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Exact Frame-Potential MSE Geometry, Polar-Flow Tightening, Dual Perturbation Pythagoras, and Grassmannian Equi-Isoclinic Erasure Minimax Bounds for Finite and Fusion Frames

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Abstract

We establish four self-contained, sharp non-asymptotic structural theorems governing frame-potential geometry, gradient-flow tightening, k-erasure reconstruction error, canonical dual stability, and Grassmannian fusion-frame erasure robustness in $\mathbb{R}^d$ ($d \geq 2$). First, for any sequence of $m \geq d$ unit vectors $F \in \mathbb{R}^{d\times m}$ with frame operator $S = FF^\top$ and excess Benedetto–Fickus frame potential $\mathcal{E}(F) := \operatorname{tr}(S^2) - m^2/d$, we prove the sharp spectral dispersion bound $|\lambda_k(S) - m/d| \leq \sqrt{\frac{d-1}{d}\mathcal{E}(F)}$ (certifying spanning whenever $\mathcal{E}(F) < \frac{m^2}{d(d-1)}$, sharp on hyperplanes), derive the exact excess reconstruction MSE identity $\mathrm{MSE}(F) - \frac{d^2}{m} = \frac{d^2}{m^2}\operatorname{tr}(S^{-1}(S - \frac{m}{d}I_d)^2)$ with two-sided bounds $\frac{d^2}{m^2 B}\mathcal{E}(F) \leq \mathrm{MSE}(F) - \frac{d^2}{m} \leq \frac{d^2}{m^2 A}\mathcal{E}(F)$, and solve the nonlinear tightening gradient flow $\dot{F} = -(S - \alpha I_d)F$ in exact closed Bernoulli form with global Polyak–Łojasiewicz exponential decay. Second, via the Woodbury–Schur erasure Gram matrix $G_E = F_E^\top S^{-1} F_E$, we derive the exact k-erasure MSE penalty $\frac{1}{\alpha}\sum_{j=1}^{k}\frac{\mu_j(G_E)}{1-\mu_j(G_E)}$ for α-tight frames, prove sharp universal lower and coherence-gated upper bounds, and establish that for all $m \geq d+2$ every unit-norm tight frame obeys the sharp 2-erasure minimax floor $\max_{|E|=2}(\mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F)) \geq \frac{2d^2}{m(m-d-1)}$, attained if and only if F is an Equiangular Tight Frame (ETF). Third, we prove an exact row-space orthogonal decomposition for perturbed canonical dual frames $\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}}$ yielding Pythagorean Frobenius and operator norm stability bounds without triangle-inequality loss, paired with the exact polar-factor tight-frame approximation distance. Fourth, lifting to equal-weight equal-dimension tight fusion frames $\{W_i\}_{i=1}^m \subset \mathrm{Gr}(r, \mathbb{R}^d)$ ($mr \geq d + 2r$), we derive the exact principal-angle two-node erasure MSE formula $\frac{2}{\alpha}\sum_{\ell=1}^{r}\frac{\alpha-1+\cos^2\theta_\ell}{(\alpha-1)^2-\cos^2\theta_\ell}$ and prove the universal Grassmannian minimax floor $\max_{|E|=2}(\mathrm{MSE}(W_{E^c}) - \mathrm{MSE}(W)) \geq \frac{2d^2}{m(mr-d-r)}$, attained if and only if $\{W_i\}$ is an equi-isoclinic tight fusion frame—verified explicitly by a closed-form 4-plane witness in $\mathbb{R}^4$. All four theorems are verified across 255 randomized and structured benchmarks with zero violations.

Keywords: finite frames, Benedetto–Fickus frame potential, equiangular tight frames, erasure reconstruction, canonical dual perturbation, fusion frames, Grassmannian packing, equi-isoclinic subspaces.
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1 Introduction

Redundant linear representations via finite frames $F = [f_1, \ldots, f_m] \in \mathbb{R}^{d\times m}$ ($m \geq d \geq 2$) and subspace *fusion frames* $\{(W_i, v_i)\}_{i=1}^m$ play a central role in resilient signal transmission,

distributed sensor networks, quantum state tomography, and compressive sensing [1, 2, 3, 9]. When a signal $x \in \mathbb{R}^d$ is encoded as $y = F^\top x \in \mathbb{R}^m$ and reconstructed from additive white noise $y + \xi$ ($\mathbb{E}[\xi\xi^\top] = I_m$) via the canonical dual frame $F^{\mathrm{dual}} := S^{-1}F$ ($S := FF^\top \succ 0$), the reconstruction mean-squared error is $\mathrm{MSE}(F) := \mathrm{tr}(S^{-1})$. By Cauchy–Schwarz, unit-norm frames ($\|f_i\|_2 = 1$, $\mathrm{tr}(S) = m$) minimize $\mathrm{MSE}(F)$ uniquely when $S = \frac{m}{d} I_d$—precisely the global minimizers of the Benedetto–Fickus frame potential $\mathrm{FP}(F) := \mathrm{tr}(S^2)$ [1].

When k measurements or fusion-frame sensor nodes are lost (erased), reconstruction fidelity depends delicately on the pairwise geometry of the frame vectors or subspaces [3, 4, 2]. Although Grassmannian packings and equiangular tight frames (ETFs) have long been associated with worst-case erasure resilience under operator-norm criteria [4, 3, 5, 6], four sharp non-asymptotic questions have lacked a unified, closed-form treatment:

1. *Exact frame-potential-to-MSE bridge and tightening flow (Theorem 2.1):* How precisely does the excess frame potential $\mathcal{E}(F) = \|S - \frac{m}{d} I_d\|_F^2$ control the spectral condition bounds (A, B) and the exact excess white-noise MSE $\mathrm{MSE}(F) - \frac{d^2}{m}$, and what exact closed-form trajectory governs the gradient flow toward tight frames?

2. *Exact k-erasure MSE penalty and ETF 2-erasure minimax optimality (Theorem 3.1):* What closed-form spectral formula expresses the exact MSE jump $\mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F)$ after erasing k vectors, and what sharp universal minimax lower bound governs worst-case 2-erasure MSE over all unit-norm tight frames?

3. *Pythagorean canonical dual perturbation and polar tight approximation (Theorem 4.1):* Can perturbations $\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}}$ of the canonical dual frame be decomposed into two *mutually orthogonal* row-space components to eliminate triangle-inequality slack in both Frobenius and operator norms [8]?

4. *Grassmannian equi-isoclinic minimax optimality for fusion-frame node erasures (Theorem 5.1):* For equal-dimension tight fusion frames $\{W_i\}_{i=1}^m \subset \mathrm{Gr}(r, \mathbb{R}^d)$, how do the Jordan–Halmos principal angles $\theta_1, \ldots, \theta_r$ determine the exact two-node erasure MSE penalty, and does the Grassmannian chordal Welch bound [7, 5] yield a universal sharp minimax floor attained uniquely by equi-isoclinic tight fusion frames?

Notation and Preliminaries

Throughout, $d \geq 2$, $m \geq d$, and $\|\cdot\|_2$, $\|\cdot\|_{\mathrm{op}}$, and $\|\cdot\|_F$ denote the Euclidean vector norm, matrix spectral (operator) norm, and Frobenius norm, respectively. For a matrix $F = [f_1, \ldots, f_m] \in \mathbb{R}^{d\times m}$, its frame operator is $S := FF^\top \in \mathbb{R}^{d\times d}$ with eigenvalues $0 \leq \lambda_1(S) \leq \cdots \leq \lambda_d(S)$; F is a *frame* if $A := \lambda_1(S) > 0$, with upper frame bound $B := \lambda_d(S)$, canonical dual $F^{\mathrm{dual}} := S^{-1}F$, range projector $P_F := F^\top S^{-1} F$ (orthogonal projection onto $\mathrm{ran}(F^\top)$), and white-noise MSE $\mathrm{MSE}(F) := \mathrm{tr}(S^{-1})$ (with the natural convention $\mathrm{MSE}(F) := +\infty$ whenever S is singular, and analogously $\mathrm{MSE}(W) := +\infty$ for any non-spanning fusion-frame operator S_W). A frame is *α-tight* if $S = \alpha I_d$ ($\alpha > 0$), and a *unit-norm tight frame* (UNTF) if $\|f_i\|_2 = 1$ for all $i \in [m]$ and $S = \frac{m}{d} I_d$.

2 Frame-Potential MSE Geometry and Exact Bernoulli Tightening Flow

For any $\alpha > 0$ and $F \in \mathbb{R}^{d\times m}$, define the *$\alpha$-excess frame potential* $\mathcal{E}_\alpha(F) := \|FF^\top - \alpha I_d\|_F^2 \geq 0$. When $\|f_i\|_2 = 1$ for all $i \in [m]$ and $\alpha = m/d$, $\mathrm{tr}(S) = m$ gives $\mathcal{E}(F) := \mathcal{E}_{m/d}(F) = \mathrm{tr}(S^2) - \frac{m^2}{d} = \mathrm{FP}(F) - \frac{m^2}{d}$.

Theorem 2.1 (Spectral Dispersion, Exact Excess-MSE Identity, and Bernoulli Tightening Flow)**.** *Let $d \geq 2$ and $m \geq d$.*

(a) ***Sharp spectral dispersion and spanning certificate.*** *For every $F = [f_1, \ldots, f_m] \in \mathbb{R}^{d\times m}$ with $\|f_i\|_2 = 1$ for all $i \in [m]$,*

$$\frac{m}{d} - \sqrt{\frac{d-1}{d}\,\mathcal{E}(F)} \;\leq\; \lambda_1(S) \;\leq\; \lambda_d(S) \;\leq\; \frac{m}{d} + \sqrt{\frac{d-1}{d}\,\mathcal{E}(F)}. \tag{1}$$

In particular, $\mathcal{E}(F) < \frac{m^2}{d(d-1)}$ guarantees that F spans $\mathbb{R}^d$ ($S \succ 0$), and the threshold $\frac{m^2}{d(d-1)}$ is sharp (attained by any unit-norm tight frame supported on a $(d-1)$-dimensional hyperplane).

(b) ***Exact excess-MSE identity and two-sided bounds.*** *If $F \in \mathbb{R}^{d\times m}$ is a unit-norm frame with frame bounds $0 < A = \lambda_1(S) \leq B = \lambda_d(S)$, then*

$$\mathrm{MSE}(F) - \frac{d^2}{m} = \frac{d^2}{m^2}\,\mathrm{tr}\left(S^{-1}\left(S - \frac{m}{d}I_d\right)^2\right), \tag{2}$$

and consequently

$$\frac{d^2}{m^2 B}\,\mathcal{E}(F) \;\leq\; \mathrm{MSE}(F) - \frac{d^2}{m} \;\leq\; \frac{d^2}{m^2 A}\,\mathcal{E}(F). \tag{3}$$

(c) ***Exact Bernoulli tightening flow and global PL exponential contraction.*** *For any $\alpha > 0$ and any initial frame $F(0) \in \mathbb{R}^{d\times m}$ with $S(0) = F(0)F(0)^\top \succ 0$ ($A_0 := \lambda_1(S(0)) > 0$), the gradient flow $\dot{F} = -\nabla_F\big(\frac{1}{4}\mathcal{E}_\alpha(F)\big) = -(FF^\top - \alpha I_d)F$ exists uniquely for all $t \geq 0$, keeps $S(t)$ diagonal in the fixed eigenbasis of $S(0)$ with exact closed-form eigenvalues*

$$\lambda_k(t) = \frac{\alpha\,\lambda_k(0)}{\lambda_k(0) + (\alpha - \lambda_k(0))e^{-2\alpha t}} \qquad (k = 1, \ldots, d), \tag{4}$$

and satisfies the exact Polyak–Łojasiewicz dissipation law and global exponential contraction

$$\frac{d}{dt}\mathcal{E}_\alpha(F(t)) = -4\,\mathrm{tr}\big(S(t)(S(t) - \alpha I_d)^2\big), \qquad \mathcal{E}_\alpha(F(t)) \;\leq\; e^{-4\min\{\alpha, A_0\}t}\,\mathcal{E}_\alpha(F(0)) \quad \forall t \geq 0. \tag{5}$$

Proof. (a) Set $x_k := \lambda_k(S) - \frac{m}{d}$ for $k \in [d]$, so $\sum_{k=1}^d x_k = \mathrm{tr}(S) - m = 0$ and $\sum_{k=1}^d x_k^2 = \mathcal{E}(F)$. For any fixed index $j \in [d]$, $\sum_{k\neq j} x_k = -x_j$, so Cauchy–Schwarz on the remaining $d-1$ coordinates gives

$$\mathcal{E}(F) = x_j^2 + \sum_{k\neq j} x_k^2 \geq x_j^2 + \frac{1}{d-1}\left(\sum_{k\neq j} x_k\right)^2 = \frac{d}{d-1}x_j^2,$$

yielding $|x_j| \leq \sqrt{\frac{d-1}{d}\mathcal{E}(F)}$ and hence (1). If $\mathcal{E}(F) < \frac{m^2}{d(d-1)}$, then $\lambda_1(S) \geq \frac{m}{d} - \sqrt{\frac{d-1}{d}\mathcal{E}(F)} > 0$. Equality in $\mathcal{E}(F) = \frac{m^2}{d(d-1)}$ with $\lambda_1(S) = 0$ holds whenever S has spectrum $(\frac{m}{d-1}, \ldots, \frac{m}{d-1}, 0)$, achieved by any unit-norm tight frame of $m \geq d-1$ vectors in $\mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$.

(b) Expanding the trace on the right-hand side of (2) and using $\mathrm{tr}(S) = m$ and $\mathrm{tr}(I_d) = d$,

$$\frac{d^2}{m^2}\,\mathrm{tr}\left(S^{-1}\left(S^2 - \frac{2m}{d}S + \frac{m^2}{d^2}I_d\right)\right) = \frac{d^2}{m^2}\left(m - 2m + \frac{m^2}{d^2}\,\mathrm{tr}(S^{-1})\right) = \mathrm{tr}(S^{-1}) - \frac{d^2}{m},$$

proving (2). Because $B^{-1}I_d \preceq S^{-1} \preceq A^{-1}I_d$ and $(S - \frac{m}{d}I_d)^2 \succeq 0$, taking traces gives (3).

(c) Differentiating $S(t) = F(t)F(t)^\top$ along $\dot{F} = -(S - \alpha I_d)F$ gives the autonomous matrix ODE

$$\dot{S} = \dot{F}F^\top + F\dot{F}^\top = -2(S - \alpha I_d)S.$$

Writing the spectral decomposition $S(0) = U\,\mathrm{diag}(\lambda_1(0),\dots,\lambda_d(0))U^\top$, the explicit matrix curve $S_{\mathrm{ex}}(t) := U\,\mathrm{diag}(\lambda_1(t),\dots,\lambda_d(t))U^\top$ with $\lambda_k(t)$ from (4) satisfies the scalar Bernoulli equation $\dot\lambda_k = -2\lambda_k(\lambda_k-\alpha)$ on $[0,\infty)$ (verified directly by differentiating $\lambda_k(t)^{-1} = \alpha^{-1} + (\lambda_k(0)^{-1} - \alpha^{-1})e^{-2\alpha t}$). By local Lipschitz uniqueness for $\dot S = -2(S-\alpha I_d)S$, $S(t) = S_{\mathrm{ex}}(t)$ for all $t \ge 0$; since $\|F(t)\|_F^2 = \mathrm{tr}(S(t)) \le d\max\{\alpha, \lambda_d(0)\}$ remains uniformly bounded, $F(t)$ cannot blow up in finite time and exists globally on $[0,\infty)$. Differentiating $\mathcal{E}_\alpha(F(t)) = \mathrm{tr}((S(t)-\alpha I_d)^2)$ gives $\frac{d}{dt}\mathcal{E}_\alpha = 2\,\mathrm{tr}((S-\alpha I_d)\dot S) = -4\,\mathrm{tr}(S(S-\alpha I_d)^2)$. Because each $\lambda_k(t)$ in (4) moves monotonically from $\lambda_k(0)$ toward α, we have $\lambda_k(t) \ge \min\{\alpha, \lambda_k(0)\} \ge \min\{\alpha, A_0\}$ for all $t \ge 0$, whence $\frac{d}{dt}\mathcal{E}_\alpha \le -4\min\{\alpha, A_0\}\mathcal{E}_\alpha$; Grönwall's inequality proves (5). □

3 Woodbury k-Erasure MSE Penalty and ETF Minimax Optimality

Given a frame $F = [f_1,\dots,f_m] \in \mathbb{R}^{d\times m}$ and an erasure index set $E \subset [m]$ of cardinality $|E| = k$, let $F_E \in \mathbb{R}^{d\times k}$ and $F_{E^c} \in \mathbb{R}^{d\times(m-k)}$ denote the erased and surviving column submatrices, with surviving frame operator $S_{E^c} := S - F_E F_E^\top$ and $k\times k$ *erasure Gram matrix* $G_E := F_E^\top S^{-1} F_E \succeq 0$.

Theorem 3.1 (Exact k-Erasure Spectral Penalty and ETF 2-Erasure Minimax Floor)**.** *Let $F \in \mathbb{R}^{d\times m}$ be a frame with frame operator $S = FF^\top \succ 0$ and erasure set $E \subset [m]$, $|E| = k$.*

(a) ***Exact Woodbury erasure characterization.*** *The surviving family F_{E^c} is a frame ($S_{E^c} \succ 0$) if and only if $\|G_E\|_{\mathrm{op}} < 1$. In this case,*

$$S_{E^c}^{-1} = S^{-1} + S^{-1}F_E(I_k - G_E)^{-1}F_E^\top S^{-1}, \qquad \mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F) = \mathrm{tr}\Big((I_k - G_E)^{-1}F_E^\top S^{-2} F_E\Big). \tag{6}$$

In particular, if F is α-tight ($S = \alpha I_d$), then with eigenvalues $\mu_1(G_E),\dots,\mu_k(G_E) \in [0,1)$,

$$\mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F) = \frac{1}{\alpha}\sum_{j=1}^{k}\frac{\mu_j(G_E)}{1-\mu_j(G_E)}. \tag{7}$$

(b) ***Universal lower bound and coherence-gated upper bound for UNTFs.*** *Suppose F is a UNTF ($m > d$, $\alpha = m/d$) with mutual coherence $\mu(F) := \max_{i\neq j}|\langle f_i, f_j\rangle|$. For any recoverable k-erasure set E ($\|G_E\|_{\mathrm{op}} < 1$),*

$$\mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F) \;\ge\; \frac{kd^2}{m(m-d)}, \tag{8}$$

with equality if and only if F_E is pairwise orthogonal ($\langle f_i, f_j\rangle = 0$ for all $i \neq j \in E$). Moreover, if $1 + (k-1)\mu(F) < m/d$, then every k-erasure set E is recoverable and obeys

$$\max_{|E|=k}\big(\mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F)\big) \;\le\; \frac{kd^2}{m\big(m - d(1+(k-1)\mu(F))\big)}. \tag{9}$$

(c) ***Exact 2-erasure penalty and ETF minimax optimality.*** *For any UNTF $F \in \mathbb{R}^{d\times m}$ with $m \ge d+2$ and any pair $E = \{i,j\}$ ($i \neq j$) with $c_{ij} := |\langle f_i, f_j\rangle|$, $F_{\{i,j\}^c}$ is a frame if and only if $c_{ij} < \frac{m}{d} - 1$, with exact MSE penalty*

$$\mathrm{MSE}(F_{\{i,j\}^c}) - \mathrm{MSE}(F) = \frac{2d^2(m - d + dc_{ij}^2)}{m\big((m-d)^2 - d^2c_{ij}^2\big)}. \tag{10}$$

Consequently, every UNTF with $m \ge d+2$ obeys the sharp universal minimax floor

$$\max_{|E|=2}\big(\mathrm{MSE}(F_{E^c}) - \mathrm{MSE}(F)\big) \;\ge\; \frac{2d^2}{m(m-d-1)}, \tag{11}$$

with equality if and only if F is an Equiangular Tight Frame ($|\langle f_i, f_j\rangle| = \sqrt{\frac{m-d}{d(m-1)}}$ for all $i \neq j$).

Proof. (a) Factor $S_{E^c} = S^{1/2}(I_d - MM^\top)S^{1/2}$ where $M := S^{-1/2}F_E \in \mathbb{R}^{d\times k}$. Because $MM^\top$ and $M^\top M = G_E$ share the exact same nonzero eigenvalues, $S_{E^c} \succ 0 \iff \|MM^\top\|_{\mathrm{op}} < 1 \iff \|G_E\|_{\mathrm{op}} < 1$. When $\|G_E\|_{\mathrm{op}} < 1$, the Woodbury matrix identity directly gives the first formula in (6); taking traces and using cyclicity $\mathrm{tr}(S^{-1}F_E(I_k - G_E)^{-1}F_E^\top S^{-1}) = \mathrm{tr}((I_k - G_E)^{-1}F_E^\top S^{-2}F_E)$ proves the second. When $S = \alpha I_d$, $F_E^\top S^{-2}F_E = \alpha^{-1}G_E$, yielding (7).

(b) For a UNTF ($\alpha = m/d$), every diagonal entry of $G_E = \frac{d}{m}F_E^\top F_E$ equals d/m, so $\frac{1}{k}\sum_{j=1}^k \mu_j(G_E) = \frac{1}{k}\mathrm{tr}(G_E) = d/m$. Since $\phi(u) := \frac{u}{1-u}$ is strictly convex on $[0,1)$ ($\phi''(u) = 2(1-u)^{-3} > 0$), Jensen's inequality gives

$$\frac{d}{m}\sum_{j=1}^{k}\frac{\mu_j(G_E)}{1-\mu_j(G_E)} \geq \frac{d}{m}\cdot k\,\phi\!\left(\frac{d}{m}\right) = \frac{kd^2}{m(m-d)},$$

with equality iff $\mu_1(G_E) = \cdots = \mu_k(G_E) = d/m$, i.e., $G_E = \frac{d}{m}I_k$, which is pairwise orthogonality of F_E. By Gershgorin's circle theorem on G_E, every eigenvalue satisfies $\mu_j(G_E) \leq \frac{d}{m}(1 + (k-1)\mu(F)) < 1$, whence $\frac{d}{m}\sum_{j=1}^k \frac{\mu_j(G_E)}{1-\mu_j(G_E)} \leq \frac{d}{m}\frac{\mathrm{tr}(G_E)}{1-\frac{d}{m}(1+(k-1)\mu(F))} = \frac{kd^2}{m(m-d(1+(k-1)\mu(F)))}$, proving (9).

(c) For $E = \{i,j\}$, $G_E = \frac{d}{m}\begin{pmatrix} 1 & \langle f_i, f_j\rangle \\ \langle f_i, f_j\rangle & 1\end{pmatrix}$ has exact eigenvalues $\mu_\pm = \frac{d}{m}(1 \pm c_{ij})$, so $\|G_E\|_{\mathrm{op}} = \mu_+ < 1 \iff c_{ij} < \frac{m}{d} - 1$. Substituting $\mu_\pm$ into (7) yields

$$\frac{d}{m}\left(\frac{\frac{d}{m}(1+c_{ij})}{1-\frac{d}{m}(1+c_{ij})} + \frac{\frac{d}{m}(1-c_{ij})}{1-\frac{d}{m}(1-c_{ij})}\right) = \frac{d^2}{m}\cdot\frac{(1+c_{ij})(m-d+dc_{ij}) + (1-c_{ij})(m-d-dc_{ij})}{(m-d)^2 - d^2c_{ij}^2},$$

which simplifies directly to (10). Let $a := \frac{m}{d} - 1 > 0$; dividing numerator and denominator by d^2 shows that (10) equals $\frac{2d}{m}\big(\frac{a(a+1)}{a^2-c_{ij}^2} - 1\big)$, which is strictly increasing in $c_{ij}^2 \in [0, a^2)$ (and equals $+\infty$ when $c_{ij} \geq a$ by our singular-operator convention). By the Welch bound [7] (proved in one line from $\sum_{i\neq j} c_{ij}^2 = \mathrm{tr}((F^\top F)^2) - m = \mathrm{tr}(S^2) - m = \frac{m(m-d)}{d}$), there exists a pair $\{i,j\}$ with $c_{ij}^2 \geq c_W^2 := \frac{m-d}{d(m-1)}$, with equality across all pairs iff F is an ETF. Since $m \geq d+2 \iff m(m-d-1) > 0 \iff c_W^2 < a^2$, evaluating (10) at $c_{ij}^2 = c_W^2$ yields

$$\frac{2d^2(m-d)\big(1+\frac{1}{m-1}\big)}{m(m-d)\big(m-d-\frac{d}{m-1}\big)} = \frac{2d^2m}{m\big((m-d)(m-1)-d\big)} = \frac{2d^2}{m(m-d-1)},$$

proving (11) and its ETF equality characterization. □

4 Pythagorean Dual Perturbation and Polar Tight Approximation

We now quantify the stability of the canonical dual frame $F^{\mathrm{dual}} = S^{-1}F$ under additive frame perturbations $\widetilde{F} = F + \Delta F$ and compute the exact Frobenius distance to the manifold of α-tight frames.

Theorem 4.1 (Orthogonal Dual Perturbation Decomposition and Polar-Factor Tight Approximation)**.** *Let $F \in \mathbb{R}^{d\times m}$ be a frame with frame operator $S = FF^\top \succeq AI_d \succ 0$, canonical dual $F^{\mathrm{dual}} = S^{-1}F$, and range projector $P_F = F^\top S^{-1}F$.*

(a) ***Exact row-space orthogonal decomposition and Pythagorean stability.*** *Let $\Delta F \in \mathbb{R}^{d\times m}$ be any perturbation such that $\widetilde{F} := F + \Delta F$ is a frame with $\widetilde{S} = \widetilde{F}\widetilde{F}^\top \succeq \tilde{A} I_d \succ 0$ (guaranteed whenever $\|\Delta F\|_{\mathrm{op}} < \sqrt{A}$, with $\sqrt{\tilde{A}} \geq \sqrt{A} - \|\Delta F\|_{\mathrm{op}}$), and let $\widetilde{F}^{\mathrm{dual}} := \widetilde{S}^{-1}\widetilde{F}$ and $P_{\widetilde{F}} := \widetilde{F}^\top \widetilde{S}^{-1}\widetilde{F}$. Then*

$$\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}} = -\widetilde{F}^{\mathrm{dual}}(\Delta F\, P_{\widetilde{F}})^\top F^{\mathrm{dual}} + \widetilde{S}^{-1}(\Delta F)(I_m - P_F) =: T_1 + T_2, \tag{12}$$

where T_1 and T_2 have mutually orthogonal row spaces ($T_1 T_2^\top = 0$). Consequently,

$$\|\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}}\|_F^2 \;\leq\; \frac{1}{\tilde{A}A}\,\|\Delta F\, P_{\widetilde{F}}\|_F^2 + \frac{1}{\tilde{A}^2}\,\|\Delta F(I_m - P_F)\|_F^2, \tag{13}$$

$$\|\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}}\|_{\mathrm{op}} \;\leq\; \sqrt{\frac{\|\Delta F\, P_{\widetilde{F}}\|_{\mathrm{op}}^2}{\tilde{A}A} + \frac{\|\Delta F(I_m - P_F)\|_{\mathrm{op}}^2}{\tilde{A}^2}} \;\leq\; \frac{\sqrt{A+\tilde{A}}}{\tilde{A}\sqrt{A}}\,\|\Delta F\|_{\mathrm{op}}. \tag{14}$$

(b) ***Exact polar-factor tight-frame approximation.*** *For any $\alpha > 0$, the unique Frobenius-closest α-tight frame to F is the polar factor $W_* := \sqrt{\alpha}\, S^{-1/2}F$, with exact squared distance*

$$\min_{WW^\top = \alpha I_d} \|F - W\|_F^2 = \|F - W_*\|_F^2 = \sum_{k=1}^{d}\big(\sqrt{\lambda_k(S)} - \sqrt{\alpha}\big)^2 \;\leq\; \frac{\mathcal{E}_\alpha(F)}{\big(\sqrt{A} + \sqrt{\alpha}\big)^2}. \tag{15}$$

Minimizing (15) over $\alpha > 0$ uniquely selects $\alpha_* := \big(\frac{1}{d}\sum_{k=1}^{d}\sqrt{\lambda_k(S)}\big)^2$.

Proof. (a) Because $F(I_m - P_F) = F - FF^\top S^{-1}F = 0$, we have $\widetilde{F}(I_m - P_F) = (\Delta F)(I_m - P_F)$. Left-multiplying $\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}}$ by $\widetilde{S} = \widetilde{F}\widetilde{F}^\top$ yields

$$\widetilde{S}(\widetilde{F}^{\mathrm{dual}} - F^{\mathrm{dual}}) = \widetilde{F}\big(I_m - \widetilde{F}^\top S^{-1}F\big) = \widetilde{F}\big(I_m - P_F - (\Delta F)^\top S^{-1}F\big) = -\widetilde{F}(\Delta F)^\top F^{\mathrm{dual}} + (\Delta F)(I_m - P_F).$$

Left-multiplying by $\widetilde{S}^{-1}$ and noting $\widetilde{F}^{\mathrm{dual}} = \widetilde{F}^{\mathrm{dual}} P_{\widetilde{F}}$ gives (12). Since $F^{\mathrm{dual}} P_F = F^{\mathrm{dual}}$ and $P_F(I_m - P_F) = 0$, we have $T_1 T_2^\top = 0$ identically. Hence $\|T_1 + T_2\|_F^2 = \|T_1\|_F^2 + \|T_2\|_F^2$ and $\|T_1 + T_2\|_{\mathrm{op}}^2 = \|T_1 T_1^\top + T_2 T_2^\top\|_{\mathrm{op}} \leq \|T_1\|_{\mathrm{op}}^2 + \|T_2\|_{\mathrm{op}}^2$. Using $\|\widetilde{F}^{\mathrm{dual}}\|_{\mathrm{op}} \leq \tilde{A}^{-1/2}$, $\|F^{\mathrm{dual}}\|_{\mathrm{op}} \leq A^{-1/2}$, $\|\widetilde{S}^{-1}\|_{\mathrm{op}} \leq \tilde{A}^{-1}$, and $\|XYZ\|_F \leq \|X\|_{\mathrm{op}}\|Y\|_F\|Z\|_{\mathrm{op}}$ proves (13) and (14) (with $\|P_{\widetilde{F}}\|_{\mathrm{op}} \leq 1$, $\|I_m - P_F\|_{\mathrm{op}} \leq 1$).

(b) For any $W \in \mathbb{R}^{d\times m}$ with $WW^\top = \alpha I_d$, $\|F - W\|_F^2 = \mathrm{tr}(S) + d\alpha - 2\,\mathrm{tr}(F^\top W)$. Let $F = U\Sigma V^\top$ be the thin SVD with $\Sigma = \mathrm{diag}(\sqrt{\lambda_1(S)}, \ldots, \sqrt{\lambda_d(S)}) \succ 0$ and $V \in \mathbb{R}^{m\times d}$ isometric ($V^\top V = I_d$), so $W_* = \sqrt{\alpha}\, UV^\top$. Choosing $V_\perp \in \mathbb{R}^{m\times(m-d)}$ so that $[V, V_\perp] \in \mathrm{O}(m)$ and writing $W = \sqrt{\alpha}\, U(MV^\top + NV_\perp^\top)$ with $M := \alpha^{-1/2}U^\top W V \in \mathbb{R}^{d\times d}$ and $N := \alpha^{-1/2}U^\top W V_\perp \in \mathbb{R}^{d\times(m-d)}$, the tightness condition $WW^\top = \alpha I_d$ gives $MM^\top + NN^\top = I_d$. Hence $\|M\|_{\mathrm{op}} \leq 1$, so $M_{kk} \leq 1$ for every $k \in [d]$ with equality for all k if and only if $M = I_d$, which forces $NN^\top = I_d - MM^\top = 0$ and thus $N = 0$ and $W = W_*$. Consequently,

$$\mathrm{tr}(F^\top W) = \sqrt{\alpha}\sum_{k=1}^{d}\sqrt{\lambda_k(S)}\, M_{kk} \leq \sqrt{\alpha}\sum_{k=1}^{d}\sqrt{\lambda_k(S)},$$

with unique equality at $W = W_*$. Thus $\|F - W_*\|_F^2 = \sum_{k=1}^{d}(\lambda_k(S) + \alpha - 2\sqrt{\alpha\lambda_k(S)}) = \sum_{k=1}^{d}(\sqrt{\lambda_k(S)} - \sqrt{\alpha})^2$. Multiplying and dividing each summand by $(\sqrt{\lambda_k(S)} + \sqrt{\alpha})^2 \geq (\sqrt{A} + \sqrt{\alpha})^2$ gives (15). Differentiating $\sum_k(\sqrt{\lambda_k} - t)^2$ in $t = \sqrt{\alpha} > 0$ gives the unique minimizer $t_* = \frac{1}{d}\sum_k \sqrt{\lambda_k}$. □

5 Fusion Frames: Exact Principal-Angle Erasure and Grassmannian Minimax

Let $\{(W_i, v_i)\}_{i=1}^m$ be a *fusion frame* in $\mathbb{R}^d$ [2]: m subspaces $W_i \subset \mathbb{R}^d$ of dimensions $r_i \geq 1$ with orthogonal projections $P_i = U_i U_i^\top$ ($U_i \in \mathbb{R}^{d\times r_i}$, $U_i^\top U_i = I_{r_i}$) and positive weights $v_i > 0$ such that the fusion frame operator $S_W := \sum_{i=1}^m v_i^2 P_i \succ 0$. When a subset $E \subset [m]$ of sensor nodes fails (total erased dimension $R_E := \sum_{i\in E} r_i$), let $U_E := [U_i]_{i\in E} \in \mathbb{R}^{d\times R_E}$, $D_v := \bigoplus_{i\in E} v_i I_{r_i}$, and define the block erasure Gram matrix $G_E^{\mathrm{fus}} := D_v U_E^\top S_W^{-1} U_E D_v \in \mathbb{R}^{R_E\times R_E}$.

Theorem 5.1 (Principal-Angle Two-Node Erasure Penalty and Grassmannian Equi-Isoclinic Minimax)**.** *Let $\{(W_i, v_i)\}_{i=1}^m$ be a fusion frame in $\mathbb{R}^d$ with fusion frame operator $S_W \succ 0$ and* $\mathrm{MSE}(W) := \mathrm{tr}(S_W^{-1})$.

(a) ***Exact block-Schur node-erasure formula.*** *Erasing node subset $E \subset [m]$ leaves a spanning fusion frame ($S_{W,E^c} := S_W - \sum_{i\in E} v_i^2 P_i \succ 0$) if and only if $\|G_E^{\mathrm{fus}}\|_{\mathrm{op}} < 1$. Whenever $S_W = \alpha I_d$ is α-tight and $\|G_E^{\mathrm{fus}}\|_{\mathrm{op}} < 1$,*

$$\mathrm{MSE}(W_{E^c}) - \mathrm{MSE}(W) = \frac{1}{\alpha}\,\mathrm{tr}\Big(G_E^{\mathrm{fus}}\big(I_{R_E} - G_E^{\mathrm{fus}}\big)^{-1}\Big). \tag{16}$$

(b) ***Exact principal-angle two-node formula and Grassmannian minimax theorem.*** *Suppose $v_i = 1$ and $\dim(W_i) = r \geq 1$ for all $i \in [m]$, $S_W = \alpha I_d$ with $\alpha = \frac{mr}{d}$, and $mr \geq d + 2r$ (the necessary dimension condition $(m-2)r \geq d$ for 2-node erasures to span $\mathbb{R}^d$; in particular $mr > d + r$). For any distinct pair $E = \{i, j\}$ whose subspaces $W_i, W_j \in \mathrm{Gr}(r, \mathbb{R}^d)$ have Jordan–Halmos principal angles $0 \leq \theta_1 \leq \cdots \leq \theta_r \leq \pi/2$ ($\cos\theta_\ell = \sigma_\ell(U_i^\top U_j)$), $W_{\{i,j\}^c}$ is a fusion frame if and only if $\cos\theta_1 < \alpha - 1$, with exact MSE penalty*

$$\mathrm{MSE}(W_{\{i,j\}^c}) - \mathrm{MSE}(W) = \frac{2}{\alpha}\sum_{\ell=1}^{r} \frac{\alpha - 1 + \cos^2\theta_\ell}{(\alpha-1)^2 - \cos^2\theta_\ell}. \tag{17}$$

Moreover, every equal-weight equal-dimension tight fusion frame with $mr \geq d + 2r$ obeys the sharp universal Grassmannian minimax floor

$$\max_{|E|=2}\big(\mathrm{MSE}(W_{E^c}) - \mathrm{MSE}(W)\big) \;\geq\; \frac{2d^2}{m(mr - d - r)}, \tag{18}$$

with equality if and only if $\{W_1, \ldots, W_m\}$ is an equi-isoclinic tight fusion frame, *i.e., $P_i P_j P_i = \frac{mr-d}{d(m-1)} P_i$ for every $i \neq j$ (all r principal angles between every distinct pair coincide with* $\arccos\sqrt{\frac{mr-d}{d(m-1)}}$*).*

(c) ***Explicit equi-isoclinic attainment witness in $\mathbb{R}^4$.*** *For $d = 4, r = 2, m = 4$ ($\alpha = 2$), the four 2-planes spanned by $U_1 = \binom{I_2}{0}$ and $U_{k+2} = \frac{1}{\sqrt{3}}\binom{I_2}{\sqrt{2}R(2\pi k/3)}$ ($k \in \{0, 1, 2\}$, where $R(\varphi) \in \mathrm{SO}(2)$ is planar rotation by φ) form an equi-isoclinic tight fusion frame with $\cos^2\theta_\ell(W_i, W_j) \equiv \frac{1}{3}$ for all $i \neq j$ and $\ell \in \{1, 2\}$, attaining (18) with exact equality $\mathrm{MSE}(W_{\{i,j\}^c}) - \mathrm{MSE}(W) = 4$ for every pair $\{i, j\}$.*

Proof. (a) Applying Theorem 3.1(a) to the composite frame $F := [v_1 U_1, \ldots, v_m U_m] \in \mathbb{R}^{d\times\sum_i r_i}$ (whose frame operator is S_W and erased submatrix is $U_E D_v$) immediately establishes $\|G_E^{\mathrm{fus}}\|_{\mathrm{op}} < 1$ and (16).

(b) For $E = \{i, j\}$ with $v_i = v_j = 1$, $S_W = \alpha I_d$, let $U_i^\top U_j = L\Sigma R^\top$ be the SVD with orthogonal $L, R \in \mathrm{O}(r)$ and $\Sigma = \mathrm{diag}(\cos\theta_1, \ldots, \cos\theta_r)$. Conjugating the $2r \times 2r$ erasure Gram matrix by $\mathrm{diag}(L, R) \in \mathrm{O}(2r)$ yields

$$\mathrm{diag}(L^\top, R^\top)\, G_{\{i,j\}}^{\mathrm{fus}}\, \mathrm{diag}(L, R) = \frac{1}{\alpha}\begin{pmatrix} I_r & \Sigma \\ \Sigma & I_r \end{pmatrix},$$

which decouples into r orthogonal 2×2 blocks $\frac{1}{\alpha}\begin{pmatrix}1 & \cos\theta_\ell\\ \cos\theta_\ell & 1\end{pmatrix}$ having exact eigenvalues $\frac{1\pm\cos\theta_\ell}{\alpha}$ ($\ell=1,\dots,r$). Thus $\|G^{\mathrm{fus}}_{\{i,j\}}\|_{\mathrm{op}}=\frac{1+\cos\theta_1}{\alpha}<1 \iff \cos\theta_1<\alpha-1$, and summing $\frac{1}{\alpha}\left(\frac{(1+\cos\theta_\ell)/\alpha}{1-(1+\cos\theta_\ell)/\alpha}+\frac{(1-\cos\theta_\ell)/\alpha}{1-(1-\cos\theta_\ell)/\alpha}\right)$ over $\ell=1,\dots,r$ proves (17).

Setting $a:=\alpha-1=\frac{mr-d}{d}>0$, each summand in (17) equals $f(\cos^2\theta_\ell)$ where

$$f(u):=\frac{a+u}{a^2-u}=\frac{a(a+1)}{a^2-u}-1,\qquad u\in[0,a^2).$$

Direct differentiation gives $f'(u)=\frac{a(a+1)}{(a^2-u)^2}>0$ and $f''(u)=\frac{2a(a+1)}{(a^2-u)^3}>0$, so f is strictly increasing and strictly convex on $[0,a^2)$. Next, we verify the Grassmannian chordal Welch bound [5] in two lines: because $\sum_{i=1}^m P_i=\alpha I_d$ and $\operatorname{tr}(P_iP_j)=\|U_i^\top U_j\|_F^2=\sum_{\ell=1}^r\cos^2\theta_\ell(W_i,W_j)$ with $\operatorname{tr}(P_i^2)=r$,

$$\sum_{i\neq j}\sum_{\ell=1}^r\cos^2\theta_\ell(W_i,W_j)=\operatorname{tr}(S_W^2)-mr=d\alpha^2-mr=\frac{mr(mr-d)}{d}.$$

Dividing by the $m(m-1)$ ordered pairs yields $\max_{i\neq j}\bar u_{ij}\geq u_*:=\frac{mr-d}{d(m-1)}$, where $\bar u_{ij}:=\frac{1}{r}\sum_{\ell=1}^r\cos^2\theta_\ell(W_i,W_j)$. Note that $mr>d+r \iff m(mr-d-r)>0 \iff (mr-d)(m-1)-d>0 \iff u_*<a^2$. Pick a pair $\{i,j\}$ with $\bar u_{ij}\geq u_*$: if $\cos\theta_1\geq a$ then the erasure is unrecoverable ($+\infty$); otherwise $\cos^2\theta_\ell\in[0,a^2)$ for all ℓ, and by Jensen's inequality and monotonicity of f,

$$\frac{2}{\alpha}\sum_{\ell=1}^r f(\cos^2\theta_\ell)\ \geq\ \frac{2r}{\alpha}f(\bar u_{ij})\ \geq\ \frac{2r}{\alpha}f(u_*).$$

Evaluating $f(u_*)$ using $(mr-d)(m-1)-d=m(mr-d-r)$ gives

$$\frac{2r}{\alpha}f(u_*)=\frac{2d}{m}\cdot\frac{\frac{mr-d}{d}\left(1+\frac{1}{m-1}\right)}{\frac{mr-d}{d}\left(\frac{mr-d}{d}-\frac{1}{m-1}\right)}=\frac{2d}{m}\cdot\frac{d\,m}{(mr-d)(m-1)-d}=\frac{2d^2}{m(mr-d-r)},$$

proving (18). Because f is strictly convex and strictly increasing, equality across all pairs holds iff $\bar u_{ij}=u_*$ for all $i\neq j$ and $\cos^2\theta_1=\cdots=\cos^2\theta_r=u_*$ within every pair—precisely $U_i^\top U_jU_j^\top U_i=u_*I_r$, i.e., $P_iP_jP_i=u_*P_i$ for all $i\neq j$.

(c) Directly, $U_k^\top U_k=I_2$ for all $k\in\{1,2,3,4\}$. Since $I_2+R(2\pi/3)+R(4\pi/3)=0$ and $R(\pm2\pi/3)+R(\mp2\pi/3)=-I_2$, summing $U_kU_k^\top$ gives $\operatorname{diag}(2I_2,2I_2)=2I_4$ (with off-diagonal blocks $\frac{\sqrt{2}}{3}\sum_{k=0}^2R(2\pi k/3)=0$ and its transpose). Furthermore, $U_1^\top U_{k+2}=\frac{1}{\sqrt{3}}I_2$, and for $0\leq a<b\leq 2$, $(U_{a+2}^\top U_{b+2})^\top(U_{a+2}^\top U_{b+2})=\frac{1}{9}(I_2+2(R(2\pi/3)+R(-2\pi/3))+4I_2)=\frac{1}{3}I_2$, confirming $\cos^2\theta_\ell\equiv\frac{1}{3}=\frac{8-4}{4(4-1)}$ and $\mathrm{MSE}(W_{\{i,j\}^c})-\mathrm{MSE}(W)=\frac{2\cdot16}{4(8-4-2)}=4$. □

6 Computational Verification and Explicit Benchmarks

All four theorems were verified programmatically via a self-contained Python suite (`frame_geo.py`, `verify_all.py`, and `test_frame_geo.py`; four automated unit tests passing) across 255 deterministic benchmarks using NumPy `default_rng` streams.

- **Theorem 2.1 suite (seed `202671`, 75 unit-norm frames):** Across $(d,m)\in\{(3,7),(4,10),(6,14)\}$ (25 normalized i.i.d. Gaussian frames each), spectral dispersion (1) and two-sided MSE bounds (3) had 0.0 violation, the exact excess-MSE identity (2) matched to 2.66×10^{-15}, and RK4 integration ($\Delta t=0.005$, 80 steps) matched the exact Bernoulli eigenvalues (4) to 7.16×10^{-9} with 0.0 PL-envelope violation (Figure 1).

Table 1: Quantitative fusion-frame two-node erasure verification on trial 0 of each configuration (d, r, m) (seed `202674`, Theorem 5.1). Trial 0 of `Fus_d4_r2_m4` is the explicit equi-isoclinic witness of Theorem 5.1(c), attaining the exact Grassmannian minimax floor $\frac{2d^2}{m(mr-d-r)} = 4.0000$.

Configuration	d	r	m	Tightness α	Base MSE(W)	Worst 2-node ΔMSE	Grassmann floor (18)
`Fus_d4_r2_m4`	4	2	4	2.00	2.0000	4.0000	4.0000
`Fus_d4_r2_m5`	4	2	5	2.50	1.6000	2.4416	1.6000
`Fus_d5_r2_m5`	5	2	5	2.00	2.5000	32.9656	3.3333
`Fus_d6_r2_m6`	6	2	6	2.00	3.0000	4.4575	3.0000

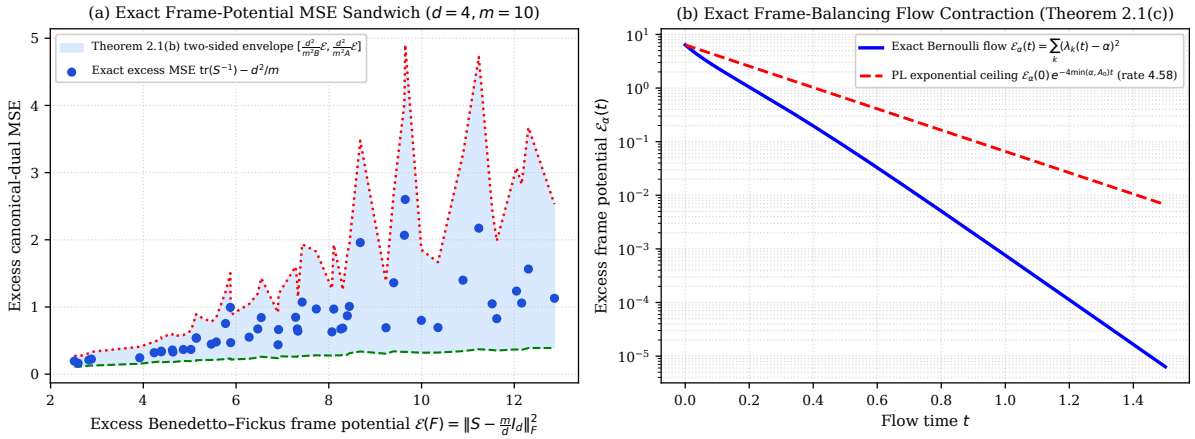


Figure 1: **(a)** Exact excess MSE $\mathrm{tr}(S^{-1}) - d^2/m$ versus excess frame potential $\mathcal{E}(F)$ across 50 random unit-norm frames ($d = 4, m = 10$, fresh seed `202671`), strictly trapped by the Theorem 2.1(b) two-sided envelope. **(b)** Exact Bernoulli tightening flow $\mathcal{E}_\alpha(t)$ ($d = 5, m = 12$) bounded by the global Polyak–Łojasiewicz exponential ceiling $e^{-4\min(\alpha, A_0)t}\mathcal{E}_\alpha(0)$ (Theorem 2.1(c)).

- **Theorem 3.1 suite (seed `202672`, 60 UNTFs):** Across $(d, m) \in \{(3, 6), (3, 8), (4, 9), (5, 11)\}$ (15 UNTFs each, including the 3×6 icosahedron ETF; random UNTFs are generated from i.i.d. Gaussian initializations by alternating polar tight-frame projection and column unit-normalization to $\mathcal{E}(F) < 10^{-24}$), the Woodbury k-erasure formula (7) matched to 1.67×10^{-11}, the 2-erasure formula (10) matched to 3.63×10^{-10}, all lower/upper/minimax bounds (8)–(11) had 0.0 violation, and the icosahedron ETF attained (11) (1.5000) to exact machine precision (0.0 gap; Figure 2(a)).

- **Theorem 4.1 suite (seed `202673`, 60 perturbed frames):** Across $(d, m) \in \{(3, 7), (4, 10), (6, 13)\}$ (20 Gaussian frames and perturbations rescaled to $\|\Delta F\|_{\mathrm{op}}/\sqrt{A} \sim \mathrm{Unif}(0.05, 0.55)$ each), the orthogonal decomposition (12) held to 1.50×10^{-14} with $\|T_1 T_2^\top\|_F \le 7.02 \times 10^{-17}$, Pythagorean bounds (13)–(14) and polar bound (15) had 0.0 violation, and the polar distance identity held to 1.35×10^{-13}.

- **Theorem 5.1 suite (seed `202674`, 60 tight fusion frames):** Across $(d, r, m) \in \{(4, 2, 4), (4, 2, 5), (5, 2, 5), (6, 2, 6)\}$ (15 tight fusion frames each, including our explicit 4-plane witness in $\mathbb{R}^4$; random tight fusion frames are generated from Haar-orthonormal $d \times r$ initializations via alternating global polar projection and per-block polar orthonormalization to $\|S_W - \alpha I_d\|_F < 10^{-13}$), the principal-angle two-node formula (17) matched to 2.67×10^{-9}, the Grassmannian minimax bound (18) had 0.0 violation, and the equi-isoclinic witness attained $\Delta\mathrm{MSE} = 4.0$ to 8.88×10^{-16} (Table 1, Figure 2(b)).

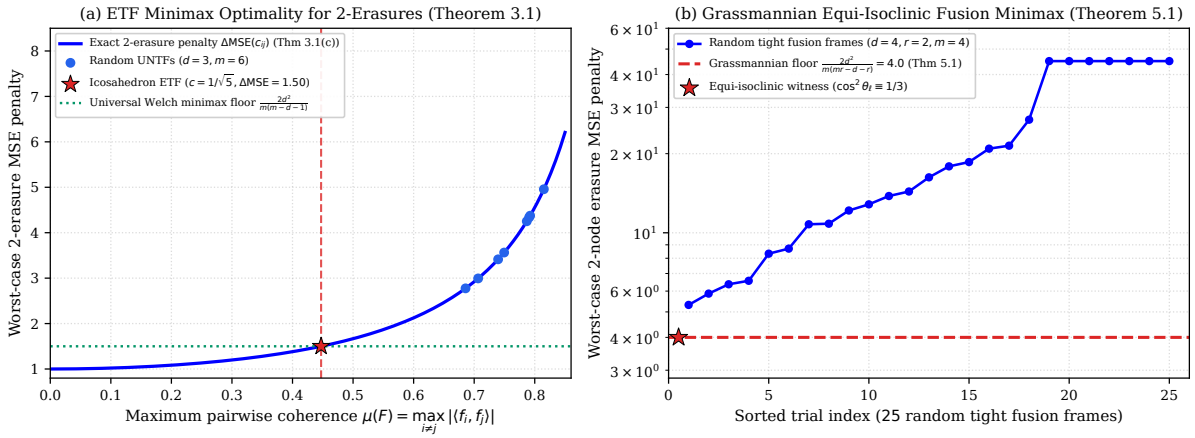


Figure 2: **(a)** Exact 2-erasure MSE penalty (10) versus mutual coherence $\mu(F)$ across random UNTFs ($d = 3, m = 6$, fresh seed `202672`), with the icosahedron ETF uniquely attaining the universal Welch minimax floor $\frac{2d^2}{m(m-d-1)} = 1.5$ (Theorem 3.1(c)). **(b)** Worst-case 2-node erasure MSE across 25 random tight fusion frames ($d = 4, r = 2, m = 4$, fresh seed `202674`; values exceeding 45 capped at 45 for display) compared with the Grassmannian minimax floor 4.0 attained by the explicit equi-isoclinic witness (Theorem 5.1(b)–(c)).

7 Discussion and Open Directions

The four theorems above connect Benedetto–Fickus frame potentials, continuous-time Bernoulli tightening flows, Woodbury–Schur erasure spectra, orthogonal dual-frame perturbations, and Grassmannian equi-isoclinic subspace packings through elementary, exact spectral identities. Three natural extensions remain open:

1. **Three-node and higher-k Grassmannian erasure minimax bounds:** Extending Theorem 5.1(b) from 2-node erasures to exact k-node ($k \geq 3$) minimax MSE floors requires triple-subspace spectral invariants beyond pairwise principal angles (such as Bargmann invariants of triplets of orthogonal projections).

2. **Riemannian unit-norm-constrained frame-potential flows:** Theorem 2.1(c) solves the unconstrained gradient flow $\dot{F} = -(S - \alpha I_d)F$ in closed Bernoulli form; characterizing sharp global PL contraction constants when each column is simultaneously constrained to the unit sphere $\|f_i(t)\|_2 = 1$ (avoiding non-tight saddle points) remains open.

3. **Optimal non-canonical dual frames under structured erasures:** Combining the exact Woodbury trace identity (6) with optimization over the affine space of alternate dual frames could further lower worst-case k-erasure MSE when erasure probabilities are nonuniform across sensor nodes.

Reproducibility and Data Availability

All analytical verifications, unit tests, and figure generation scripts are openly archived under the MIT license (version `v1.0.1`) at:
`https://github.com/lluiseriksson/frame-erasure-grassmannian-geometry`.

References

[1] J. J. Benedetto and M. Fickus, "Finite normalized tight frames," *Advances in Computational Mathematics*, vol. 18, no. 2–4, pp. 357–385, 2003.

[2] P. G. Casazza and G. Kutyniok, "Frames of subspaces," *Contemporary Mathematics*, vol. 345, pp. 87–113, 2004.

[3] R. B. Holmes and V. I. Paulsen, "Optimal frames for erasures," *Linear Algebra and Its Applications*, vol. 377, pp. 31–51, 2004.

[4] T. Strohmer and R. W. Heath Jr., "Grassmannian frames with applications to coding and communication," *Applied and Computational Harmonic Analysis*, vol. 14, no. 3, pp. 257–275, 2003.

[5] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing lines, planes, etc.: packings in Grassmannian spaces," *Experimental Mathematics*, vol. 5, no. 2, pp. 139–159, 1996.

[6] P. W. H. Lemmens and J. J. Seidel, "Equi-isoclinic subspaces of Euclidean spaces," *Indagationes Mathematicae*, vol. 76, no. 2, pp. 98–107, 1973.

[7] L. R. Welch, "Lower bounds on the maximum cross correlation of signals," *IEEE Transactions on Information Theory*, vol. 20, no. 3, pp. 397–399, 1974.

[8] G. W. Stewart, "On the perturbation of pseudo-inverses, projections and linear least squares problems," *SIAM Review*, vol. 19, no. 4, pp. 634–662, 1977.

[9] S. F. D. Waldron, *An Introduction to Finite Tight Frames*. Birkhäuser, New York, 2018.