

Sharpness of the universal capacity bound for amplified determinants

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Abstract

For a finite tuple of square complex matrices, let F_∞ be the supremum of the normalized determinant over all matrix amplifications and Hilbert–Schmidt unit coefficient tuples, and let $\mathcal{C}$ be the maximum operator capacity over coefficient covariances of trace one. The previously established comparison $\mathcal{C}/e \leq F_\infty \leq \mathcal{C}$ has the optimal universal constant: we construct finite tuples with $\mathcal{C}/F_\infty$ arbitrarily close to e . The tuples may have exactly identity Hilbert–Schmidt Gram matrix; their capacities tend to one and their amplified determinant values tend to e^{-1} . The construction uses independent complex Gaussian coefficients with a fixed number of coefficients before the matrix dimension tends to infinity. A balance property of determinant maximizers makes polynomial moment comparison uniform over every amplification. A separate cut criterion turns Gaussian expansion estimates into a dimension-independent capacity lower bound. For each fixed number of Gaussian coefficients, scalar and arbitrarily amplified optima both converge in probability to e^{-1} .

1 Definitions and statement

Write $\tau_d = d^{-1} \text{Tr}$ on $M_d(\mathbb{C})$ and $\|X\|_2 = \tau_d(X^*X)^{1/2}$. Operator norms are denoted by $\|\cdot\|_{\text{op}}$. For a tuple $A = (A_1, \dots, A_s) \in M_d(\mathbb{C})^s$, put

$$L_A(Y) = \sum_{i=1}^s A_i \otimes Y_i, \tag{1}$$

$$F_m(A) = \max_{\substack{Y_i \in M_m(\mathbb{C}) \\ \sum_i \tau_m(Y_i^* Y_i) = 1}} |\det L_A(Y)|^{2/(dm)}, \quad F_\infty(A) = \sup_{m \geq 1} F_m(A). \tag{2}$$

The maximum in F_m exists by compactness. Singular evaluations contribute zero.

For a completely positive map $\Phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$, define

$$\text{cap}(\Phi) = \inf_{X > 0} \frac{\det \Phi(X)}{\det X} = \inf_{\substack{X > 0 \\ \det X = 1}} \det \Phi(X). \tag{3}$$

For $Q \in M_s(\mathbb{C})$ positive semidefinite with $\text{Tr } Q = 1$, let

$$\Phi_{A,Q}(X) = \sum_{i,j=1}^s Q_{ij} A_i X A_j^*, \quad \mathcal{C}(A) = \max_{\substack{Q \geq 0 \\ \text{Tr } Q = 1}} \text{cap}(\Phi_{A,Q})^{1/d}. \tag{4}$$

The convention in (4) makes $\Phi_{A,Q}$ completely positive. The maximum exists: capacity is an infimum of continuous functions of Q , hence is upper semicontinuous on the compact covariance set. The Gram matrix is

$$G(A)_{ij} = \tau_d(A_i^* A_j). \quad (5)$$

The universal comparison proved in [1, Theorem *Universal comparison*] is

$$e^{-1}\mathcal{C}(A) \leq F_\infty(A) \leq \mathcal{C}(A) \quad (6)$$

for arbitrary finite square complex tuples. Our construction below does not use either inequality to bound its examples. We import (6) only when identifying the optimal universal constant and, consequently, the limits of the constructed values. The cited item is the published version ARR-2026-1DGGW5KX2M9R4VJT v1, not its earlier Hermitian-only predecessor.

Theorem 1. *For every integer $s \geq 4096$ and every $\theta > 0$, there exist $d \in \mathbb{N}$ and $B \in M_d(\mathbb{C})^s$ such that*

$$G(B) = I_s, \quad 1 - \frac{23}{\sqrt{s}} \leq \mathcal{C}(B) \leq 1, \quad 0 < F_\infty(B) \leq \exp(-1 + \theta). \quad (7)$$

In particular there is a sequence of finite tuples with identity Gram matrix for which $\mathcal{C}(B) \rightarrow 1$ and $F_\infty(B) \rightarrow e^{-1}$.

Corollary 2. *The optimal universal comparison constant is e :*

$$\inf_{\substack{d,s \geq 1, \\ A \in M_d(\mathbb{C})^s \\ \mathcal{C}(A) > 0}} \frac{F_\infty(A)}{\mathcal{C}(A)} = \frac{1}{e}. \quad (8)$$

The same infimum is obtained after imposing $G(A) = I_s$. Thus $F_\infty(A) \geq \mathcal{C}(A)/2$ is false in general.

Proof of the corollary from Theorem 1. The lower bound for the infimum is (6). The theorem gives

$$\frac{F_\infty(B)}{\mathcal{C}(B)} \leq \frac{e^{-1+\theta}}{1 - 23/\sqrt{s}}.$$

First choose s large, and then choose θ small. For example, $s = 65536$ and $\theta = 1/20$ give $\mathcal{C}(B)/F_\infty(B) \geq (233/256)e^{19/20} > 2$. This specifies parameters of an existence construction, rather than an explicit small matrix tuple. $\square$

The inputs from random matrix theory are standard. Gaussian convergence is used in the form of Haagerup–Thorbjørnsen [6, Theorem A and (1.2)]. The Gaussian superoperator estimate is the square specialization of González-Guillén–Junge–Nechita [4, Lemma 5.3 and Proposition 5.4]; the logarithmic moment choice needed here is detailed in Section 4. The broader connection between free determinants and operator capacity was established by Mai–Speicher [9]. Spectral-gap bounds for operator capacity also precede this work; see Kwok–Lau–Ramachandran [8, Theorem 1.8]. We use an elementary sufficient cut criterion, rather than asserting novelty for the general connection between expansion and capacity. Balancing determinant maximizers and combining their moments with a logarithmic majorant were already used for the three-dimensional antisymmetric channel in [2, Lemmas *Balancing at a nonsingular maximum* and *Logarithmic moment bound*]. We restate the balance argument and extend the finite-moment comparison to Gaussian coefficient tuples. The resulting examples establish sharpness of the unrestricted constant. Uniform trace-polynomial approximation on bounded matrix tuples is also established background; see [7, Section 3.3, Definition 3.4 and Lemma 3.6]. Our use is the concrete finite-word estimate in Proposition 5.

2 Balance and a uniform determinant upper bound

Lemma 3 (Balance at a positive maximum). *If $F_m(A) > 0$, every maximizing tuple in (2) satisfies*

$$\sum_i Y_i^* Y_i = I_m, \quad \sum_i Y_i Y_i^* = I_m. \quad (9)$$

In particular $\|Y_i\|_{\text{op}} \leq 1$ for each i .

Proof. Let Y be a maximizer and put $S = \sum_i Y_i^* Y_i$. If $Sv = 0$ for $v \neq 0$, then $Y_i v = 0$ for every i . Consequently $L_A(Y)$ has a nonzero kernel, contrary to positivity of the maximum. Thus $S > 0$. The tuple $\tilde{Y}_i = Y_i S^{-1/2}$ is feasible, since $\sum_i \tilde{Y}_i^* \tilde{Y}_i = I_m$. Moreover

$$L_A(\tilde{Y}) = L_A(Y)(I_d \otimes S^{-1/2}), \quad \left| \det L_A(\tilde{Y}) \right|^{2/(dm)} = |\det L_A(Y)|^{2/(dm)} (\det S)^{-1/m}.$$

Since $\tau_m(S) = 1$, the arithmetic–geometric mean inequality gives $\det S \leq 1$, with equality exactly when $S = I_m$. A strict inequality would contradict maximality.

The same argument on the left, with $T = \sum_i Y_i Y_i^*$ and $\hat{Y}_i = T^{-1/2} Y_i$, proves $T = I_m$. Here T is positive definite because a common left kernel would again make $L_A(Y)$ singular. Finally $Y_i^* Y_i \leq I_m$. $\square$

Let $(c_1, \dots, c_s)$ be a free standard circular family in a tracial C^* -probability space $(\mathcal{A}, \tau)$ with faithful trace. Thus $c_i = (x_i + iz_i)/\sqrt{2}$, where the $2s$ self-adjoint variables x_i, z_i form a free semicircular family of variance one. In particular $\|c_i\|_{\text{op}} = 2$ and $\tau(c_i^* c_j) = \delta_{ij}$.

Lemma 4 (Balanced circular evaluation). *Suppose that $Y_i \in M_m(\mathbb{C})$ satisfy (9), and put $H = \sum_i c_i \otimes Y_i$. The law of $H^* H$ under $\tau \otimes \tau_m$ is*

$$d\mu(x) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} \mathbf{1}_{(0,4)}(x) dx, \quad (10)$$

the Marchenko–Pastur law of parameter one. If

$$\ell(t) = \int_0^4 \log(x+t) d\mu(x) \quad (t > 0),$$

then

$$\ell(0) := \int_0^4 \log x d\mu(x) = -1, \quad 0 \leq \ell(t) + 1 \leq 2\sqrt{t}. \quad (11)$$

Proof. Hermitize H :

$$\mathbf{H} = \begin{pmatrix} 0 & H \\ H^* & 0 \end{pmatrix}.$$

Conditional expectation over the free variables makes $\mathbf{H}$ a semicircular element with values in $M_{2m}(\mathbb{C})$. Its covariance on a block matrix is

$$\eta \begin{pmatrix} U & V \\ W & Z \end{pmatrix} = \begin{pmatrix} \sum_i Y_i Z Y_i^* & 0 \\ 0 & \sum_i Y_i^* U Y_i \end{pmatrix}. \quad (12)$$

This also follows directly by expanding the free Wick formula: only the pairings c_i with c_i^* and c_i^* with c_i have nonzero second cumulant. Balance gives $\eta(I_{2m}) = I_{2m}$.

For completeness, let M_k be the conditional expectation of $\mathbf{H}^k$. Noncrossing pairings have zero contribution to odd moments. Pairing the first letter of an even word with its partner gives the recurrence

$$M_0 = I_{2m}, \quad M_{2k} = \sum_{j=0}^{k-1} \eta(M_{2j}) M_{2(k-1-j)}.$$

Induction yields $M_{2k} = C_k I_{2m}$, where $C_0 = 1$ and $C_k = \sum_{j=0}^{k-1} C_j C_{k-1-j}$ are the Catalan numbers. The lower diagonal block of H^{2k} is $(H^*H)^k$. Its moments are therefore C_k , which are the moments of (10). Both laws have compact support, so the moments identify the law.

The logarithmic integral can be computed from

$$\int x^\alpha d\mu(x) = \frac{4^\alpha \Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi} \Gamma(\alpha + 2)} = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1) \Gamma(\alpha + 2)} \quad (\alpha > -\frac{1}{2}).$$

Differentiating at zero and using $\Gamma(u + 1) = u\Gamma(u)$ gives -1 . For $t > 0$, integration of (10) gives

$$\ell'(t) = \int \frac{d\mu(x)}{x+t} = \frac{\sqrt{1+4/t} - 1}{2} \leq t^{-1/2}.$$

Integrating from zero proves (11). $\square$

A word in s variables and their adjoints has a *length* equal to its number of letters. The following finite condition is the step that controls every auxiliary dimension at once.

Proposition 5 (Uniform polynomial transfer). *Fix $s \in \mathbb{N}$, $t > 0$, and $\zeta > 0$. There exist an integer $k \geq 1$ and a number $a > 0$ such that the following implication holds for every d and every tuple $A \in M_d(\mathbb{C})^s$. If $\|A_i\|_{\text{op}} \leq 3$ and*

$$|\tau_d(w(A)) - \tau(w(c))| \leq a \quad \text{for every word } w \text{ of length at most } 2k, \quad (13)$$

then

$$F_\infty(A) \leq \exp(\ell(t) + 3\zeta). \quad (14)$$

The parameters k, a do not depend on the amplification dimension.

Proof. Choose a real polynomial $p(x) = \sum_{j=0}^k p_j x^j$ such that

$$\sup_{0 \leq x \leq 9s^2} |p(x) - \log(x+t)| \leq \zeta,$$

using polynomial approximation of a continuous function on a compact interval. Increase k to one if necessary and choose $a > 0$ with

$$a \sum_{j=1}^k |p_j| s^{2j} \leq \zeta. \quad (15)$$

If the sum is zero, any $a > 0$ suffices.

Fix m with $F_m(A) > 0$ and choose a maximizing tuple Y . Lemma 3 makes every Y_i a contraction. Writing $L = L_A(Y)$ and $H = \sum_i c_i \otimes Y_i$, we have $\|L\|_{\text{op}} \leq 3s$ and $\|H\|_{\text{op}} \leq 2s$. For each $j \geq 1$, expand $(L^*L)^j$ and $(H^*H)^j$. There are s^{2j} summands. Each is a coefficient word of length $2j$ tensored with a word in Y_i, Y_i^* , and the absolute normalized trace of the latter word is at most one. Consequently (13) and (15) give

$$|(\tau_d \otimes \tau_m)p(L^*L) - (\tau \otimes \tau_m)p(H^*H)| \leq \zeta.$$

The constant terms agree. Functional calculus and two polynomial approximation errors now give

$$\begin{aligned} \log F_m(A) &= (\tau_d \otimes \tau_m) \log(L^*L) \\ &\leq (\tau_d \otimes \tau_m) \log(L^*L + tI) \\ &\leq (\tau \otimes \tau_m) \log(H^*H + tI) + 3\zeta \\ &= \ell(t) + 3\zeta, \end{aligned}$$

where Lemma 4 applies because Y is balanced. If $F_m(A) = 0$, the desired upper bound is automatic. Taking the supremum over all m proves (14). $\square$

Remark 6. Moment convergence for each fixed m would not, by itself, justify an upper bound for F_∞ . Proposition 5 avoids that interchange: a single finite list of scalar moments controls every m . Also, only the inequality $\log x \leq \log(x + t)$ is used. No lower-tail estimate for the smallest singular value, or convergence of unregularized logarithmic determinants, is required.

3 A dimension-independent capacity criterion

For a linear map on $M_d(\mathbb{C})$, let $\|\Phi\|_{2 \rightarrow 2}$ be its operator norm on the Hilbert space with norm $\|\cdot\|_2$. Using unnormalized Hilbert–Schmidt norms on both sides gives the same map norm. Let Φ^* be the trace adjoint and set $\mathcal{D}_d(X) = \tau_d(X)I_d$.

Lemma 7 (Capacity from cut estimates). *Let $\Phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$ be completely positive. Suppose $\epsilon, q \geq 0$ satisfy*

$$\Phi(I_d), \Phi^*(I_d) \geq (1 - \epsilon)I_d, \quad \|\Phi - \mathcal{D}_d\|_{2 \rightarrow 2} \leq q, \quad c := 1 - \epsilon - 2q \geq \frac{1}{2}. \quad (16)$$

Then

$$\text{cap}(\Phi)^{1/d} \geq c. \quad (17)$$

Proof. Choose any two orthonormal bases (u_i) and (v_j) of $\mathbb{C}^d$, and define a nonnegative matrix

$$B_{ij} = \langle u_i, \Phi(v_j v_j^*) u_i \rangle.$$

Every row sum and column sum is at least $1 - \epsilon$. For subsets $S, T \subseteq \{1, \dots, d\}$ of sizes a, b , write $B(S, T) = \sum_{i \in S, j \in T} B_{ij}$. Applying the map norm in (16) to the projections onto the two coordinate subspaces gives

$$|B(S, T) - ab/d| \leq q\sqrt{ab}. \quad (18)$$

Indeed the normalized-trace estimate is $q\sqrt{a/d}\sqrt{b/d}$, and multiplication by d gives (18).

We claim that B dominates an entrywise nonnegative matrix Z whose row and column sums are all c . The max-flow/min-cut criterion [3] says this is equivalent to

$$B(S, T^c) \geq c(a - b) \quad \text{for all } S, T. \quad (19)$$

To see the exact orientation, place an edge of capacity c from a source to each row, an edge of capacity B_{ij} from row i to column j , and an edge of capacity c from each column to the sink. The cut with rows S and columns T on the source side has capacity $c(d - a) + B(S, T^c) + cb$. Thus (19) gives a flow of total value cd , and the row-to-column flows form Z .

If $a \leq b$, (19) follows from nonnegativity. Suppose $a > b$. There are three cases.

If $a \leq cd$, the row sums and (18) imply

$$B(S, T^c) \geq (1 - \epsilon)a - ab/d - q\sqrt{ab} \geq (1 - \epsilon - q)a - ab/d.$$

After subtracting $c(a - b)$, the last expression leaves $qa + b(c - a/d) \geq 0$.

If $v := d - b \leq cd$, put $u = d - a < v$ and use the column sums over T^c , subtracting $B(S^c, T^c)$. This gives

$$B(S, T^c) \geq (1 - \epsilon - q)v - uv/d.$$

The difference from $c(v - u) = c(a - b)$ is $qv + u(c - v/d) \geq 0$.

In the remaining case $a > cd$ and $v = d - b > cd$, we have

$$B(S, T^c) \geq av/d - q\sqrt{av} \geq (1 - q/c)av/d.$$

Here $\sqrt{av} \geq cd$ justifies the second inequality. Since $1 - c = \epsilon + 2q$ and $c \geq 1/2$, $c(1 - c) \geq q$, and hence $1 - q/c \geq c$. Finally $av/d - (a + v - d) = (d - a)(d - v)/d \geq 0$. This proves (19).

Write $Z = cW$, where W is doubly stochastic. For an arbitrary $X > 0$, now choose the v_j to diagonalize X , with eigenvalues x_j , and the u_i to diagonalize $\Phi(X)$, with eigenvalues y_i . The latter matrix is positive definite because $\Phi(X) \geq \lambda_{\min}(X)\Phi(I_d) > 0$. These basis choices are legitimate because all preceding estimates hold for every pair of bases. We obtain

$$y_i = \sum_j B_{ij}x_j \geq c \sum_j W_{ij}x_j.$$

Concavity of the logarithm and the two stochasticity conditions yield

$$\sum_i \log y_i \geq d \log c + \sum_{i,j} W_{ij} \log x_j = d \log c + \sum_j \log x_j.$$

Taking the infimum in (3) proves the result. $\square$

Remark 8. The expansion hypothesis in (16) is essential to this argument. Approximate unitality and trace preservation alone are not being used as a dimension-independent continuity principle for capacity. The cut proof supplies the missing control over subspaces of every dimension.

Remark 9 (Relation to spectral scaling bounds). Theorem 1.8 of [8] gives a capacity estimate under the assumptions of its Theorem 1.5, including $\lambda^2 \geq C\epsilon \log d$ in the square case. With s fixed in our Gaussian model, the marginal error is of order $s^{-1/2}$ as $d \rightarrow \infty$; that displayed hypothesis does not give the dimension-independent estimate required here. Lemma 7 is an elementary sufficient bound tailored to this order of limits, not a claim of superiority under the different hypotheses of [8].

4 Gaussian coefficients and expansion

Fix s . For each d , let $A_1^{(d)}, \dots, A_s^{(d)}$ be independent complex Ginibre matrices: their entries are independent centered complex Gaussian variables with $\mathbb{E} |(A_i^{(d)})_{ab}|^2 = 1/d$. Set

$$\Psi_d(X) = \frac{1}{s} \sum_{i=1}^s A_i^{(d)} X (A_i^{(d)})^*, \quad P_d(X) = X - \tau_d(X) I_d. \quad (20)$$

Throughout this section s is fixed as $d \rightarrow \infty$.

Lemma 10 (Gaussian convergence facts). *For fixed s , the following hold almost surely:*

$$\tau_d(w(A^{(d)})) \longrightarrow \tau(w(c)) \quad \text{for every fixed } *- \text{word } w, \quad (21)$$

$$\|A_i^{(d)}\|_{\text{op}} \longrightarrow 2 \quad (1 \leq i \leq s), \quad (22)$$

$$\|\Psi_d(I_d) - I_d\|_{\text{op}}, \quad \|\Psi_d^*(I_d) - I_d\|_{\text{op}} \longrightarrow r_s := \frac{2}{\sqrt{s}} + \frac{1}{s}. \quad (23)$$

Proof. Write $A_i^{(d)} = (H_i^{(d)} + iK_i^{(d)})/\sqrt{2}$, where the $2s$ matrices $H_i^{(d)}, K_i^{(d)}$ are independent GUE matrices normalized by $\mathbb{E}\tau_d((H_i^{(d)})^2) = 1$. This representation has exactly the Ginibre entry distribution. The moment convergence and strong convergence theorem [6, (1.2) and Theorem A] apply to these $2s$ matrices, and hence to all fixed $*$ -polynomials in the $A_i^{(d)}$. They give (21) and (22).

For (23), the two limiting polynomials are $s^{-1} \sum_i c_i c_i^* - 1$ and $s^{-1} \sum_i c_i^* c_i - 1$. Each positive sum has the free Poisson law of rate s , scaled by $1/s$. Its support is $[(1 - s^{-1/2})^2, (1 + s^{-1/2})^2]$; the rate is at least one, so there is no separated atom at zero. The trace is faithful, and the distance of the furthest endpoint from one is r_s . Strong convergence gives the asserted norms. Equivalently, these are the two rectangular Wishart edge limits obtained by horizontal and vertical concatenation of the Ginibre blocks. $\square$

The next statement is a Gaussian estimate, not a claim that the maps Ψ_d are channels. In particular no conversion from Haar isometries is necessary.

Lemma 11 (Gaussian restriction estimate). *For every fixed s and every $h > 8/\sqrt{s}$,*

$$\mathbb{P}\{\|\Psi_d P_d\|_{2 \rightarrow 2} > h\} \longrightarrow 0. \quad (24)$$

Proof. We give the parameter substitution and the moment argument. Under Hilbert–Schmidt vectorization, P_d is the projection $I_{d^2} - \omega_d$, where ω_d projects onto $\text{vec}(I_d)/\sqrt{d}$. Put $U_i = A_i^{(d)}/\sqrt{s}$, so the entries of U_i have variance $(ds)^{-1}$. The superoperator matrix of Ψ_d is, up to the vectorization convention, $M_d = \sum_i U_i \otimes \bar{U}_i$.

The direct Gaussian statements in [4, Lemma 5.3 and Proposition 5.4], with their $n = d$, $k = s$, and aspect ratio $\lambda = 1$, give the following. If V_i are independent copies of U_i and $p \geq 2$ is even, then

$$\mathbb{E} \|M_d(I - \omega_d)\|_{\text{op}}^p \leq 2^p \mathbb{E} \|\sum_i U_i \otimes V_i\|_{\text{op}}^p, \quad (25)$$

$$\mathbb{E} \|\sum_i U_i \otimes V_i\|_{\text{op}}^p \leq d^2 s^{p/2} (\mathbb{E} \|U_1\|_{\text{op}}^p)^2. \quad (26)$$

The second inequality is the trace-moment estimate (15) in that proof, followed by $\text{Tr} |U_1|^p \leq d \|U_1\|_{\text{op}}^p$. The first uses the Gaussian decoupling factor 2 only. The factor involving χ in their Proposition 5.2 concerns Haar isometries and does not enter (25).

Gaussian concentration, as used in [4, (14)], supplies the scalar estimate

$$(\mathbb{E} \|U_1\|_{\text{op}}^p)^{1/p} \leq \frac{2}{\sqrt{s}} + a_d + K_s \sqrt{p/d}, \quad a_d \longrightarrow 0,$$

with K_s independent of d, p . We retain the square in the resulting bound:

$$\mathbb{E} \|M_d(I - \omega_d)\|_{\text{op}}^p \leq d^2 \left[2\sqrt{s} \left(\frac{2}{\sqrt{s}} + a_d + K_s \sqrt{p/d} \right)^2 \right]^p. \quad (27)$$

For any fixed $a > 0$ and the even integer $p_d = 2\lceil a \log d/2 \rceil$, the bracket is $8/\sqrt{s} + o(1)$. Terms of order p_d/d tend to zero; no estimate uniform for arbitrarily large p/d is asserted.

Choose b strictly between $8/\sqrt{s}$ and h . For all sufficiently large d , Markov's inequality gives

$$\mathbb{P}\{\|M_d(I - \omega_d)\|_{\text{op}} > h\} \leq d^2 (b/h)^{p_d}.$$

Taking $a > 3/\log(h/b)$ makes the right-hand side tend to zero. Vectorization preserves the relevant operator norm, proving (24). $\square$

Remark 12. The factor d^2 in (27) is the reason for choosing a logarithmic moment order. A fixed moment bound would not establish (24).

Proposition 13 (Capacity of the Gaussian family). *For every fixed $s \geq 4096$,*

$$\mathbb{P}\left\{\text{cap}(\Psi_d)^{1/d} \geq 1 - \frac{23}{\sqrt{s}}\right\} \longrightarrow 1. \quad (28)$$

Proof. Lemmas 10 and 11 imply, with probability tending to one,

$$\|\Psi_d(I_d) - I_d\|_{\text{op}}, \|\Psi_d^*(I_d) - I_d\|_{\text{op}} \leq \frac{3}{\sqrt{s}}, \quad \|\Psi_d P_d\|_{2 \rightarrow 2} \leq \frac{9}{\sqrt{s}}.$$

For $X = \alpha I_d + X_0$, where $\tau_d(X_0) = 0$, the domain decomposition is orthogonal and $\|X\|_2^2 = |\alpha|^2 + \|X_0\|_2^2$. Therefore

$$\begin{aligned} \|(\Psi_d - \mathcal{D}_d)(X)\|_2 &\leq \frac{3}{\sqrt{s}} |\alpha| + \frac{9}{\sqrt{s}} \|X_0\|_2 \\ &\leq \frac{\sqrt{90}}{\sqrt{s}} \|X\|_2 < \frac{10}{\sqrt{s}} \|X\|_2. \end{aligned} \quad (29)$$

The last step is scalar Cauchy–Schwarz; it does not require the two images to be orthogonal. Apply Lemma 7 with $\epsilon = 3/\sqrt{s}$, $q = 10/\sqrt{s}$, and $c = 1 - 23/\sqrt{s} \geq 1/2$. $\square$

5 Exact Gram normalization and the proof of sharpness

Lemma 14 (Stability under Gram normalization). *For fixed s , let $A^{(d)}$ be the Gaussian family above and define*

$$G_d = (\tau_d((A_i^{(d)})^* A_j^{(d)}))_{i,j=1}^s, \quad B_j^{(d)} = \sum_{i=1}^s A_i^{(d)} (G_d^{-1/2})_{ij} \quad (30)$$

whenever $G_d > 0$. Then $G(B^{(d)}) = I_s$. The probability that $G_d > 0$ tends to one, and

$$\max_j \|B_j^{(d)} - A_j^{(d)}\|_{\text{op}} \longrightarrow 0 \quad \text{almost surely.} \quad (31)$$

The moment limits (21) and norm limits (22) hold with $B^{(d)}$ in place of $A^{(d)}$. Moreover, for $s \geq 4096$ fixed, the maps

$$\tilde{\Psi}_d(X) = s^{-1} \sum_i B_i^{(d)} X (B_i^{(d)})^*$$

satisfy, with probability tending to one,

$$\|\tilde{\Psi}_d(I_d) - I_d\|_{\text{op}}, \|\tilde{\Psi}_d^*(I_d) - I_d\|_{\text{op}} \leq 3/\sqrt{s}, \quad (32)$$

$$\|\tilde{\Psi}_d P_d\|_{2 \rightarrow 2} \leq 9/\sqrt{s}. \quad (33)$$

Proof. By (21), $G_d \rightarrow I_s$ almost surely. Since s is fixed, G_d is eventually positive definite and $T_d := G_d^{-1/2} \rightarrow I_s$ in every matrix norm. The orientation of the coefficient indices in (30) gives $G(B^{(d)}) = T_d^* G_d T_d = I_s$. The boundedness of the $\|A_i^{(d)}\|_{\text{op}}$ and the finite sum

$$B_j^{(d)} - A_j^{(d)} = \sum_i A_i^{(d)} (T_d - I_s)_{ij}$$

give (31). Telescoping each fixed word shows that its normalized trace changes by $o(1)$; norms likewise change by $o(1)$.

Put

$$\delta_d = \frac{1}{s} \sum_i (\|B_i^{(d)}\|_{\text{op}} + \|A_i^{(d)}\|_{\text{op}}) \|B_i^{(d)} - A_i^{(d)}\|_{\text{op}}.$$

Then $\delta_d \rightarrow 0$ almost surely, and expanding the difference of the two quadratic expressions gives

$$\begin{aligned}\|\tilde{\Psi}_d - \Psi_d\|_{2 \rightarrow 2} &\leq \delta_d, \\ \|\tilde{\Psi}_d(I_d) - \Psi_d(I_d)\|_{\text{op}} &\leq \delta_d, \\ \|\tilde{\Psi}_d^*(I_d) - \Psi_d^*(I_d)\|_{\text{op}} &\leq \delta_d.\end{aligned}$$

For $s \geq 4096$, $r_s < (5/2)/\sqrt{s}$. Thus (23) and Lemma 11 give, with probability tending to one, marginal errors at most $(5/2)/\sqrt{s}$ and a restriction norm at most $(17/2)/\sqrt{s}$ for Ψ_d . Intersect this event with $\delta_d \leq 1/(2\sqrt{s})$. The last three inequalities imply (32)–(33). These strict preliminary margins justify using the same final constants after Gram normalization. $\square$

Proof of Theorem 1. Fix $s \geq 4096$ and $\theta > 0$. Choose $t, \zeta > 0$ such that $2\sqrt{t} + 3\zeta \leq \theta$, and let k, a be as in Proposition 5. Lemma 14 shows that, with probability tending to one, the tuple $B^{(d)}$ is defined, has exactly identity Gram matrix, has coefficient norms at most three, and satisfies (13). Hence

$$F_\infty(B^{(d)}) \leq \exp(\ell(t) + 3\zeta) \leq \exp(-1 + \theta).$$

On another event with probability tending to one, the two estimates (32)–(33) hold. The argument (29) and Lemma 7 then give

$$\text{cap}(\tilde{\Psi}_d)^{1/d} \geq 1 - \frac{23}{\sqrt{s}}.$$

Since $\tilde{\Psi}_d = \Phi_{B^{(d)}, I_s/s}$, its covariance is admissible in (4). This proves the desired lower bound for $\mathcal{C}(B^{(d)})$.

For the upper bound, identity Gram gives, for every admissible Q ,

$$\tau_d(\Phi_{B^{(d)}, Q}(I_d)) = \sum_{i,j} Q_{ij} \tau_d(B_i^{(d)}(B_j^{(d)})^*) = \text{Tr } Q = 1.$$

Evaluating capacity at $X = I_d$ and using the arithmetic–geometric mean inequality therefore gives $\text{cap}(\Phi_{B^{(d)}, Q})^{1/d} \leq 1$. Taking the maximum over Q yields $\mathcal{C}(B^{(d)}) \leq 1$.

The positive-value condition also holds. Almost surely $A_1^{(d)}$ is invertible. An invertible change of coefficient coordinates preserves their linear span, so some scalar combination of the $B_i^{(d)}$ is invertible. Normalizing its scalar coefficient vector shows $F_1(B^{(d)}) > 0$.

For all sufficiently large d , the intersection of the required events has positive probability. Select a realization in that intersection to obtain (7). Finally let $s \rightarrow \infty$ and $\theta \downarrow 0$ along a sequence. The capacities tend to one, and (6) together with the determinant upper bound gives $F_\infty(B) \rightarrow e^{-1}$. $\square$

Proposition 15 (No asymptotic gain from amplification at fixed s). *For every fixed $s \geq 1$, the Gaussian family satisfies*

$$F_1(A^{(d)}) \rightarrow e^{-1}, \quad F_\infty(A^{(d)}) \rightarrow e^{-1} \quad \text{in probability as } d \rightarrow \infty.$$

The same statements hold for the Gram-normalized tuples $B^{(d)}$. Consequently $F_\infty/F_1 \rightarrow 1$ in probability for both families. This statement does not require the capacity comparison (6).

Proof. Lemmas 10 and 14, together with Proposition 5 and (11), already give the upper bound in probability for every fixed s . For a matching scalar lower bound, use

$$F_1(A^{(d)}) \geq \left| \det A_1^{(d)} \right|^{2/d}.$$

We recall the elementary complex Gaussian determinant calculation underlying the complex Wishart formula of Goodman [5]. Apply Gram–Schmidt to the independent columns of $\sqrt{d} A_1^{(d)}$. Conditional on the preceding columns, the squared norm of the orthogonal residual is a sum of r independent unit-mean exponential variables, where $r = d, d-1, \dots, 1$. Its conditional distribution does not depend on the preceding columns. The residual squared norms are therefore independent, and

$$\left| \det A_1^{(d)} \right|^2 \stackrel{\text{law}}{=} d^{-d} \prod_{r=1}^d Z_r, \quad Z_r \text{ independent with law } \text{Gamma}(r, 1).$$

Differentiating $\mathbb{E} Z_r^u = \Gamma(r+u)/\Gamma(r)$ at zero gives $\mathbb{E} \log Z_r = \psi(r) = H_{r-1} - \gamma_E$ and $\text{Var}(\log Z_r) = \psi_1(r) = \sum_{k=r}^{\infty} k^{-2}$. Here $H_d = \sum_{k=1}^d k^{-1}$, $H_0 = 0$, and γ_E is Euler’s constant. Hence, for $W_d = d^{-1} \log \left| \det A_1^{(d)} \right|^2$,

$$\begin{aligned} \mathbb{E} W_d &= H_d - 1 - \gamma_E - \log d \rightarrow -1, \\ \text{Var}(W_d) &= \frac{1}{d^2} \left(H_d + d \sum_{k>d} k^{-2} \right) \leq \frac{H_d + 1}{d^2} \rightarrow 0. \end{aligned}$$

Thus $\left| \det A_1^{(d)} \right|^{2/d} \rightarrow e^{-1}$ in probability.

For the normalized tuple let $z = G_d^{1/2} e_1$. The coefficient convention (30) gives $\sum_j B_j^{(d)} z_j = A_1^{(d)}$ and $\|z\|_{\ell^2}^2 = (G_d)_{11} \rightarrow 1$. Normalizing this scalar coefficient vector proves

$$F_1(B^{(d)}) \geq \frac{\left| \det A_1^{(d)} \right|^{2/d}}{(G_d)_{11}}.$$

The common upper bound, these lower bounds, and $F_1 \leq F_\infty$ prove all claims, including the ratio limit. On the event that G_d is singular, define $B^{(d)}$ arbitrarily; the probability of this event tends to zero. $\square$

6 Scope of the construction

The order of choices is part of the proof. For prescribed accuracy one first fixes a sufficiently large, finite number s of coefficients, fixes the logarithmic regularization and its polynomial approximation, and only then increases d . The auxiliary size m is unrestricted throughout. After selecting a finite tuple one may increase s to improve the ratio. No convergence estimate uniform in a simultaneously growing s and d is needed.

The result settles the unrestricted comparison constant, including the identity-Gram class. It does not assert that a finite tuple attains the ratio e , that the number of coefficients can be held at a prescribed small value while approaching e , or that the tuples have Hermitian coefficients. Gram normalization gives $\mathcal{C} \leq 1$ and capacities converging to one, rather than an identity proving $\mathcal{C} = 1$ for each tuple. The argument is an existence construction; it provides neither small explicit matrices nor a quantitative dimension bound in terms of the desired error.

The free value e^{-1} alone would not establish sharpness: larger matrix amplifications could in principle improve it. The balance lemma and uniform finite-moment transfer rule out that improvement for these tuples. Conversely, the near-balanced Gaussian marginals alone would not provide the required capacity estimate. The restriction bound and cut argument supply that independent step.

Development note. This draft was developed with AI assistance, including mathematical checks using reused context. Such checks are not independent peer review. The cited primary results are external inputs; no claim of an exhaustive priority search or historical precedence is made here.

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