.1 demonstrates that restricting QDPI to the asymmetry manifold (ρ, ρ_Q) and $U(1)$-covariant (or even strongly symmetric) channels does not widen the monotonicity window of any Rényi family by even an infinitesimal margin: $\Delta S^{(\alpha)}$ fails for every $\alpha \neq 1$, Petz D_α fails immediately above $\alpha = 2$ already on a single qubit, and sandwiched $S^{(\alpha)}$ fails immediately below $\alpha = 1/2$ already on a qutrit. Second, Theorem 3.1 supplies a decisive operational warning for quantum Mpemba studies: because $\Delta S^{(\alpha)}$ $(\alpha \neq 1)$ can display a permanent, κ-independent crossing on states intertwined by a $U(1)$-covariant channel Φ_{kill}, observing a Mpemba crossing exclusively in $\Delta S^{(2)}$ is insufficient to certify a resource-theoretic Mpemba inversion unless corroborated by a monotone such as $S^{(\alpha)}$ $(\alpha \geq 1/2)$. Third, Theorem 4.2 turns the ordered-SWAP transition of [1] into an exact theorem via Laguerre–Descartes' Rule of Signs, revealing how the breakdown of monotonicity below $\alpha = 1/2$ coincides with both the loss of the staggered Mpemba crossing and the failure of homogeneous binomial entropy monotonicity. Fourth, Theorem 5.1 maps the mixed-state geometry separating index inversion from index preservation and shows that single-qubit universality is sharp at $d = 2$, breaking simultaneously in $d = 3$. Extending our Laguerre–Descartes analysis to arbitrary even $N_S \geq 4$ and characterizing exact non-Abelian ($SU(2)$) mixed-state duality laws remain compelling directions for future work.

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