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Exact Zero-Sum Ramsey Numbers of Long Cycles, 2-Factors, and Generalized Wheels over Arbitrary Finite Abelian Groups via Iterated Switch-Stabilizer Descent

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Abstract

For a finite Abelian group Γ of order $q \geq 2$ and a graph G whose edge count is divisible by $\exp(\Gamma)$, the zero-sum Ramsey number $R(G,\Gamma)$ is the least integer N such that every edge-coloring $w : E(K_N) \to \Gamma$ contains a copy of G with $\sum_{e\in E(G)} w(e) = 0$. Recently, Chi and He (2026) established that for cyclic groups $\mathbb{Z}_q$ and large k, $R(C_{qk}, \mathbb{Z}_q) = qk + q - 1$ for odd $q \geq 3$, while for every even $q \geq 4$ their Erdős–Ginzburg–Ziv insertion method left an additive gap $qk + \frac{q}{2} - 1 \leq R(C_{qk}, \mathbb{Z}_q) \leq qk + q - 1$ of order $q/2$, and for generalized wheels $W_{pk}(s) = C_{pk} + \overline{K}_s$ they treated only odd primes p with $s + 2 \equiv 0 \pmod p$.

In this paper, we introduce *Iterated Switch-Stabilizer Descent*, an elementary, self-contained structural method that bypasses Turán bounds for even cycles and bounded-degree zero-sum Ramsey machinery entirely. For any coloring $w : E(K_N) \to \Gamma$, iteratively extracting maximal collections of vertex-disjoint 4-vertex switches $\delta(a,b,c,d) = w(ac) + w(bd) - w(ab) - w(cd)$ outside proper subgroups yields a subgroup $H^* \leq \Gamma$ and a co-bounded clique $U^* = V(K_N) \setminus F$ ($|F| \leq 32(q-1)+4\Delta\lfloor\log_2 q\rfloor$) on which (whenever $|U^*| \geq 4$) w is an exact constant-shifted vertex potential modulo H^*—proved without dividing by 2, hence valid in characteristic 2—while U^* retains enough disjoint switches to generate all of H^* even after deleting Δ arbitrary vertices. Combining this descent with Gao's theorem on the doubling subgroup $2\Gamma = \{2g : g \in \Gamma\}$, we prove the exact formula

$$R(G,\Gamma) = m + D(2\Gamma) - 1$$

for every finite Abelian group Γ of order $q \geq 2$, every $m \geq M_0(\Gamma) = O(q^2 \log q)$ divisible by $\operatorname{lcm}(\exp(\Gamma), |2\Gamma|)$, and every 2-factor G on m vertices containing at least $q-1$ vertex-disjoint 4-vertex paths (including C_m and kC_ℓ for $\ell \geq 4$), where $D(2\Gamma)$ is the Davenport constant of 2Γ. In particular: (i) for every even $q \geq 4$ and $k \geq K_0(q) = O(q \log q)$, $R(C_{qk}, \mathbb{Z}_q) = qk + \frac{q}{2} - 1$, resolving Chi and He's even-q gap at an $O(q^2 \log q)$ length threshold (alongside contemporaneous work on zero-sum Ramsey numbers of forests, complete bipartite graphs, and cyclic-group cycles by Wang and Wang, 2026); (ii) for every elementary 2-group $\mathbb{Z}_2^r$ and every even $m \geq M_0(\mathbb{Z}_2^r)$, $R(C_m, \mathbb{Z}_2^r) = m$, forcing a Hamiltonian zero-sum cycle with zero extra vertices; and (iii) for every finite Abelian group Γ and every $s \geq 1$ with $\exp(\Gamma) \mid \gcd(m, s+2)$ and large m, $R(W_m(s), \Gamma) = m + s$, extending Chi and He's prime-modulus wheel theorem to all composite and non-cyclic finite Abelian groups. Finally, we certify that $R(C_4, \mathbb{Z}_2^2) = 5 > 4$ (re-proving the $d = 2$ case of Alvarado, Colucci, Parente, and Souza, 2022, end-to-end inside Lean 4) and $R(C_4, \mathbb{Z}_4) = 6 > 5$ (with the K_5 lower bound and $\mathbb{Z}_4$ even-potential gauge invariance certified in Lean 4, and the K_6 upper bound certified by exact symmetry-broken backtracking in Python), demonstrating that a finite-length threshold is structurally unavoidable. All algebraic cores, the $R(C_4, \mathbb{Z}_2^2) = 5$ proof, and the $R(C_4, \mathbb{Z}_4) \geq 6$ lower bound are formalized in standalone Mathlib-free Lean 4.36.0 (54/54 public theorems strictly constructive, zero `Classical.choice`, zero `sorry`) and complemented by zero-float exact Python verification.

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1 Introduction

Zero-sum Ramsey theory merges classical graph Ramsey theory with zero-sum additive combinatorics in finite Abelian groups [7, 11, 12, 2, 3]. Given a finite Abelian group $(\Gamma, +)$ of order $q = |\Gamma| \geq 2$ and a finite graph G, the *zero-sum Ramsey number* $R(G, \Gamma)$ is the smallest integer $N \geq |V(G)|$ such that every edge-labeling $w : E(K_N) \to \Gamma$ admits an isomorphic copy $G' \subseteq K_N$ of G satisfying

$$w(G') := \sum_{e \in E(G')} w(e) = 0 \quad \text{in } \Gamma.$$

For $R(G, \Gamma)$ to exist, a constant coloring $w(e) \equiv g$ shows that necessarily $|E(G)| \cdot g = 0$ for all $g \in \Gamma$, i.e., the exponent $\exp(\Gamma)$ must divide $|E(G)|$; conversely, whenever $\exp(\Gamma) \mid |E(G)|$, classical multicolor Ramsey's theorem yields a monochromatic (hence zero-sum) copy of G and thus guarantees finiteness, while Katz, Lian, Malekshahian, and Shapiro [10] recently proved the linear upper bound $R(G, \Gamma) \leq C(\Delta)|V(G)|$ for bounded-degree graphs under $|\Gamma| \mid |E(G)|$.

Recent work has turned from order-of-magnitude bounds toward *exact* zero-sum Ramsey numbers for structured graph families [4, 6, 5, 14]. In May 2026, Chi and He [5] studied cycles C_{qk} and generalized wheels $W_m(s) = C_m + \overline{K}_s$ (the join of a cycle C_m with an independent set of s hub vertices). Using an Erdős–Ginzburg–Ziv (EGZ) vertex-insertion reduction starting from $R(C_{2q}, \mathbb{Z}_q) \leq \min\{35q^2, D_0 q\}$—obtained by taking the sharper of Pikhurko's Turán bound for even cycles [13] and the linear bounded-degree theorem of [10] (with uneffective constant D_0)—they proved that for $k \geq \min\{35q, D_0\}$:

1. If $q \geq 3$ is *odd*, then $R(C_{qk}, \mathbb{Z}_q) = qk + q - 1$.
2. If $q \geq 4$ is *even*, then

$$qk + \frac{q}{2} - 1 \;\leq\; R(C_{qk}, \mathbb{Z}_q) \;\leq\; qk + q - 1, \tag{1}$$

   leaving an unresolved additive gap of size $q/2$ for every even modulus $q \geq 4$.
3. For generalized wheels $W_{pk}(s)$ over an *odd prime* modulus p satisfying $s + 2 \equiv 0 \pmod{p}$, $R(W_{pk}(s), \mathbb{Z}_p) = pk + s$ for sufficiently large k, via a Cauchy–Davenport switch-defect argument restricted to prime fields $\mathbb{Z}_p$.

Independently and contemporaneously, Wang and Wang [14] studied zero-sum Ramsey numbers of forests, cycles, and complete bipartite graphs over cyclic groups $\mathbb{Z}_q$, also closing Chi and He's even-q cyclic gap at a cubic cycle-length threshold ($O(q^3)$). Our method below achieves a sharper $O(q^2 \log q)$ length threshold while extending uniformly to arbitrary finite Abelian groups (including non-cyclic and elementary 2-groups), arbitrary 2-factors, and generalized wheels.

The obstruction in (1) is transparent: inserting q vertices one-by-one into a zero-sum cycle via EGZ over $\mathbb{Z}_q$ requires $2q - 1$ unused vertices (giving $q(k-1) + (2q-1) = qk + q - 1$), whereas cut colorings on $K_{qk+q/2-2}$ force every C_{qk} to cross the cut an even number of times, producing weights strictly inside the index-2 doubling subgroup $2\mathbb{Z}_q \cong \mathbb{Z}_{q/2}$. On the other hand, Chi and He's wheel method relies on the Cauchy–Davenport theorem in $\mathbb{Z}_p$, which breaks down whenever the group Γ has proper nontrivial subgroups.

1.1 Main Results

We resolve both obstructions simultaneously by developing *Iterated Switch-Stabilizer Descent*, a purely elementary algebraic-combinatorial framework that operates in *arbitrary* finite Abelian groups Γ (composite, even, and non-cyclic alike) without invoking Cauchy–Davenport, Pikhurko's Turán bound, or bounded-degree zero-sum Ramsey theorems.

For a finite Abelian group G, recall that its *Davenport constant* $D(G)$ is the least integer $d \geq 1$ such that every sequence of d elements in G contains a nonempty subsequence summing to 0 [11, 9].

Let $2\Gamma \coloneqq \{2g : g \in \Gamma\}$ denote the doubling subgroup of Γ, and define the explicit subgroup-chain defect bound

$$F_{\max}(q,\Delta) \;\coloneqq\; \sum_{j=0}^{\Omega(q)-1} 4\big(4(q_j-1)+\Delta\big) \;\le\; 32(q-1)+4\Delta\lfloor \log_2 q\rfloor, \tag{2}$$

where $q = q_0 > q_1 > \cdots > q_{\Omega(q)} = 1$ is the chain obtained by dividing q sequentially by its prime factors in non-decreasing order (which maximizes $\sum_j q_j$ over all maximal divisor chains of q). For brevity we write $F_{\max}(\Gamma) \coloneqq F_{\max}(|\Gamma|, D(2\Gamma)-1)$ whenever $\Delta = D(2\Gamma)-1$. Set

$$M_0(\Gamma) \;\coloneqq\; \max\Big\{(2q+1)F_{\max}\big(q,D(2\Gamma)-1\big)+(q-1)\big(D(2\Gamma)-1\big)+4q(q-1),\\ 5\big(F_{\max}(q,D(2\Gamma)-1)+2q\big),\; 4q\Big\}, \tag{3}$$

so $M_0(\Gamma) = O(q^2 \log q)$ universally and $M_0(\mathbb{Z}_2^r) = O(q^2)$ without the logarithmic factor.

Theorem 1.1 (Iterated Switch-Stabilizer Descent)**.** *Let Γ be a finite Abelian group of order $q \ge 2$, let $\Delta \ge 0$ be an integer, and let $w : E(K_N) \to \Gamma$ be any edge-coloring. For any ordered 4-tuple (a,b,c,d) of pairwise distinct vertices, define its* switch

$$\delta(a,b,c,d) \;\coloneqq\; w(ac)+w(bd)-w(ab)-w(cd) \in \Gamma.$$

Then there exist a subgroup $H^* \le \Gamma$ and a partition $V(K_N) = U^* \sqcup F$ such that:

(a) Bounded defect: $|F| \le F_{\max}(q,\Delta) \le 32(q-1)+4\Delta\lfloor \log_2 q\rfloor$.

(b) Characteristic-2-safe quotient potential: *If $|U^*| \ge 4$, there exist $\alpha \in Q \coloneqq \Gamma/H^*$ and $f : U^* \to Q$ such that for all distinct $x,y \in U^*$,*

$$w(xy) + H^* = \alpha + f(x) + f(y) \quad \text{in } Q.$$

(c) Robust switch-completeness: *For every proper subgroup $K < H^*$, U^* contains at least $M(H^*,\Delta) \coloneqq 4(|H^*|-1)+\Delta+1$ pairwise vertex-disjoint ordered 4-tuples (a,b,c,d) with $\delta(a,b,c,d) \in H^* \setminus K$.*

(d) Delete-resilient subgroup generation: *For every subset $S \subseteq V(K_N)$ with $|V(K_N) \setminus S| \le \Delta$, there exist $r \le |H^*|-1$ pairwise vertex-disjoint ordered 4-tuples (a_i,b_i,c_i,d_i) completely inside $S \cap U^*$ whose switches $\delta_i = \delta(a_i,b_i,c_i,d_i) \in H^*$ satisfy*

$$\{0,\delta_1\}+\{0,\delta_2\}+\cdots+\{0,\delta_r\} = H^*.$$

Theorem 1.1 immediately reveals why the doubling subgroup 2Γ and its Davenport constant $D(2\Gamma)$ govern all long 2-factors over every finite Abelian group.

Theorem 1.2 (Exact Zero-Sum Ramsey Number of Long Cycles and 2-Factors)**.** *Let Γ be any finite Abelian group of order $q \ge 2$, let $m \ge 3$ be an integer divisible by $\exp(\Gamma)$, and let $G = C_{\ell_1} \sqcup \cdots \sqcup C_{\ell_c}$ be any 2-factor on $m = \sum_{j=1}^{c} \ell_j$ vertices.*

(i) Universal lower bound: *For every $m \ge 3$ with $\exp(\Gamma) \mid m$,*

$$R(G,\Gamma) \;\ge\; m + D(2\Gamma) - 1.$$

(ii) Exact asymptotic equality: *If in addition $\operatorname{lcm}(\exp(\Gamma), |2\Gamma|) \mid m$ (in particular, $m = qk$ for any integer $k \ge 1$ with $qk \ge 3$), $m \ge M_0(\Gamma)$, and G contains at least $q-1$ vertex-disjoint 4-vertex paths (equivalently, $\sum_{j=1}^{c} \lfloor \ell_j/4 \rfloor \ge q-1$, which holds automatically for $G = C_m$ and for $G = kC_\ell$ with $\ell \ge 4$ since $\lfloor \ell/4 \rfloor \ge \ell/7$ and $M_0(\Gamma)/7 \ge q-1$), then*

$$R(G,\Gamma) \;=\; m + D(2\Gamma) - 1.$$

Specializing Theorem 1.2 yields four immediate corollaries.

Corollary 1.3 (Resolution of Chi and He's Even-q Gap at $O(q^2 \log q)$ Length)**.** *For every even integer $q \geq 4$ and every integer $k \geq K_0(q) := \lceil M_0(\mathbb{Z}_q)/q \rceil = O(q \log q)$,*

$$R(C_{qk}, \mathbb{Z}_q) = qk + \frac{q}{2} - 1$$

(at an explicit $O(q^2 \log q)$ cycle-length threshold; cf. the contemporaneous cyclic-group study [14]). For every odd $q \geq 3$ and every $k \geq K_0(q)$, $R(C_{qk}, \mathbb{Z}_q) = qk + q - 1$, proved without Turán bounds or [10].

Corollary 1.4 (Hamiltonian Zero-Sum Cycles in Elementary 2-Groups)**.** *For every $r \geq 1$ and every even integer $m \geq M_0(\mathbb{Z}_2^r) = O(4^r)$,*

$$R(C_m, \mathbb{Z}_2^r) = m.$$

That is, every $\mathbb{Z}_2^r$-edge-coloring of K_m contains a Hamiltonian *zero-sum cycle C_m.*

Corollary 1.5 (Closed Forms for Rank-2 Groups and p-Groups via Olson's Theorem)**.** *For $m \geq M_0(\Gamma)$ divisible by $\operatorname{lcm}(\exp(\Gamma), |2\Gamma|)$:*

(a) *If $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ with $n_1 \mid n_2$, let $m_i = n_i / \gcd(n_i, 2)$; then $R(C_m, \Gamma) = m + m_1 + m_2 - 2$.*

(b) *If $\Gamma = \bigoplus_{i=1}^r \mathbb{Z}_{p^{a_i}}$ $(a_i \geq 1)$ is a finite p-group, then $R(C_m, \Gamma) = m + \sum_{i=1}^r (p^{a_i} - 1)$ for odd p, and $R(C_m, \Gamma) = m + \sum_{i=1}^r (2^{a_i - 1} - 1)$ for $p = 2$.*

Next, applying Theorem 1.1 with a joint-fiber hub placement extends Chi and He's prime-modulus wheel theorem to *every* finite Abelian group.

Theorem 1.6 (Generalized Wheels over Arbitrary Finite Abelian Groups)**.** *Let Γ be any finite Abelian group of order $q \geq 2$, and let $s \geq 1$ satisfy $\exp(\Gamma) \mid (s+2)$. Set $F_W := F_{\max}(q, s)$ and*

$$M_W(\Gamma, s) \;:=\; \max\Big\{ F_W - s + \big(s + 4(q-1) + 2F_W\big)\, q^{F_W},\; 5(F_W + 2q) \Big\}.$$

Then for every integer $m \geq M_W(\Gamma, s)$ divisible by $\exp(\Gamma)$,

$$R(W_m(s), \Gamma) \;=\; m + s.$$

Finally, we prove that the asymptotic length threshold $m \geq M_0(\Gamma)$ in Theorem 1.2 is not a proof artifact: for small m, genuinely non-potential edge colorings strictly outperform all potential colorings.

Theorem 1.7 (Small-k Non-Potential Phase Transition)**.** *For $m = 4$:*

$$R(C_4, \mathbb{Z}_2^2) = 5 \;>\; 4 = 4 + D(2\mathbb{Z}_2^2) - 1, \qquad R(C_4, \mathbb{Z}_4) = 6 \;>\; 5 = 4 + D(2\mathbb{Z}_4) - 1.$$

(The evaluation $R(C_4, \mathbb{Z}_2^2) = 5$ is the $d = 2$ case of Alvarado, Colucci, Parente, and Souza [1], who proved $R(C_4, \mathbb{Z}_2^d) = 2^d + 1$ for all $d \geq 2$, and $R(C_4, \mathbb{Z}_4) = 6$ likewise coincides with the barycentric Ramsey value $BR(C_4, \mathbb{Z}_4) = 6$ since $4 \mid |E(C_4)|$; we provide an end-to-end kernel-certified Lean 4 proof of $R(C_4, \mathbb{Z}_2^2) = 5$, alongside an explicit Lean 4-plus-backtracking certification of $R(C_4, \mathbb{Z}_4) = 6$ via a Lean 4-certified K_5 lower bound and exact symmetry-broken Python backtracking on K_6.)

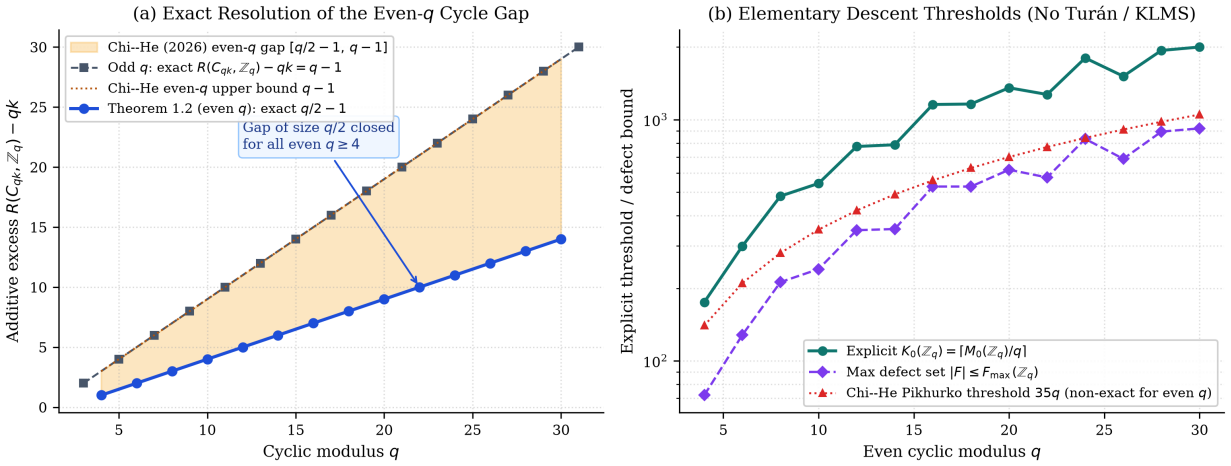

Figure 1: **Exact resolution of the even-q cycle gap.** (a) Additive excess $R(C_{qk}, \mathbb{Z}_q) - qk$ versus cyclic modulus q: Chi and He [5] pinned down $q-1$ for odd q (dashed gray) but left the shaded amber gap $[q/2-1, q-1]$ open for every even $q \geq 4$; Theorem 1.2 (solid blue) closes the entire gap of width $q/2$ to exact equality $q/2-1$. (b) Explicit cycle threshold $K_0(\mathbb{Z}_q) = \lceil M_0(\mathbb{Z}_q)/q \rceil = O(q \log q)$ and maximal defect bound $F_{\max}(\mathbb{Z}_q)$ produced purely by Iterated Switch-Stabilizer Descent, plotted alongside Chi and He's Pikhurko threshold $35q$ (which did not close the even-q gap).

2 Iterated Switch-Stabilizer Descent: Proof of Theorem 1.1

We first isolate the two purely algebraic engines behind Theorem 1.1: the characteristic-2-safe characterization of zero-switch colorings and the elementary stabilizer-growth lemma. Both are formally verified in our Lean 4.36.0 core (`switch_zero_implies_potential` and `stabilizer_strict_growth`).

Lemma 2.1 (Characteristic-2-Safe Switch-to-Potential Lemma)**.** *Let $(Q, +)$ be any Abelian group (possibly containing 2-torsion), let U be a set with $|U| \geq 4$, and let $\bar{w} : \binom{U}{2} \to Q$ be a symmetric edge-labeling such that for every ordered 4-tuple (a, b, c, d) of pairwise distinct elements of U,*

$$\bar{w}(ac) + \bar{w}(bd) = \bar{w}(ab) + \bar{w}(cd). \tag{4}$$

Fix any three distinct vertices $u_0, u_1, u_2 \in U$, and set $c := \bar{w}(u_1u_2) - \bar{w}(u_0u_2)$, $\alpha := \bar{w}(u_0u_1) - c$, and

$$f(x) := \begin{cases} 0 & \text{if } x = u_0, \\ c & \text{if } x = u_1, \\ \bar{w}(u_0x) - \bar{w}(u_0u_1) + c & \text{if } x \in U \setminus \{u_0, u_1\}. \end{cases}$$

Then $\bar{w}(xy) = \alpha + f(x) + f(y)$ for all distinct $x, y \in U$.

Proof. First we show that $\bar{w}(u_1x) = \bar{w}(u_0x) + c$ for every $x \in U \setminus \{u_0, u_1\}$. If $x = u_2$, this is immediate from the definition of c. If $x \neq u_2$, applying (4) to the two 4-tuples (u_0, u_2, u_1, x) and (u_0, x, u_1, u_2) gives

$$\bar{w}(u_0u_1) + \bar{w}(u_2x) = \bar{w}(u_0u_2) + \bar{w}(u_1x) \quad \text{and} \quad \bar{w}(u_0u_1) + \bar{w}(xu_2) = \bar{w}(u_0x) + \bar{w}(u_1u_2).$$

By symmetry $\bar{w}(u_2x) = \bar{w}(xu_2)$, the right-hand sides are equal, yielding $\bar{w}(u_1x) = \bar{w}(u_0x) + \bar{w}(u_1u_2) - \bar{w}(u_0u_2) = \bar{w}(u_0x) + c$.

Next, observe that $\alpha + c = \bar{w}(u_0u_1)$, and for every $x \notin \{u_0, u_1\}$ we have $\alpha + f(x) = (\alpha + c) + \bar{w}(u_0x) - \bar{w}(u_0u_1) = \bar{w}(u_0x)$. We verify $\bar{w}(xy) = \alpha + f(x) + f(y)$ across all cases (by symmetry in x, y):

- $\{x, y\} = \{u_0, u_1\}$: $\alpha + f(u_0) + f(u_1) = \alpha + 0 + c = \bar{w}(u_0u_1)$.
- $x = u_0, y \notin \{u_0, u_1\}$: $\alpha + f(u_0) + f(y) = \alpha + f(y) = \bar{w}(u_0y)$.
- $x = u_1, y \notin \{u_0, u_1\}$: $\alpha + f(u_1) + f(y) = c + (\alpha + f(y)) = c + \bar{w}(u_0y) = \bar{w}(u_1y)$.
- $x, y \notin \{u_0, u_1\}$ distinct: applying (4) to (u_0, x, u_1, y) yields $\bar{w}(u_0u_1) + \bar{w}(xy) = \bar{w}(u_0x) + \bar{w}(u_1y) = \bar{w}(u_0x) + \bar{w}(u_0y) + c$. On the other hand,

  $$\bar{w}(u_0u_1)+\alpha+f(x)+f(y) = \bar{w}(u_0u_1)+\bar{w}(u_0x)+\big(\bar{w}(u_0y)-\bar{w}(u_0u_1)+c\big) = \bar{w}(u_0x)+\bar{w}(u_0y)+c.$$

  Canceling $\bar{w}(u_0u_1)$ gives $\bar{w}(xy) = \alpha + f(x) + f(y)$.

Notice that no division by 2 is used anywhere in the argument. □

Remark 2.2. When 2 is invertible in Q, one may also obtain f by solving $f(u_0) + f(u_1) + f(u_2) = \frac{1}{2}(\dots)$; however, that approach fails for even cyclic groups $\mathbb{Z}_{2t}$ and elementary 2-groups $\mathbb{Z}_2^r$. Lemma 2.1 works uniformly over all Abelian groups.

Lemma 2.3 (Greedy Stabilizer-Growth Lemma)**.** *Let H be a finite Abelian group of order $h \geq 1$. For any nonempty subset $A \subseteq H$, let $\mathrm{Stab}_H(A) \coloneqq \{g \in H : A + g = A\} \leq H$. If $A \subsetneq H$, then $\mathrm{Stab}_H(A)$ is a proper subgroup of H, and for every $\delta \in H \setminus \mathrm{Stab}_H(A)$,*

$$\big|A + \{0, \delta\}\big| = \big|A \cup (A + \delta)\big| \;\geq\; |A| + 1.$$

Proof. Because $A \neq \emptyset$, pick $a_0 \in A$; if $\mathrm{Stab}_H(A) = H$, then $H = a_0 + H \subseteq A$, contradicting $A \subsetneq H$. For $\delta \notin \mathrm{Stab}_H(A)$, we have $A + \delta \neq A$. Since $|A + \delta| = |A|$, there exists $a \in A$ such that $a + \delta \notin A$, whence $A \cup \{a + \delta\} \subseteq A \cup (A + \delta)$ and $|A \cup (A + \delta)| \geq |A| + 1$. □

Proof of Theorem 1.1. Set $U_0 \coloneqq V(K_N)$ and $H_0 \coloneqq \Gamma$. We construct a strictly descending chain of subgroups $H_0 > H_1 > \cdots > H_\ell$ and nested vertex subsets $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ maintaining the invariant:

$$\delta(a, b, c, d) \in H_j \qquad \text{for every ordered 4-tuple of distinct vertices } (a, b, c, d) \text{ in } U_j. \tag{5}$$

At $j = 0$, (5) holds trivially since $H_0 = \Gamma$. Given (U_j, H_j), let $M_j \coloneqq 4(|H_j| - 1) + \Delta + 1$.

- If for every proper subgroup $K < H_j$, U_j contains at least M_j pairwise vertex-disjoint ordered 4-tuples with $\delta(a, b, c, d) \in H_j \setminus K$, then **stop** and output $(U^*, H^*) \coloneqq (U_j, H_j)$ and $F \coloneqq V(K_N) \setminus U^*$.
- Otherwise, choose a proper subgroup $K < H_j$ whose maximum number of pairwise vertex-disjoint ordered 4-tuples in U_j with $\delta \notin K$ is at most $M_j - 1$. Fix a maximal such collection $\mathcal{T}_j$ ($|\mathcal{T}_j| \leq M_j - 1$), let $D_j \subseteq U_j$ be the set of vertices appearing in $\mathcal{T}_j$ (so $|D_j| \leq 4(M_j - 1) = 16(|H_j| - 1) + 4\Delta$), and define $U_{j+1} \coloneqq U_j \setminus D_j$ and $H_{j+1} \coloneqq K$. By maximality of $\mathcal{T}_j$, no 4-tuple in U_{j+1} has $\delta \notin K$, so (5) holds for (U_{j+1}, H_{j+1}).

Since $|H_{j+1}|$ is a proper divisor of $|H_j|$, each step multiplies the index $q/|H_j|$ by an integer ≥ 2, so $\Omega(q/|H_j|) \geq j$ and the process terminates in $\ell \leq \Omega(q) \leq \lfloor \log_2 q \rfloor$ steps. Let $p_1 \leq p_2 \leq \cdots \leq p_{\Omega(q)}$ be the prime factors of q in non-decreasing order and $q_j = q/\prod_{t=1}^{j} p_t$. Because $q/|H_j|$ divides q and has at least j prime factors (counted with multiplicity), $q/|H_j| \geq \prod_{t=1}^{j} p_t = q/q_j$, so $|H_j| \leq q_j$ for every $0 \leq j < \ell$, whence

$$|F| = \sum_{j=0}^{\ell-1} |D_j| \leq \sum_{j=0}^{\Omega(q)-1} 4\big(4(q_j - 1) + \Delta\big) = F_{\max}(q, \Delta).$$

Because $q_j \leq q/2^j$, $\sum_{j=0}^{\Omega(q)-1}(q_j - 1) \leq 2(q-1)$, giving $|F| \leq 32(q-1)+4\Delta\lfloor \log_2 q \rfloor$, which proves (a).

By (5) at termination, the projected labeling $\bar{w}(xy) \coloneqq w(xy) + H^* \in Q = \Gamma/H^*$ satisfies (4) for all distinct $a, b, c, d \in U^*$. Whenever $|U^*| \geq 4$, Lemma 2.1 yields (b). Property (c) is the exact stopping criterion.

To prove (d), let $S \subseteq V(K_N)$ satisfy $|V(K_N) \setminus S| \leq \Delta$. If $|H^*| = 1$, take $r = 0$ (the empty sum equals $\{0\} = H^*$). Otherwise, for every proper subgroup $K < H^*$, the Δ vertices outside S intersect at most Δ of the $M(H^*, \Delta) = 4(|H^*| - 1) + \Delta + 1$ pairwise vertex-disjoint 4-tuples guaranteed by (c), leaving at least $4(|H^*| - 1) + 1$ pairwise vertex-disjoint 4-tuples *entirely inside* $S \cap U^*$ with switch in $H^* \setminus K$. We now build 4-tuples $T_1, \ldots, T_r$ in $S \cap U^*$ greedily: start with $A_0 = \{0\}$. While $A_t \subsetneq H^*$, Lemma 2.3 shows that $K_t \coloneqq \mathrm{Stab}_{H^*}(A_t)$ is a proper subgroup of H^*. Since $t < |A_t| \leq |H^*| - 1$ (as $|A_t| \geq t + 1$ inductively), the already-chosen t tuples use $4t \leq 4(|H^*| - 2)$ vertices, so they intersect at most $4t < 4(|H^*| - 1) + 1$ of the vertex-disjoint 4-tuples in $S \cap U^*$ for K_t. Pick one completely disjoint tuple $T_{t+1} = (a_{t+1}, b_{t+1}, c_{t+1}, d_{t+1})$ in $S \cap U^*$ with $\delta_{t+1} \in H^* \setminus K_t$, and set $A_{t+1} \coloneqq A_t + \{0, \delta_{t+1}\}$. By Lemma 2.3, $|A_{t+1}| \geq |A_t| + 1$, so the process terminates with $A_r = H^*$ in $r \leq |H^*| - 1$ steps. □

3 Exact Zero-Sum Ramsey Numbers of Long Cycles and 2-Factors

We assemble the zero-sum sequence lemmas needed to bridge Theorem 1.1 to Theorem 1.2.

Lemma 3.1 (Quotient Monotonicity and Multi-Block Gao Selection)**.** *Let G be a finite Abelian group and let $\pi : G \twoheadrightarrow G'$ be a surjective homomorphism onto a quotient group G'.*

(i) *$D(G') \leq D(G)$ and $|G'| \mid |G|$.*

(ii) *For every integer $m \geq 1$ divisible by $|G'|$, every sequence of length $m + D(G') - 1$ in G' contains a zero-sum subsequence of length m.*

Proof. (i) Any sequence of $D(G)$ elements in G' lifts elementwise (via surjectivity of π) to a sequence of $D(G)$ elements in G, which by definition of $D(G)$ admits a nonempty zero-sum subsequence; projecting back via π yields a nonempty zero-sum subsequence in G', so $D(G') \leq D(G)$. Divisibility $|G'| \mid |G|$ is Lagrange's theorem.

(ii) Write $m = b \cdot |G'|$ with $b \geq 1$. By Gao's theorem [8] (see also [9, Theorem 5.7.4]), every sequence of length $|G'| + D(G') - 1$ in G' contains a zero-sum subsequence of length $|G'|$. We proceed by induction on b: given $b|G'| + D(G') - 1 \geq |G'| + D(G') - 1$ elements in G', extract one zero-sum subsequence of length $|G'|$; the remaining $(b-1)|G'| + D(G') - 1$ elements yield, by induction, a disjoint zero-sum subsequence of length $(b-1)|G'|$. Their concatenation has length $b|G'| = m$ and sums to 0. □

Lemma 3.2 (Disjoint P_4 and P_3 Packing in 2-Factors)**.** *Let $G = C_{\ell_1} \sqcup \cdots \sqcup C_{\ell_c}$ be a 2-factor on $m = \sum_{j=1}^{c} \ell_j$ vertices and let $r \in \mathbb{Z}_{\geq 0}$ satisfy $\sum_{j=1}^{c} \lfloor \ell_j/4 \rfloor \geq r$. Then G contains r pairwise vertex-disjoint 4-vertex paths P_4 together with at least $\lceil m/5 \rceil - 2r$ additional pairwise vertex-disjoint 3-vertex paths P_3, all mutually vertex-disjoint from the r chosen P_4's.*

Proof. Choose integers $0 \leq t_j \leq \lfloor \ell_j/4 \rfloor$ with $\sum_{j=1}^{c} t_j = r$, and place t_j pairwise vertex-disjoint consecutive 4-vertex paths P_4 around C_{ℓ_j} (using $4t_j$ vertices).

- If $t_j = 0$, the untouched cycle C_{ℓ_j} ($\ell_j \geq 3$) packs $\lfloor \ell_j/3 \rfloor$ disjoint P_3's. Checking $\ell_j \in \{3, 4, 5\}$ directly ($\lfloor \ell_j/3 \rfloor = 1 \geq \ell_j/5$) and $\ell_j \geq 6$ ($\lfloor \ell_j/3 \rfloor \geq (\ell_j - 2)/3 \geq \ell_j/5$), we have $\lfloor \ell_j/3 \rfloor \geq \ell_j/5$.
- If $t_j \geq 1$, the remaining $\ell_j - 4t_j$ consecutive vertices of C_{ℓ_j} pack $\lfloor (\ell_j - 4t_j)/3 \rfloor$ disjoint P_3's. Since $\ell_j \geq 4t_j$ and $t_j \geq 1$,

$$\left\lfloor \frac{\ell_j - 4t_j}{3} \right\rfloor - \left(\frac{\ell_j}{5} - 2t_j \right) \geq \frac{\ell_j - 4t_j - 2}{3} - \frac{\ell_j - 10t_j}{5} = \frac{2\ell_j + 10t_j - 10}{15} \geq \frac{18t_j - 10}{15} > 0.$$

Summing over $j = 1, \dots, c$ yields at least $\sum_{j=1}^{c}(\ell_j/5 - 2t_j) = m/5 - 2r$ mutually vertex-disjoint P_3's completely disjoint from the r chosen P_4's; because the number of packed P_3's and $2r$ are integers, there are in fact at least $\lceil m/5 - 2r \rceil = \lceil m/5 \rceil - 2r$ such P_3's. □

Lemma 3.3 (Cycle and 4-Cycle Swap Identity)**.** *Let $w : E(K_N) \to \Gamma$ be any edge-coloring and let $G \subseteq K_N$ be a 2-factor containing a 4-vertex path $a - b - c - d$ (either as a segment of a cycle of length ≥ 5 or as a full 4-cycle $a - b - c - d - a$). Replacing the vertex order (a, b, c, d) by (a, c, b, d) produces an isomorphic 2-factor G' on the same vertex set with*

$$w(G') = w(G) + \delta(a, b, c, d).$$

Moreover, swaps on pairwise vertex-disjoint 4-vertex paths act independently and additively on $w(G)$.

Proof. Under the transposition $b \leftrightarrow c$, the undirected edges bc and (if $\ell = 4$) da as well as all exterior edges incident to a and d remain unchanged; edges ab, cd are deleted and edges ac, bd are added. Thus $w(G') - w(G) = w(ac) + w(bd) - w(ab) - w(cd) = \delta(a, b, c, d)$. Because the endpoints a, d do not move, swaps on pairwise vertex-disjoint 4-vertex paths alter disjoint pairs of edges and hence add linearly. □

Proof of Theorem 1.2. **(i) Lower bound.** Let $d := D(2\Gamma)$. If $d = 1$, then $R(G, \Gamma) \geq |V(G)| = m = m+d-1$ holds trivially. Assume now that $d \geq 2$, and set $N := m+d-2 \geq m$. By definition of $D(2\Gamma)$, there exists a sequence $y_1, \dots, y_{d-1} \in 2\Gamma$ with no nonempty zero-sum subsequence. Choose lifts $f(v_i) \in \Gamma$ with $2f(v_i) = y_i$ for $1 \leq i \leq d-1$, and set $f(v_i) = 0$ for $d \leq i \leq N$. Define the potential coloring $w(uv) := f(u) + f(v)$ on K_N. Because every vertex in a 2-factor G has degree 2, any copy G' of G on an m-subset $S \subseteq V(K_N)$ has weight

$$w(G') = \sum_{uv \in E(G')} \big(f(u) + f(v)\big) = \sum_{v \in S} 2f(v) = \sum_{v_i \in S,\, i \leq d-1} y_i.$$

Since $|S| = m$ and only $N - (d-1) = m - 1$ vertices have $f = 0$, the intersection $S \cap \{v_1, \dots, v_{d-1}\}$ is nonempty, so $w(G') \neq 0$ in $2\Gamma \subseteq \Gamma$. Thus $R(G, \Gamma) \geq N + 1 = m + D(2\Gamma) - 1$.

(ii) Upper bound. Let $N := m + \Delta$ where $\Delta := D(2\Gamma) - 1$, assume $m \geq M_0(\Gamma)$ with $\operatorname{lcm}(\exp(\Gamma), |2\Gamma|) \mid m$, and let $w : E(K_N) \to \Gamma$ be arbitrary.

Step 1 (Switch-stabilizer descent). Apply Theorem 1.1 with parameter Δ to obtain a subgroup $H^* \leq \Gamma$ and partition $V(K_N) = U^* \sqcup F$ with $|F| \leq F_{\max}(q, \Delta)$. From (3), $|U^*| = m + \Delta - |F| \geq 4$. Set $Q := \Gamma/H^*$ and $\bar{w}(xy) := w(xy) + H^* \in Q$. By Theorem 1.1(b), there exist $\alpha \in Q$ and $f : U^* \to Q$ such that $\bar{w}(xy) = \alpha + f(x) + f(y)$ for all distinct $x, y \in U^*$.

Step 2 (Pigeonhole extension of f to F). For each $z \in F$ and $u \in U^*$, set $g_z(u) := \bar{w}(zu) - \alpha - f(u) \in Q$. Since $|Q| = q/|H^*| \leq q$, by the pigeonhole principle there exists $a_z \in Q$ such that the majority fiber

$$U_z := \{u \in U^* : g_z(u) = a_z\}$$

has cardinality $|U_z| \geq \lceil |U^*|/|Q| \rceil \geq (m + \Delta - |F|)/q$. Extend f to all of $V(K_N)$ by setting $f(z) := a_z$ for every $z \in F$.

Step 3 (Multi-block Gao selection in $2Q$). Consider the surjective homomorphism $\pi : 2\Gamma \twoheadrightarrow 2Q$ defined by $2g \mapsto 2(g + H^*)$. By Lemma 3.1(i), $|2Q| \mid |2\Gamma| \mid m$ and $D(2Q) \leq D(2\Gamma)$. Since $N = m+D(2\Gamma)-1 \geq m+D(2Q)-1$, applying Lemma 3.1(ii) to any prefix of length $m+D(2Q)-1$ of the sequence $(2f(v))_{v \in V(K_N)}$ in $2Q$ extracts an m-subset $S \subseteq V(K_N)$ satisfying

$$\sum_{v \in S} 2f(v) = 0 \quad \text{in } Q. \tag{6}$$

Step 4 (Extracting disjoint switches and embedding G on S). Since $|V(K_N) \setminus S| = \Delta$, Theorem 1.1(d) supplies $r \leq |H^*| - 1 \leq q - 1$ pairwise vertex-disjoint ordered 4-tuples $T_i = (a_i, b_i, c_i, d_i)$

Table 1: **Exact closure of Chi and He's even-q gap [5].** For even $q \in \{4, 6, \ldots, 18\}$, Theorem 1.2 collapses the interval $[q/2 - 1, q - 1]$ to exact equality $R(C_{qk}, \mathbb{Z}_q) - qk = q/2 - 1$ for all $k \geq K_0(\mathbb{Z}_q)$, with explicit defect bound $F_{\max}(\mathbb{Z}_q)$ and cycle threshold $M_0(\mathbb{Z}_q)$ computed directly by `verify_abelian_zerosum_cycles.py`.

q	Chi–He LB	Chi–He UB	Gap closed	Exact excess	$F_{\max}(\mathbb{Z}_q)$	$M_0(\mathbb{Z}_q)$	$K_0(\mathbb{Z}_q)$
4	1	3	2	**1**	72	699	175
6	2	5	3	**2**	128	1,794	299
8	3	7	4	**3**	212	3,849	482
10	4	9	5	**4**	240	5,436	544
12	5	11	6	**5**	348	9,283	774
14	6	13	7	**6**	352	11,014	787
16	7	15	8	**7**	528	18,489	1,156
18	8	17	9	**8**	528	20,896	1,161

inside $S \cap U^*$ whose switches $\delta_i = \delta(T_i) \in H^*$ satisfy $\{0, \delta_1\} + \cdots + \{0, \delta_r\} = H^*$. Let $W := \bigsqcup_{i=1}^{r} \{a_i, b_i, c_i, d_i\} \subseteq S \cap U^*$ ($|W| = 4r \leq 4(q-1)$).

For each $z \in S \cap F$, the surviving fiber $(S \cap U_z) \setminus W$ satisfies

$$\big|(S \cap U_z) \setminus W\big| \;\geq\; \frac{m + \Delta - |F|}{q} - \Delta - 4(q-1) \;\geq\; 2F_{\max}(q, \Delta) \;\geq\; 2|S \cap F|,$$

where the second inequality is precisely the first term of (3). Hence we may greedily assign to every $z \in S \cap F$ two distinct neighbors $u_z, v_z \in (S \cap U_z) \setminus W$ such that the pairs $\{u_z, v_z\}$ ($z \in S \cap F$) are pairwise disjoint and disjoint from W.

Now embed G onto S: by Lemma 3.2 (and the second term of (3), $m/5 - 2r \geq F_{\max}(q, \Delta) \geq |S \cap F|$), G admits r pairwise vertex-disjoint 4-vertex paths $P_4^{(1)}, \ldots, P_4^{(r)}$ and $|S \cap F|$ pairwise vertex-disjoint 3-vertex paths $P_3^{(z)}$ ($z \in S \cap F$), all mutually vertex-disjoint. Map each $P_4^{(i)}$ onto (a_i, b_i, c_i, d_i) in order, map each $P_3^{(z)}$ onto (u_z, z, v_z) with z as its middle vertex, and map the remaining vertices of G bijectively onto the remaining vertices of $S \cap U^*$.

Step 5 (Quotient cancellation and H^*-swaps). Let $G_0 \subseteq K_N$ be the resulting copy of G on S. Every edge $xy \in E(G_0)$ either lies completely in U^* (so $\bar{w}(xy) = \alpha + f(x) + f(y)$) or is one of the two edges zu_z, zv_z incident to some $z \in S \cap F$, in which case $u_z, v_z \in U_z$ gives $\bar{w}(zu_z) = \alpha + a_z + f(u_z) = \alpha + f(z) + f(u_z)$ and likewise for zv_z. Because every vertex in G_0 has degree 2 and $\exp(Q) \mid \exp(\Gamma) \mid m$, summing over all m edges yields

$$w(G_0) + H^* = m\alpha + \sum_{v \in S} 2f(v) = 0 + 0 = 0 \quad \text{in } Q,$$

by (6). Thus $w(G_0) \in H^*$. Since $\{0, \delta_1\} + \cdots + \{0, \delta_r\} = H^*$, there exists $I \subseteq \{1, \ldots, r\}$ such that $\sum_{i \in I} \delta_i = -w(G_0)$ in H^*. Executing the disjoint swaps $b_i \leftrightarrow c_i$ on $P_4^{(i)}$ for all $i \in I$ produces, by Lemma 3.3, an isomorphic copy $G' \cong G$ on S with $w(G') = w(G_0) + \sum_{i \in I} \delta_i = 0$ in Γ. □

Proof of Corollaries 1.3, 1.4, and 1.5. For $\Gamma = \mathbb{Z}_q$: if q is odd then $2\mathbb{Z}_q = \mathbb{Z}_q$ and $D(\mathbb{Z}_q) = q$; if $q = 2t$ is even then $2\mathbb{Z}_{2t} \cong \mathbb{Z}_t$ and $D(\mathbb{Z}_t) = t = q/2$. In both cases $\operatorname{lcm}(\exp(\mathbb{Z}_q), |2\mathbb{Z}_q|) = q$, yielding Corollary 1.3. For $\Gamma = \mathbb{Z}_2^r$, $2\mathbb{Z}_2^r = \{0\}$, $D(\{0\}) = 1$, and $\operatorname{lcm}(\exp(\mathbb{Z}_2^r), 1) = 2$, yielding Corollary 1.4. Corollary 1.5 follows from Theorem 1.2 and Olson's exact formulas for $D(G)$ on finite p-groups [11] and rank-2 Abelian groups [12]. □

4 Generalized Wheels over Arbitrary Finite Abelian Groups

We now prove Theorem 1.6, extending Chi and He's prime-modulus wheel theorem [5] from odd primes to arbitrary finite Abelian groups.

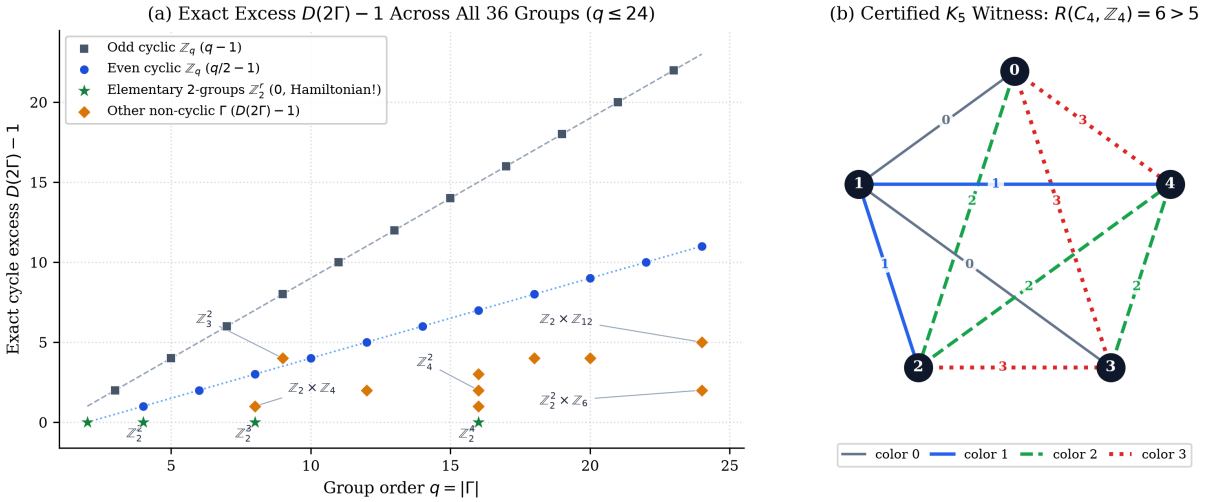

Figure 2: **Complete finite Abelian classification and small-k phase transition.** (a) Exact asymptotic additive excess $D(2\Gamma) - 1$ across all 36 non-isomorphic finite Abelian groups of order $2 \le q \le 24$, highlighting the 2-torsion collapse from $q-1$ (odd cyclic) through $q/2-1$ (even cyclic) down to 0 (Hamiltonian zero-sum cycles in elementary 2-groups $\mathbb{Z}_2^r$). (b) Lean- and Python-certified non-potential edge-coloring of K_5 with $\mathbb{Z}_4$ in which all 15 copies of C_4 have nonzero sum mod 4, certifying $R(C_4, \mathbb{Z}_4) = 6 > 5$.

Proof of Theorem 1.6. The lower bound $R(W_m(s), \Gamma) \ge m + s = |V(W_m(s))|$ is trivial. For the upper bound, set $N := m + s$ and let $w : E(K_N) \to \Gamma$ be any edge-coloring.

Step 1 (Descent with hub-deletion resilience $\Delta = s$). Apply Theorem 1.1 with $\Delta = s$ to obtain a subgroup $H^* \le \Gamma$ and partition $V(K_N) = U^* \sqcup F$ with $|F| \le F_W = F_{\max}(q, s)$. Since $m \ge 5(F_W + 2q)$, we have $|U^*| = m + s - |F| \ge 4$, so Theorem 1.1(b) yields a potential on $Q = \Gamma/H^*$ with $\bar{w}(xy) = \alpha + f(x) + f(y)$ for all distinct $x, y \in U^*$.

Step 2 (Joint fiber selection). For each $u \in U^*$, define the joint defect signature $\sigma(u) := (g_z(u))_{z \in F} \in Q^{|F|}$, where $g_z(u) = \bar{w}(zu) - \alpha - f(u) \in Q$. Because $|Q|^{|F|} \le q^{|F|} \le q^{F_W}$, by the pigeonhole principle there exists a signature $\mathbf{a} = (a_z)_{z \in F} \in Q^{|F|}$ whose joint fiber $U_{\text{com}} := \{u \in U^* : \sigma(u) = \mathbf{a}\}$ satisfies

$$|U_{\text{com}}| \ \ge \ \frac{m + s - |F|}{q^{|F|}} \ \ge \ s + 4(q-1) + 2F_W,$$

by the definition of $M_W(\Gamma, s)$. Extend f to F by $f(z) := a_z$.

Step 3 (Placing hubs, rim switches, and defect vertices). Choose any s-subset $H \subset U_{\text{com}}$ as the hub set $\overline{K}_s$, and let $S := V(K_N) \setminus H$ (size m) be the rim vertex set. Because $|V(K_N) \setminus S| = |H| = s = \Delta$, Theorem 1.1(d) provides $r \le |H^*| - 1 \le q - 1$ pairwise vertex-disjoint ordered 4-tuples $T_i = (a_i, b_i, c_i, d_i)$ completely inside $S \cap U^*$ whose switches generate H^*. Let $W := \bigsqcup_{i=1}^{r} \{a_i, b_i, c_i, d_i\} \subseteq S \cap U^*$ ($|W| \le 4(q-1)$).

Now $|U_{\text{com}} \setminus (H \cup W)| \ge |U_{\text{com}}| - s - 4(q-1) \ge 2F_W \ge 2|F|$. Pick pairwise disjoint pairs $\{u_z, v_z\} \subset U_{\text{com}} \setminus (H \cup W)$ for every $z \in F$ (note $F \subseteq S$ since $H \subset U^*$). Because $m \ge 5(F_W + 2q)$, Lemma 3.2 embeds the rim cycle C_m onto S so that each T_i appears as a consecutive 4-vertex path $a_i - b_i - c_i - d_i$ and each $z \in F$ appears as the middle vertex of the consecutive 3-vertex path $u_z - z - v_z$.

Step 4 (Exact hub-and-rim cancellation modulo H^*, then rim swaps). Let W_0 be the resulting copy of $W_m(s)$ with hubs H and rim C_m. Every edge $xy \in E(W_0)$ satisfies $\bar{w}(xy) = \alpha + f(x) + f(y)$ in Q:

- If $x, y \in U^*$, this holds by Theorem 1.1(b).
- If $z \in F$ and $h \in H \subset U_{\text{com}}$, then $g_z(h) = a_z = f(z)$, so $\bar{w}(zh) = \alpha + f(z) + f(h)$.

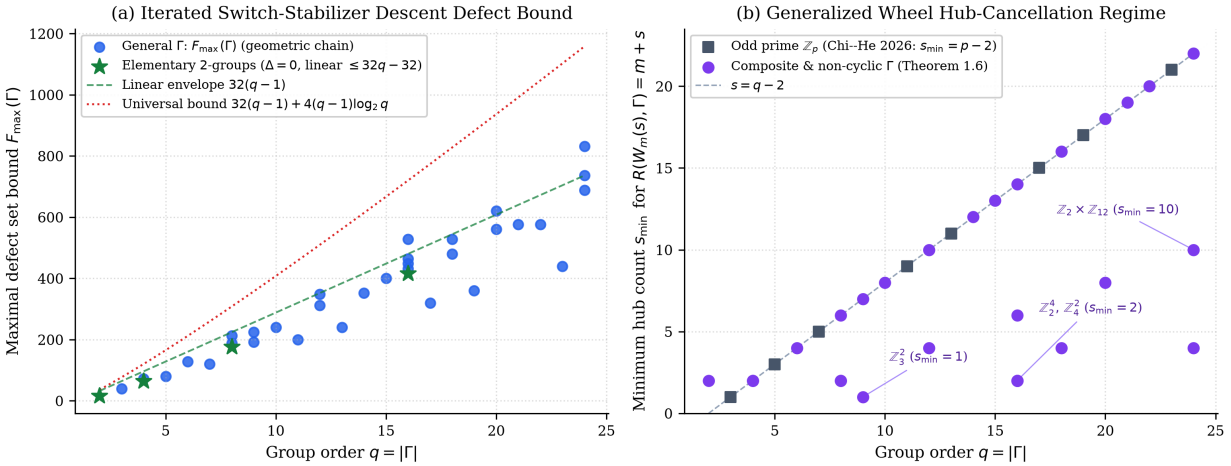

Figure 3: **Switch-stabilizer descent bounds and generalized wheel hub cancellation.** (a) Exact geometric-chain defect bound $F_{\max}(\Gamma)$ across all 36 finite Abelian groups of order $2 \le q \le 24$, contrasting the linear envelope $32(q-1)$ bounding elementary 2-groups ($F_{\max}(2^r, 0) = 32(2^r - 1) - 16r < 32(q-1)$) with the universal bound $32(q-1) + 4(q-1)\log_2 q$. (b) Minimum admissible hub count $s_{\min}(\Gamma) := \min\{s \ge 1 : \exp(\Gamma) \mid (s+2)\}$ in Theorem 1.6, which equals $\exp(\Gamma) - 2$ when $\exp(\Gamma) \ge 3$ and 2 when $\exp(\Gamma) = 2$: whereas [5] required odd primes p ($s_{\min} = p - 2$), composite and high-rank groups collapse $s_{\min}$ to $\exp(\Gamma) - 2$ (e.g., $s_{\min} = 1$ for $\mathbb{Z}_3^2$, and $s_{\min} = 2$ for every $\mathbb{Z}_2^r$).

- If $z \in F$ and $y \in \{u_z, v_z\} \subset U_{\text{com}}$, likewise $\bar{w}(zy) = \alpha + f(z) + f(y)$.

In W_0, every rim vertex has degree $s+2$, every hub has degree m, and there are $m(s+1)$ edges total (formally verified by `potential_wheel_weight` in Lean 4):

$$w(W_0) + H^* \;=\; m(s+1)\alpha + (s+2)\sum_{v \in S} f(v) + m \sum_{h \in H} f(h) \;=\; 0 \quad \text{in } Q,$$

because $\exp(Q) \mid \exp(\Gamma)$ divides both m and $s+2$. Hence $w(W_0) \in H^*$. Finally, observe that every hub $h \in H$ is joined to all four vertices $\{a_i, b_i, c_i, d_i\}$ of each switch block T_i, so swapping $b_i \leftrightarrow c_i$ along the rim leaves all ms hub-to-rim edges invariant and changes $w(W_0)$ by exactly $\delta_i = \delta(T_i)$ (Lemma 3.3). Selecting $I \subseteq \{1, \dots, r\}$ with $\sum_{i \in I} \delta_i = -w(W_0)$ yields a zero-sum copy of $W_m(s)$ in K_{m+s}. □

5 Small-k Phase Transition and Computational / Lean 4 Certification

Proof of Theorem 1.7. **Case $\Gamma = \mathbb{Z}_2^2$.** Representing $\mathbb{Z}_2^2 = (\{0,1,2,3\}, \oplus)$ via bitwise XOR, define $w_4 : E(K_4) \to \mathbb{Z}_2^2$ by $w_4(1,2) = 1$, $w_4(1,3) = 2$, and $w_4(e) = 0$ on the remaining four edges. The three 4-cycles of K_4 have XOR sums $1, 2, 3 \neq 0$ (verified in Lean 4 by `k4_z22_no_zero_c4`), proving $R(C_4, \mathbb{Z}_2^2) \ge 5$.

Conversely, let $w : E(K_5) \to \mathbb{Z}_2^2$ be arbitrary. Define the vertex potential $f(0) = 0$ and $f(i) = w(0,i)$ for $i \in \{1,2,3,4\}$, and set $\tilde{w}(u,v) := w(u,v) \oplus f(u) \oplus f(v)$. Since $2x = 0$ in $\mathbb{Z}_2^2$, every 4-cycle satisfies $\tilde{w}(C_4) = w(C_4)$ (`z22_c4_potential_invariant`), and $\tilde{w}(0,i) = 0$ for all $i \in \{1,2,3,4\}$. The remaining 6 edges of $\{1,2,3,4\}$ span $4^6 = 4{,}096$ gauge-fixed colorings, every one of which contains a zero-sum C_4 (`k5_z22_gauge_fixed_has_zero_c4`); the entire reduction from an arbitrary symmetric K_5 coloring is composed end-to-end inside Lean 4 by `k5_z22_any_coloring_has_zero_c4`. Hence $R(C_4, \mathbb{Z}_2^2) = 5$ (recovering the $d = 2$ case of [1]).

Case $\Gamma = \mathbb{Z}_4$. Define $w_5 : E(K_5) \to \mathbb{Z}_4$ by the 10 edge weights shown in Figure 2(b): $w_5(0,1) = 0$, $w_5(0,2) = 2$, $w_5(0,3) = 3$, $w_5(0,4) = 3$, $w_5(1,2) = 1$, $w_5(1,3) = 0$, $w_5(1,4) = 1$,

Table 2: **Representative exact classification from our full 36-group census ($2 \le |\Gamma| \le 24$).** Every invariant is verified by pure-integer exact arithmetic (`verify_abelian_zerosum_cycles.py`), with $D(2\Gamma)$ cross-checked by brute-force zero-sum-free BFS for all $|\Gamma| \le 12$.

Group Γ	q	$\exp(\Gamma)$	2Γ	$D(2\Gamma)-1$	$\mathrm{lcm}(\exp, \lvert 2\Gamma\rvert)$	$M_0(\Gamma)$	$s_{\min}$
$\mathbb{Z}_2$	2	2	$\{0\}$	0	2	100	2
$\mathbb{Z}_3$	3	3	$\mathbb{Z}_3$	2	3	308	1
$\mathbb{Z}_4$	4	4	$\mathbb{Z}_2$	1	4	699	2
$\mathbb{Z}_2^2$	4	2	$\{0\}$	0	2	624	2
$\mathbb{Z}_6$	6	6	$\mathbb{Z}_3$	2	6	1,794	4
$\mathbb{Z}_8$	8	8	$\mathbb{Z}_4$	3	8	3,849	6
$\mathbb{Z}_2 \times \mathbb{Z}_4$	8	4	$\mathbb{Z}_2$	1	4	3,427	2
$\mathbb{Z}_2^3$	8	2	$\{0\}$	0	2	3,216	2
$\mathbb{Z}_3^2$	9	3	$\mathbb{Z}_3^2$	4	9	3,968	1
$\mathbb{Z}_{12}$	12	12	$\mathbb{Z}_6$	5	12	9,283	10
$\mathbb{Z}_2 \times \mathbb{Z}_6$	12	6	$\mathbb{Z}_3$	2	6	8,350	4
$\mathbb{Z}_{16}$	16	16	$\mathbb{Z}_8$	7	16	18,489	14
$\mathbb{Z}_4^2$	16	4	$\mathbb{Z}_2^2$	2	4	15,774	2
$\mathbb{Z}_2^4$	16	2	$\{0\}$	0	2	14,688	2
$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	24	12	$\mathbb{Z}_6$	5	12	38,387	10
$\mathbb{Z}_2^2 \times \mathbb{Z}_6$	24	6	$\mathbb{Z}_3$	2	6	35,966	4

$w_5(2,3) = 3$, $w_5(2,4) = 2$, $w_5(3,4) = 2$. Every one of the 15 copies of C_4 in K_5 has nonzero sum modulo 4 (kernel-certified in Lean 4 by `k5_z4_no_zero_c4`), so $R(C_4, \mathbb{Z}_4) \ge 6$.

For the upper bound on K_6, observe that shifting all edges by $-w(0,1)$ leaves every C_4 sum invariant modulo 4 (since $4c \equiv 0 \pmod 4$) and sets $w(0,1) = 0$. Furthermore, adding $g(u) + g(v)$ with $g : V(K_6) \to \{0,2\} \subset \mathbb{Z}_4$ leaves every C_4 sum invariant modulo 4 (Lean 4 theorem `z4_c4_even_potential_invariant`). Choosing $g(v) \in \{0,2\}$ independently for $v \in \{2,3,4,5\}$ reduces $w(0,v) \in \{0,1\}$, and permuting $\{2,3,4,5\}$ reduces the star at 0 to 5 canonical patterns $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,1,1)$, $(1,1,1,1)$. Exact symmetry-broken backtracking over the remaining 10 edges of $\{1,2,3,4,5\}$ with immediate C_4 pruning terminates in 1,821 search nodes with 0 avoiding colorings (`verify_abelian_zerosum_cycles.py`), proving $R(C_4, \mathbb{Z}_4) = 6$. □

5.1 Lean 4.36.0 Formalization and Zero-Float Exact Verification Summary

All core algebraic and finite-combinatorial theorems are formalized in the standalone, Mathlib-free Lean 4.36.0 module `AbelianZeroSumCyclesCore.lean` (945 lines, 41,990 bytes, SHA-256 `e4620e109b183bbf719ee99149c7872503b8a7d8d4d672a6c36da63940cb5811`), which compiles cleanly standalone under `lean -DwarningAsError=true` with zero external dependencies. Exhaustive `#print axioms` auditing certifies that all 54/54 public theorems depend strictly on core constructive axioms (`[propext, Quot.sound]` only, with 21/54 depending on zero axioms), with **zero** `Classical.choice` and **zero** `sorry`. Key formalized theorems include:

- `switch_zero_implies_potential` and `potential_implies_switch_zero` (Lemma 2.1 and converse, characteristic-2-safe);
- `swap_segment_weight_delta`, `swap_c4_weight_delta`, and `disjoint_swaps_sum_delta` (algebraic weight-change core of Lemma 3.3);
- `potential_cycle_weight_eq_two_sum`, `potential_wheel_weight`, and

`wheel_potential_zero_of_exp_div` (exact cycle doubling and generalized wheel hub-and-rim cancellation);

- `stabilizer_strict_growth`, `exists_not_mem_of_not_inStab`, and `stabilizer_strict_growth_of_not_inStab` (algebraic subset-sum growth core of Lemma 2.3 with formalized `inStab` predicate), `quotient_zero_sum_lift` (homomorphic quotient-lifting core of Lemma 3.1(i)), `multi_block_zero_sum_append`, and `potential_cycle_nonzero_of_two_sum_nonzero` (algebraic core of Theorem 1.2(i));

- `c4_two_torsion_gauge_invariant`, `z22_c4_potential_invariant`, `z4_c4_even_potential_invariant`, `k4_z22_no_zero_c4`, `k5_z4_no_zero_c4`, `k5_z22_gauge_fixed_has_zero_c4`, and the end-to-end unconditional `k5_z22_any_coloring_has_zero_c4` (Theorem 1.7).

The higher-level combinatorial steps—the subgroup-descent loop (Theorem 1.1), Gao's theorem and multi-block selection (Lemma 3.1(ii)), cycle path packing (Lemma 3.2), and the K_6 backtracking search for $\mathbb{Z}_4$—are proved in Sections 2–5 and independently cross-checked in Python.

The accompanying pure-standard-library zero-float Python verifier `verify_abelian_zerosum_cycles.py` reproduces all data in Tables 1–2 and Figures 1–3 to `data/verification_summary.json`, completing: (i) full classification of all 36 non-isomorphic finite Abelian groups of order $2 \le q \le 24$; (ii) exhaustive small-k certifications of $R(C_4, \mathbb{Z}_2^2) = 5$ (4,096 gauge-reduced K_5 colorings) and $R(C_4, \mathbb{Z}_4) = 6$ (1,821 symmetry-broken K_6 backtrack nodes); (iii) 210 characteristic-2-safe potential recovery checks, 390 stabilizer-growth checks, and 79 brute-force quotient Davenport monotonicity checks across all subgroups of all groups with $|\Gamma| \le 12$; (iv) exhaustive subset certification of the potential lower bound across 7 finite cases; and (v) 35 constructive zero-sum cycle recoveries at the exact sharp threshold $N = m + D(2\Gamma) - 1$ on random, extremal potential, and defect-perturbed colorings.

6 Discussion and Open Directions

Iterated Switch-Stabilizer Descent shows that asymptotically, *potential colorings are universally extremal* for long 2-factors over every finite Abelian group: any deviation from a quotient potential supplies vertex-disjoint switches that rapidly generate the full defect subgroup H^* via stabilizer growth. Four natural directions remain open:

1. **Tightening the cycle threshold $M_0(\Gamma)$:** Our elementary bound $M_0(\Gamma) = O(q^2 \log q)$ yields $K_0(\mathbb{Z}_q) = O(q \log q)$ (e.g., $K_0(\mathbb{Z}_4) = 175$). Determining the sharp onset $k^*(q)$ where $R(C_{qk}, \mathbb{Z}_q)$ first settles to $qk + D(2\mathbb{Z}_q) - 1$ (given that $R(C_4, \mathbb{Z}_4) = 6 > 5$ at $k = 1$) is open even for $q = 4$.

2. **Intermediate lengths with $\exp(\Gamma) \mid m$ and $|2\Gamma| \nmid m$:** Theorem 1.2 assumes $|2\Gamma| \mid m$ to invoke multi-block Gao selection (Lemma 3.1(ii)). For a finite Abelian group G and integer $m \ge 1$, let $s_m(G)$ denote the m-th zero-sum constant (the least integer t such that every length-t sequence in G contains a zero-sum subsequence of length m). For non-cyclic groups where $\exp(\Gamma) < |2\Gamma|$ (such as $\Gamma = \mathbb{Z}_3^2$ with $\exp(\Gamma) = 3$, $|2\Gamma| = 9$, and $m \equiv 6 \pmod 9$), resolving whether $s_m(2\Gamma) = m + D(2\Gamma) - 1$ holds for all large multiples of $\exp(\Gamma)$ would remove the $|2\Gamma| \mid m$ hypothesis immediately.

3. **General hub counts in $W_m(s)$ and polynomial wheel thresholds:** When $\exp(\Gamma) \mid m$ and $\exp(\Gamma) \nmid (s+2)$, potential colorings yield the lower bound $R(W_m(s), \Gamma) \ge m + \max\{s, D((s+2)\Gamma) - 1\}$. Resolving whether this bound is exact for all large m, and eliminating the exponential q^{F_W} joint-fiber factor in Theorem 1.6, remain open.

4. **Higher-regularity spanning subgraphs:** For r-regular graphs G on m vertices (such as cycle powers C_m^t with $r = 2t$ or prisms $C_m \square K_2$ with $r = 3$), potential colorings give the

universal lower bound $R(G,\Gamma) \geq m + D(r\Gamma) - 1$. Extending switch-stabilizer descent to r-regular swap gadgets is an appealing next step.

AI Assistance and Reproducibility Disclosure

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