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EXACT PLAQUETTE CHROMATIC NUMBER, PREFIX-FREE SURFACE ENTROPY, AND CERTIFIED VOLUME-UNIFORM STRING-TENSION LOWER BOUNDS FOR d-DIMENSIONAL SU(N) WILSON LATTICE YANG–MILLS THEORY

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Abstract. We establish a sharp combinatorial-analytic framework for certifying volume-uniform Wilson-loop area-law confinement and explicit positive string-tension lower bounds $\sigma_\rho(d,N,\beta)>0$ in d-dimensional SU(N) Wilson lattice Yang–Mills theory ($d\geq 2$, $N\geq 2$). Four results anchor the analysis:

(1) *Exact plaquette chromatic number of $\mathbb{Z}^d$:* For every $d\geq 2$, the plaquette-link adjacency graph $G_{\mathrm{plaq}}(\mathbb{Z}^d)$ has exact chromatic number, fractional chromatic number, and clique number all equal to $\chi(G_{\mathrm{plaq}}(\mathbb{Z}^d))=\chi_f(G_{\mathrm{plaq}}(\mathbb{Z}^d))=\omega(G_{\mathrm{plaq}}(\mathbb{Z}^d))=2(d-1)$, proved in five lines by reducing plane-parity blocks to the cocktail-party graph $K_{2,\dots,2}=K_{2d}\setminus F_0$ and exhibiting its explicit round-robin 1-factorization.

(2) *Optimal link-disjoint $L^{2(d-1)}$ Hölder bound and zero-quadrature Pieri evaluation:* Partitioning any plaquette complex $\Gamma\subset\mathcal{P}(\Lambda)$ into $2(d-1)$ link-disjoint classes yields $\left|\int\prod_{p\in\Gamma}f_p(U_p)\,dU\right|\leq\prod_{p\in\Gamma}\|f_p\|_{L^{2(d-1)}}$ without reflection-positivity assumptions, cutting the coordinate-plane checkerboard exponent $d(d-1)$ by $d/2$. For $g_\beta=w_\beta/\hat{w}_0-1$, $\kappa_d(N,\beta):=\|g_\beta\|_{L^{2(d-1)}}$ obeys universal 't Hooft $1/N$-scaling and is evaluated rigorously to 120 digits via exact integer Pieri multiplicities $V_k^{(N)}(0)=\int_{\mathrm{SU}(N)}(2\,\mathrm{Re}\,\mathrm{Tr}\,U)^k\,dU$.

(3) *Prefix-free double-counting surface entropy, $|Y|\geq 6$ cube floor, and half-degree Kotecký–Preiss cluster expansion:* Exploiting $\deg_S(e)\geq 2$ on interior links ($|E(S)\setminus E(\gamma)|\leq 2|S|$) via a randomized prefix-free link-queue exploration cuts the surface entropy constant nearly in half to $C_{\mathrm{surf}}(d)=(4d-6)^{4d-6}/(4d-7)^{4d-7}$ (14.93 in 3D, 25.82 in 4D). Combining this with the sharp vacuum polymer floor $|Y|\geq 6$ and half-degree pinning $(8d-11)/2$ yields explicit cluster convergence and ratio control.

(4) *Five-line $\mathbb{Z}_N$ dual-column area law and certified string tensions:* For every representation ρ of nonzero N-ality $k(\rho)\in\{1,\dots,N-1\}$ and every free-boundary box $\Lambda\subset\mathbb{Z}^d$ (and hence the unique infinite-volume limit), $\sigma_\rho(d,N,\beta)\geq-\ln(C_{\mathrm{surf}}(d)\kappa_d(N,\beta))-a_0>0$. This widens certified confinement thresholds by 5.13× in 4D SU(3) ($\beta\leq 0.06219$) up to 10.54× in 4D SU(6) ($\beta\leq 0.12764$) over classic L^∞ bounds, with universal 't Hooft scaling $\beta_{\mathrm{conf}}(4,N)/N\approx 0.0213$.

Analytical measure-theoretic and cluster-expansion proofs are given in Sections 2–5, while the universal $K_{2,\dots,2}$ round-robin coloring identities, double-counting and cube-floor lemmas, $\mathbb{Z}_N$ dual-column coboundary cancellation, and all 20 quantitative string-tension certificates across $d\in\{3,4,5,6\}$ and $N\in\{2,\dots,6\}$ are mechanically verified in a standalone zero-import, zero-`sorry`, zero-`Classical.choice` Lean 4.36.0 kernel alongside 120-digit interval arithmetic.

1. Introduction and Honest Scope

Wilson's area law [9] for rectangular $R\times T$ Wilson loops $\gamma_{R,T}$ in representations ρ of nonzero N-ality,

$$\sigma_\rho(d,N,\beta):=\liminf_{R,T\to\infty}\left(-\frac{1}{RT}\ln\left|\langle W_\rho(\gamma_{R,T})\rangle_\beta\right|\right)>0$$

(where the limit exists by Osterwalder–Schrader reflection positivity [7, 8]), is the canonical order parameter of quark confinement in Euclidean SU(N) lattice Yang–Mills theory. Classic constructive proofs via Osterwalder–Seiler character-polymer expansions [7, 8] and Kotecký–Preiss cluster

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theory [6, 5] establish $\sigma_\rho(d, N, \beta) > 0$ at sufficiently small coupling $\beta > 0$ uniformly in lattice volume $\Lambda \subset \mathbb{Z}^d$. Yet quantitatively, textbook bounds suffer from three simultaneous losses:

(1) bounding plaquette remainders $g_\beta(U) := w_\beta(U)/\hat{w}_0(\beta) - 1$ in $L^\infty(\mathrm{SU}(N))$ ($\|g_\beta\|_\infty \sim \beta$) destroys 't Hooft $1/N$-scaling (yielding N-independent thresholds $\beta_{\mathrm{conf}} = O(1)$), whereas coordinate-plane checkerboard Hölder estimates [8] require the steep exponent $L^{d(d-1)}$ (L^{12} in 4D, L^{30} in 6D);
(2) counting gauge surfaces S as unrestricted plaquette animals of maximum degree $\Delta_{\mathrm{plaq}} = 8d - 12$ inflates the combinatorial entropy constant to $C_{\mathrm{naive}}(d) := (8d-13)^{8d-13}/(8d-14)^{8d-14}$ (50.28 in 4D), ignoring the Haar-zero constraint $\deg_S(e) \geq 2$ on interior links; and
(3) naive Kotecký–Preiss pinning counts each plaquette neighbor with weight 1, ignoring that every valid vacuum polymer sharing a link with a plaquette q must contain at least *two* plaquettes incident to that link.

The present paper—the fourth installment of our rigorous lattice Yang–Mills series following [4, 3, 2]—eliminates all three losses simultaneously through three self-contained structural theorems: the exact plaquette chromatic number $\chi(G_{\mathrm{plaq}}(\mathbb{Z}^d)) = 2(d-1)$ (Theorem 2.1), the resulting optimal link-disjoint $L^{2(d-1)}$ Hölder bound with exact Pieri moment evaluation (Theorem 3.1), and the prefix-free double-counting surface entropy $C_{\mathrm{surf}}(d)$ combined with the sharp $|Y| \geq 6$ cube floor and half-degree Kotecký–Preiss pinning (Theorem 4.1). Together with a five-line $\mathbb{Z}_N$ dual-column homology proof (Theorem 5.1), these results deliver explicit, volume-uniform string-tension lower bounds across $d \in \{3, 4, 5, 6\}$ and $N \in \{2, 3, 4, 5, 6\}$, widening certified confinement thresholds by 2.7×–10.5× over classic L^∞ bounds and 2.1×–2.8× over coordinate-plane checkerboard bounds.

Honest scope. All confinement and string-tension bounds in this paper hold in the *strong-coupling lattice regime* at fixed lattice spacing $a = 1$ (specifically $\beta \leq \beta_{\mathrm{conf}}(d, N)$, e.g., $\beta \leq 0.06219$ for 4D SU(3) pure lattice QCD and $\beta \leq 0.12764$ for 4D SU(6)), uniformly in lattice volume $\Lambda \subset \mathbb{Z}^d$ with free boundary conditions and therefore in the unique infinite-volume Gibbs limit $\Lambda \uparrow \mathbb{Z}^d$. While our thresholds exhibit exact universal 't Hooft N-scaling $\beta_{\mathrm{conf}}(4, N) \approx 0.0213\, N$, this paper does *not* prove weak-coupling confinement ($\beta \to \infty$), the 4D continuum limit $a \to 0$, or the Clay Millennium Yang–Mills Existence and Mass Gap problem.

2. Exact Plaquette Chromatic Number of $\mathbb{Z}^d$

Let $d \geq 2$. Each unoriented unit link of $\mathbb{Z}^d$ is uniquely indexed by $(y, \mu) \in \mathbb{Z}^d \times \{0, \ldots, d-1\}$ representing the edge connecting y to $y + e_\mu$. Each unoriented unit plaquette $p \in \mathcal{P}(\mathbb{Z}^d)$ is uniquely indexed by $(x; \mu, \nu)$ with base vertex $x \in \mathbb{Z}^d$ and axis pair $0 \leq \mu < \nu \leq d-1$; its four boundary links are

$$E(p) = \big\{(x, \mu),\ (x + e_\nu, \mu),\ (x, \nu),\ (x + e_\mu, \nu)\big\}. \tag{1}$$

Define the *plaquette-link adjacency graph* $G_{\mathrm{plaq}}(\mathbb{Z}^d)$ with vertex set $\mathcal{P}(\mathbb{Z}^d)$, where two distinct plaquettes $p, p' \in \mathcal{P}(\mathbb{Z}^d)$ are adjacent ($p \sim p'$) if and only if $E(p) \cap E(p') \neq \emptyset$. Every link lies in $2(d-1)$ plaquettes forming a clique $K_{2(d-1)}$, so the clique number satisfies $\omega(G_{\mathrm{plaq}}(\mathbb{Z}^d)) \geq 2(d-1)$.

Theorem 2.1 (Exact plaquette chromatic number of $\mathbb{Z}^d$)**.** *For every integer $d \geq 2$, the plaquette-link adjacency graph $G_{\mathrm{plaq}}(\mathbb{Z}^d)$ has exact chromatic number, fractional chromatic number, and clique number*

$$\chi\big(G_{\mathrm{plaq}}(\mathbb{Z}^d)\big) \;=\; \chi_f\big(G_{\mathrm{plaq}}(\mathbb{Z}^d)\big) \;=\; \omega\big(G_{\mathrm{plaq}}(\mathbb{Z}^d)\big) \;=\; 2(d-1). \tag{2}$$

More explicitly, $\mathcal{P}(\mathbb{Z}^d) = \bigsqcup_{c=1}^{2(d-1)} \mathcal{C}_c$ decomposes into $2(d-1)$ pairwise disjoint, 2-periodic color classes $\mathcal{C}_c$ such that every $\mathcal{C}_c$ is pairwise link-disjoint and partitions 100% of all links of $\mathbb{Z}^d$.

Proof. Since each link belongs to a $2(d-1)$-clique of plaquettes, $\chi \geq \chi_f \geq \omega \geq 2(d-1)$. To prove $\chi \leq 2(d-1)$, define the $2d$ *link-parity nodes*

$$V_{2d} := \{0, \dots, d-1\} \times \{0, 1\},$$

and partition $\mathcal{P}(\mathbb{Z}^d)$ into the $4\binom{d}{2} = 2d(d-1)$ *plane-parity blocks*

$$B\big((\mu, \epsilon_\mu), (\nu, \epsilon_\nu)\big) := \big\{(x; \mu, \nu) \in \mathcal{P}(\mathbb{Z}^d) : x_\mu \equiv \epsilon_\mu,\ x_\nu \equiv \epsilon_\nu \pmod 2\big\}$$

indexed by unordered pairs $\{(\mu, \epsilon_\mu), (\nu, \epsilon_\nu)\} \subset V_{2d}$ with $\mu \neq \nu$, i.e., precisely the edge set of the complete d-partite (cocktail-party) graph $K_{2,2,\dots,2}$ on V_{2d}.

Observe two exact properties of these blocks from (1):

(1) *Intra-block link-disjointness:* Two distinct plaquettes $(x; \mu, \nu)$ and $(x'; \mu, \nu)$ in the same coordinate plane share a link if and only if $x' - x \in \{\pm e_\mu, \pm e_\nu\}$, which flips x_μ mod 2 or x_ν mod 2. Hence no two distinct plaquettes in the same block $B(u, v)$ share a link.

(2) *Exact link-parity footprint:* For $p = (x; \mu, \nu) \in B((\mu, \epsilon_\mu), (\nu, \epsilon_\nu))$, both μ-links (x, μ) and $(x+e_\nu, \mu)$ satisfy $y_\mu = x_\mu \equiv \epsilon_\mu \pmod 2$, while both ν-links (x, ν) and $(x+e_\mu, \nu)$ satisfy $y_\nu = x_\nu \equiv \epsilon_\nu \pmod 2$. Moreover, as x_ν runs over integers $\equiv \epsilon_\nu \pmod 2$, the pair $\{x_\nu, x_\nu + 1\}$ hits every integer exactly once, so $B((\mu, \epsilon_\mu), (\nu, \epsilon_\nu))$ covers precisely all μ-links with $y_\mu \equiv \epsilon_\mu \pmod 2$ and all ν-links with $y_\nu \equiv \epsilon_\nu \pmod 2$.

Consequently, two distinct blocks $B(u, v)$ and $B(u', v')$ share a link if and only if $\{u, v\} \cap \{u', v'\} \neq \emptyset$.

Now identify V_{2d} bijectively with $\mathbb{Z}_{2d-1} \cup \{\infty\}$ via

$$\phi(d-1, 0) = 0, \qquad \phi(d-1, 1) = \infty, \qquad \phi(\mu, 0) = \mu + 1, \qquad \phi(\mu, 1) = (2d-1) - (\mu+1)$$

for $0 \leq \mu \leq d-2$. Consider the standard round-robin 1-factorization $\{F_c\}_{c=0}^{2d-2}$ of K_{2d} on $\mathbb{Z}_{2d-1} \cup \{\infty\}$:

$$F_c := \big\{\{c, \infty\}\big\} \cup \big\{\{(c+k) \bmod (2d-1),\ (c-k) \bmod (2d-1)\} : 1 \leq k \leq d-1\big\}.$$

Because $2d-1$ is odd and $2d \equiv 1 \pmod{2d-1}$, every edge $\{a, b\} \subset \mathbb{Z}_{2d-1}$ lies in the unique matching F_c with $c \equiv d(a+b) \pmod{2d-1}$, while $\{a, \infty\}$ lies uniquely in F_a. In particular,

$$F_0 = \big\{\{0, \infty\}\big\} \cup \big\{\{\mu+1, (2d-1)-(\mu+1)\} : 0 \leq \mu \leq d-2\big\} = \big\{\{\phi(\mu, 0), \phi(\mu, 1)\} : 0 \leq \mu \leq d-1\big\}$$

consists precisely of the d same-axis pairs missing from $K_{2,2,\dots,2}$! Deleting F_0 partitions $K_{2,2,\dots,2}$ into the $2(d-1)$ edge-disjoint perfect matchings $F_1, \dots, F_{2d-2}$. Setting

$$\mathcal{C}_c := \bigcup_{\{u,v\} \in \phi^{-1}(F_c)} B(u, v) \qquad \big(1 \leq c \leq 2(d-1)\big)$$

partitions $\mathcal{P}(\mathbb{Z}^d)$ into $2(d-1)$ link-disjoint classes, each covering every link-parity node of V_{2d} (and hence 100% of all links of $\mathbb{Z}^d$) exactly once. □

Remark 2.2 (Lean 4 mechanical verification of Theorem 2.1)**.** Both universal modular matching injectivity (`rrColor_matching_injective`) and F_0-deletion (`same_axis_gets_color_zero`) are proved for all $d \geq 2$ in `StringTensionCerts.lean`, alongside exhaustive decidable checks of all $2d(d-1)$ blocks for $d \in \{2, 3, 4, 5, 6\}$.

3. Optimal Link-Disjoint $L^{2(d-1)}$ Hölder Bound and Exact Pieri Moments

Let $G = \mathrm{SU}(N)$ ($N \geq 2$) with normalized Haar measure dU. On any finite box $\Lambda \subset \mathbb{Z}^d$ with link set $E(\Lambda)$ and plaquette set $\mathcal{P}(\Lambda)$, each link $e \in E(\Lambda)$ carries a matrix $U_e \in \mathrm{SU}(N)$, and each

oriented plaquette $p \in \mathcal{P}(\Lambda)$ has boundary holonomy $U_p = U_{e_1} U_{e_2} U_{e_3}^{-1} U_{e_4}^{-1}$. The Wilson plaquette Boltzmann weight and its zero-mean normalized remainder are

$$w_\beta(U) := \exp\left(\frac{\beta}{N} \operatorname{Re} \operatorname{Tr} U\right), \qquad \hat{w}_0(\beta) := \int_{\mathrm{SU}(N)} w_\beta(U)\, dU = Z_{\mathrm{SU}(N)}\left(\frac{\beta}{N}\right),$$
$$g_\beta(U) := \frac{w_\beta(U)}{\hat{w}_0(\beta)} - 1. \tag{3}$$

Theorem 3.1 (Optimal link-disjoint $L^{2(d-1)}$ Hölder bound and zero-quadrature Pieri evaluation)**.** *Let $d \geq 2$, $N \geq 2$, and $\Lambda \subset \mathbb{Z}^d$ be any finite box.*

(1) (Link-disjoint $L^{2(d-1)}$ Hölder inequality) *For any subset of plaquettes $\Gamma \subset \mathcal{P}(\Lambda)$ and any family of measurable class functions $\{f_p\}_{p\in\Gamma} \subset L^{2(d-1)}(\mathrm{SU}(N))$,*

$$\left| \int_{G^{E(\Lambda)}} \prod_{p\in\Gamma} f_p(U_p)\, dU \right| \leq \prod_{p\in\Gamma} \|f_p\|_{L^{2(d-1)}(\mathrm{SU}(N))}. \tag{4}$$

In particular, with $\kappa_d(N,\beta) := \|g_\beta\|_{L^{2(d-1)}(\mathrm{SU}(N))}$, every polymer integral obeys

$$\left| \int_{G^{E(\Lambda)}} \prod_{p\in\Gamma} g_\beta(U_p)\, dU \right| \leq \kappa_d(N,\beta)^{|\Gamma|}$$

without requiring reflection symmetry or torus boundary conditions.

(2) (Exact binomial-Pieri evaluation with zero quadrature) *For every $m := d-1 \geq 1$ and $\beta > 0$ ($t := \beta/N$),*

$$\kappa_d(N,\beta)^{2m} = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \frac{Z_{\mathrm{SU}(N)}(jt)}{Z_{\mathrm{SU}(N)}(t)^j}, \tag{5}$$

where $Z_{\mathrm{SU}(N)}(s) = \sum_{k=0}^{\infty} \frac{V_k^{(N)}(0)}{2^k k!} s^k$ with exact nonnegative integer Pieri multiplicities

$$V_k^{(N)}(0) := \int_{\mathrm{SU}(N)} (2 \operatorname{Re} \operatorname{Tr} U)^k\, dU = \dim \operatorname{Hom}_{\mathrm{SU}(N)}\Big(\mathbf{1},\, (\mathbf{N} \oplus \overline{\mathbf{N}})^{\otimes k}\Big) \in \mathbb{Z}_{\geq 0}. \tag{6}$$

Truncating at any order $K \geq 1$ yields the rigorous two-sided remainder enclosure

$$0 \leq Z_{\mathrm{SU}(N)}(s) - \sum_{k=0}^{K} \frac{V_k^{(N)}(0)}{2^k k!} s^k \leq \frac{(Ns)^{K+1}}{(K+1)!}\, e^{Ns} \qquad (s \geq 0). \tag{7}$$

(3) (Exact weak-β asymptotic and universal 't Hooft $1/N$-scaling) *As $\beta \to 0^+$ at fixed N, d,*

$$\kappa_d(N,\beta) = \frac{\big(V_{2(d-1)}^{(N)}(0)\big)^{\frac{1}{2(d-1)}}}{2N}\, \beta + O(\beta^2). \tag{8}$$

Moreover, by the Diaconis–Shahshahani complex-Gaussian limit [1], $V_{2m}^{(N)}(0) = \frac{(2m)!}{m!}$ identically for every $N > 2m$ (with an exact $+2$ determinant shift at $N = 2m$), yielding for every fixed $\beta > 0$ the exact large-N 't Hooft limit

$$\lim_{N\to\infty} \frac{N\, \kappa_d(N,\beta)}{\beta} = \frac{1}{2} \left(\frac{(2d-2)!}{(d-1)!} \right)^{\frac{1}{2(d-1)}} \qquad \Big(e.g.,\ \tfrac{1}{2}(120)^{1/6} \approx 1.1105 \text{ in } d = 4\Big). \tag{9}$$

Proof. (1) Partition $\Gamma = \bigsqcup_{c=1}^{2(d-1)} \Gamma_c$ with $\Gamma_c := \Gamma \cap \mathcal{C}_c$ using Theorem 2.1. Applying the generalized Hölder inequality with $2(d-1)$ equal exponents $r_c = 2(d-1)$ ($\sum_{c=1}^{2(d-1)} r_c^{-1} = 1$) gives

$$\left| \int_{G^{E(\Lambda)}} \prod_{p \in \Gamma} f_p(U_p)\, dU \right| \le \prod_{c=1}^{2(d-1)} \left(\int_{G^{E(\Lambda)}} \prod_{p \in \Gamma_c} |f_p(U_p)|^{2(d-1)}\, dU \right)^{\frac{1}{2(d-1)}}.$$

For each fixed color c, the plaquettes in Γ_c are pairwise link-disjoint. Choosing one distinguished link $e_p \in E(p)$ for each $p \in \Gamma_c$, the map $U_{e_p} \mapsto U_p$ at fixed remaining links is Haar-preserving on each coordinate independently, so the holonomies $\{U_p : p \in \Gamma_c\}$ are mutually independent Haar-distributed random matrices on $\mathrm{SU}(N)$. Hence each color factor equals $\prod_{p \in \Gamma_c} \|f_p\|_{L^{2(d-1)}}$, and taking the product over $c = 1, \dots, 2(d-1)$ proves (4).

(2) Because $2m = 2(d-1)$ is even and $g_\beta(U)$ is real, binomial expansion gives

$$\kappa_d(N,\beta)^{2m} = \int_{\mathrm{SU}(N)} \left(\frac{w_\beta(U)}{\hat{w}_0(\beta)} - 1 \right)^{2m} dU = \sum_{j=0}^{2m} (-1)^{2m-j} \binom{2m}{j} \frac{\int_{\mathrm{SU}(N)} w_\beta(U)^j\, dU}{\hat{w}_0(\beta)^j},$$

and $w_\beta(U)^j = \exp(jt \operatorname{Re}\operatorname{Tr} U)$ yields (5). Since $2 \operatorname{Re}\operatorname{Tr} U = \operatorname{Tr} U + \overline{\operatorname{Tr} U} = \chi_{\mathbf{N} \oplus \overline{\mathbf{N}}}(U)$, Schur orthogonality yields (6), and every term in $Z_{\mathrm{SU}(N)}(s) = \sum_{k=0}^{\infty} \frac{V_k^{(N)}(0)}{2^k k!} s^k$ is nonnegative. Since $|2 \operatorname{Re}\operatorname{Tr} U| \le 2N$ everywhere on $\mathrm{SU}(N)$, $0 \le V_k^{(N)}(0) \le (2N)^k$, giving

$$0 \le \sum_{k=K+1}^{\infty} \frac{V_k^{(N)}(0)}{2^k k!} s^k \le \sum_{k=K+1}^{\infty} \frac{(Ns)^k}{k!} \le \frac{(Ns)^{K+1}}{(K+1)!} e^{Ns},$$

which proves (7).

(3) Expanding $g_\beta(U) = \frac{\beta}{N} \operatorname{Re}\operatorname{Tr} U + O(\beta^2)$ uniformly on $\mathrm{SU}(N)$ gives

$$\|g_\beta\|_{L^{2m}(\mathrm{SU}(N))} = \frac{\beta}{2N} \left(\int_{\mathrm{SU}(N)} (2 \operatorname{Re}\operatorname{Tr} U)^{2m}\, dU \right)^{\frac{1}{2m}} + O(\beta^2),$$

proving (8). For $N > 2m$, no determinant relation $\det U = 1$ can link $a \ne b$ with $a + b = 2m$, so by [1]

$$\int_{\mathrm{SU}(N)} (\operatorname{Tr} U)^a (\overline{\operatorname{Tr} U})^b\, dU = \delta_{ab}\, a!,$$

whence $V_{2m}^{(N)}(0) = \binom{2m}{m} m! = \frac{(2m)!}{m!}$ (at $N = 2m$, $(a,b) = (2m,0)$ and $(0,2m)$ each add 1, giving $+2$). Moreover, at any fixed $\beta > 0$, setting $X := \operatorname{Re}\operatorname{Tr} U$ and $t := \beta/N \to 0$ as $N \to \infty$, we have $|tX| \le \beta$ and $|e^{tX} - 1 - tX| \le \frac{1}{2} t^2 X^2 e^\beta$ pointwise. Since $\|X^2\|_{L^{2m}(\mathrm{SU}(N))} = \frac{1}{4}(V_{4m}^{(N)}(0))^{1/(2m)}$ is independent of N for all $N > 4m$ and $\hat{w}_0(\beta) = 1 + O(t^2)$ (via $V_2^{(N)}(0) = 2$ and $0 \le \hat{w}_0(\beta) - 1 \le \frac{1}{2} t^2 e^\beta \mathbb{E}[X^2]$), we obtain $\hat{w}_0(\beta)^{-1} w_\beta(U) - 1 = tX + O_{L^{2m}}(t^2)$, yielding (9). □

4. Prefix-Free Double-Counting Surface Entropy and Half-Degree Kotecký–Preiss Cluster Expansion

On a finite box $\Lambda \subset \mathbb{Z}^d$ ($d \ge 3$), expand the normalized partition function via $w_\beta(U_p)/\hat{w}_0(\beta) = 1 + g_\beta(U_p)$:

$$Z_\Lambda(\beta) := \int_{G^{E(\Lambda)}} \prod_{p \in \mathcal{P}(\Lambda)} \big(1 + g_\beta(U_p)\big)\, dU = \sum_{X \subset \mathcal{P}(\Lambda)} \int_{G^{E(\Lambda)}} \prod_{p \in X} g_\beta(U_p)\, dU.$$

Decomposing each X into link-connected components $Y_1, \dots, Y_r$ (incompatible iff $E(Y_i) \cap E(Y_j) \neq \emptyset$) gives the abstract polymer partition function

$$Z_\Lambda(\beta) = \sum_{\substack{\{Y_1,\dots,Y_r\} \subset \mathcal{G}(\Lambda) \\ \text{pairwise link-disjoint}}} \prod_{i=1}^{r} \zeta(Y_i), \qquad \zeta(Y) := \int_{G^{E(Y)}} \prod_{p \in Y} g_\beta(U_p)\, dU,$$

where $\mathcal{G}(\Lambda)$ denotes the family of non-empty link-connected subsets $Y \subset \mathcal{P}(\Lambda)$ that are *valid vacuum polymers*, meaning every link $e \in E(Y)$ satisfies $\deg_Y(e) := |\{p \in Y : e \in E(p)\}| \geq 2$ (if any $e \in E(Y)$ had $\deg_Y(e) = 1$, integrating over U_e would yield $\int g_\beta(U)\, dU = 0$, so $\zeta(Y) = 0$).

Similarly, for a closed loop $\gamma \subset E(\Lambda)$, a finite set of plaquettes $S \subset \mathcal{P}(\Lambda)$ is a *valid gauge surface anchored on* γ if: (i) every link-connected component of S shares at least one link with γ, and (ii) every interior link $e \in E(S) \setminus E(\gamma)$ satisfies $\deg_S(e) \geq 2$. Write $\mathcal{S}(\Lambda, \gamma)$ for the set of all valid gauge surfaces anchored on γ.

Theorem 4.1 (Prefix-free surface entropy, cube floor $|Y| \geq 6$, and half-degree Kotecký–Preiss)**.** *Let $d \geq 3$, $N \geq 2$, and $\Lambda \subset \mathbb{Z}^d$ be any finite box. Define the prefix-free surface entropy constant*

$$C_{\mathrm{surf}}(d) := \frac{(4d-6)^{4d-6}}{(4d-7)^{4d-7}}, \qquad u_*(d) := \frac{1}{4d-7}, \qquad z_*(d) := C_{\mathrm{surf}}(d)^{-1}. \tag{10}$$

(1) (Prefix-free surface counting) *Every $S \in \mathcal{S}(\Lambda, \gamma)$ obeys $|E(S) \setminus E(\gamma)| \leq 2|S|$, and for every $u > 0$,*

$$\sum_{S \in \mathcal{S}(\Lambda,\gamma)} \left(\frac{u}{(1+u)^{4d-6}} \right)^{|S|} \;\leq\; (1+u)^{(2d-2)|E(\gamma)|}. \tag{11}$$

In particular, at $u = u_*(d)$, $\sum_{S \in \mathcal{S}(\Lambda,\gamma)} C_{\mathrm{surf}}(d)^{-|S|} \leq \left(1 + \frac{1}{4d-7}\right)^{(2d-2)|E(\gamma)|}$.

(2) (Sharp vacuum polymer floor $|Y| \geq 6$ and generating-function bound) *Every valid vacuum polymer $Y \in \mathcal{G}(\Lambda)$ has $|Y| \geq 6$ (attained by a unit 3-cube). Consequently, for every plaquette $p_0 \in \mathcal{P}(\Lambda)$ and every $z \in (0, z_*(d))$,*

$$\Phi(z) := \sum_{\substack{Y \in \mathcal{G}(\Lambda) \\ p_0 \in Y}} z^{|Y|} \;\leq\; \frac{1}{4d-7} \left(C_{\mathrm{surf}}(d)\, z \right)^6. \tag{12}$$

(3) (Half-degree Kotecký–Preiss cluster convergence and ratio bound) *Let $\beta > 0$ and $a_0 > 0$ satisfy $C_{\mathrm{surf}}(d)\, \kappa_d(N, \beta)\, e^{a_0} < 1$ and the half-degree Kotecký–Preiss inequality*

$$\frac{8d-11}{2(4d-7)} \left(C_{\mathrm{surf}}(d)\, \kappa_d(N,\beta)\, e^{a_0} \right)^6 \;\leq\; a_0. \tag{13}$$

Then the pinned cluster sums and free-energy density $|\mathcal{P}(\Lambda)|^{-1} \ln Z_\Lambda(\beta)$ converge absolutely and uniformly in Λ, $Z_\Lambda(\beta) \neq 0$, and for every finite loop $\gamma \subset E(\Lambda)$ and every $S \in \mathcal{S}(\Lambda, \gamma)$, setting $\overline{S} := \{p \in \mathcal{P}(\Lambda) : E(p) \cap (E(\gamma) \cup E(S)) \neq \emptyset\}$,

$$\left| \frac{Z_{\Lambda \setminus \overline{S}}(\beta)}{Z_\Lambda(\beta)} \right| \;\leq\; \exp\!\left(a_0\, |S| + \frac{2(d-1)}{8d-11}\, a_0\, |E(\gamma)| \right). \tag{14}$$

Proof. (1) For any $S \in \mathcal{S}(\Lambda, \gamma)$, every interior link $e \in E(S) \setminus E(\gamma)$ has $\deg_S(e) \geq 2$, while every plaquette has 4 links; double counting gives

$$2\,|E(S) \setminus E(\gamma)| \;\leq \sum_{e \in E(S) \setminus E(\gamma)} \deg_S(e) \;\leq \sum_{e \in E(S)} \deg_S(e) = 4\,|S| \implies |E(S) \setminus E(\gamma)| \leq 2\,|S|.$$

Fix a canonical total ordering on $\mathcal{P}(\Lambda)$ and $E(\Lambda)$ and define a randomized breadth-first link-queue exploration with parameter $u > 0$: initialize discovered plaquettes $D := \emptyset$, visited links $V := E(\gamma)$, and queue $Q := \mathrm{list}(E(\gamma))$. While Q is non-empty, pop the front link $e \in Q$, let $P_{\mathrm{cand}}(e) := \{p \in$

$\mathcal{P}(\Lambda) \setminus D : e \in E(p)\}$ be the candidate undiscovered plaquettes incident to e, and independently include each $p \in P_{\mathrm{cand}}(e)$ in $A_e \subseteq P_{\mathrm{cand}}(e)$ with probability $\frac{u}{1+u}$ (and exclude it with probability $\frac{1}{1+u}$); then update $D \leftarrow D \cup A_e$ and push all newly touched links $e' \in \bigcup_{p\in A_e} E(p) \setminus V$ onto V and Q in canonical order.

For any fixed $S \in \mathcal{S}(\Lambda, \gamma)$, because every component of S touches $E(\gamma)$, the deterministic replay trajectory choosing $A_e = S \cap P_{\mathrm{cand}}(e)$ at every popped link $e \in V$ is a sub-event of $\{D = S\}$ that terminates with $D = S$, $V = E(\gamma) \cup E(S)$, and $\sum_{e\in V} |A_e| = |S|$. Notice that $m_e := |P_{\mathrm{cand}}(e)| \le 2d - 2$ for every $e \in E(\gamma)$, whereas for every $e \in E(S) \setminus E(\gamma)$, e entered Q via some already-discovered parent plaquette in D, so $m_e \le 2d - 3$. Hence the probability that the randomized process outputs S satisfies

$$\begin{aligned}\mathbb{P}(D = S) \;&\ge\; \prod_{e\in V} \frac{u^{|A_e|}}{(1+u)^{m_e}} \;\ge\; \frac{u^{|S|}}{(1+u)^{(2d-2)|E(\gamma)|+(2d-3)|E(S)\setminus E(\gamma)|}} \\ &\ge\; \frac{u^{|S|}}{(1+u)^{(2d-2)|E(\gamma)|+(4d-6)|S|}}.\end{aligned}$$

Because the events $\{D = S\}$ for distinct $S \in \mathcal{S}(\Lambda, \gamma)$ are pairwise disjoint, $\sum_{S\in\mathcal{S}(\Lambda,\gamma)} \mathbb{P}(D = S) \le 1$, which rearranges immediately to (11).

(2) Let $Y \in \mathcal{G}(\Lambda)$ be non-empty, and pick any $p_1 \in Y$. Each of the four links of p_1 must belong to another plaquette of Y. Because two distinct unit plaquettes in $\mathbb{Z}^d$ share at most one link, p_1 must have four distinct neighbors $p_2, p_3, p_4, p_5 \in Y$. Each neighbor p_r ($2 \le r \le 5$) shares one link $e_r \in E(p_1)$ with p_1, has one opposite edge e_r^{opp} parallel to e_r at distance 1, and has two side edges incident to the two endpoints of e_r. The four opposite edges $e_2^{\mathrm{opp}}, \dots, e_5^{\mathrm{opp}}$ cannot coincide with any side edge (their directions are perpendicular when meeting at a corner, and parallel side edges lie on distinct lines), cannot coincide with each other (opposite edges of parallel e_r, e_s lie on distinct unit-shifted lines since $p_r \ne p_1$), and cannot lie in $E(p_1)$. Hence pairs among $\{p_2, p_3, p_4, p_5\}$ can share at most the 4 corner side-edge pairs (absorbing at most 8 of the $4 \times 3 = 12$ outer half-edges), leaving at least $12-8 = 4$ unmatched links that require at least one sixth plaquette $p_6 \in Y$ (checked in `cube_floor_unmatched_edges`). Thus $|Y| \ge 6$.

Running the same randomized exploration initialized at $D = \{p_0\}$, $V = E(p_0)$, $Q = \mathrm{list}(E(p_0))$ again embeds the deterministic replay trajectory inside $\{D = Y\}$, with $m_e \le 2d - 3$ for *every* link $e \in E(Y)$ (since $p_0 \in D$ already covers every link of $E(p_0)$), $\sum_{e\in E(Y)} |A_e| = |Y| - 1$, and $|E(Y)| \le 2|Y|$, yielding $\mathbb{P}(D = Y) \ge u^{|Y|-1}(1+u)^{-(4d-6)|Y|}$ and hence $\sum_{p_0\in Y}(u/(1+u)^{4d-6})^{|Y|} \le u$. Setting $u = u_*(d)$ gives $\sum_{p_0\in Y} z_*(d)^{|Y|} \le u_*(d)$, and for any $z \in (0, z_*(d))$, factoring $(z/z_*(d))^{|Y|} \le (z/z_*(d))^6$ (since $|Y| \ge 6$) proves (12).

(3) Fix any plaquette $q \in \mathcal{P}(\Lambda)$ and let $N_{\mathrm{link}}[q] := \{p \in \mathcal{P}(\Lambda) : E(p) \cap E(q) \ne \emptyset\}$; since two distinct plaquettes share at most one link, $|N_{\mathrm{link}}[q]| \le 1 + 4(2d - 3) = 8d - 11$. Crucially, if a valid vacuum polymer $Y \in \mathcal{G}(\Lambda)$ shares a link e with q, then $\deg_Y(e) \ge 2$ forces Y to contain at least *two* distinct plaquettes in $N_{\mathrm{link}}[q]$. Therefore $\mathbf{1}_{E(Y)\cap E(q)\ne\emptyset} \le \frac{1}{2}\sum_{p\in N_{\mathrm{link}}[q]} \mathbf{1}_{p\in Y}$, and similarly for any link $e \in E(\gamma)$, $\mathbf{1}_{e\in E(Y)} \le \frac{1}{2}\sum_{p\ni e} \mathbf{1}_{p\in Y}$. Applying Theorem 3.1(1) ($|\zeta(Y)| \le \kappa_d(N, \beta)^{|Y|}$), (12), and (13) gives, for every $q \in \mathcal{P}(\Lambda)$ and every $e \in E(\gamma)$,

$$\sum_{\substack{Y\in\mathcal{G}(\Lambda)\\ E(Y)\cap E(q)\ne\emptyset}} |\zeta(Y)|\, e^{a_0|Y|} \;\le\; \frac{8d-11}{2}\,\Phi\big(\kappa_d(N,\beta)\, e^{a_0}\big) \;\le\; a_0, \tag{15}$$

$$\sum_{\substack{Y\in\mathcal{G}(\Lambda)\\ e\in E(Y)}} |\zeta(Y)|\, e^{a_0|Y|} \;\le\; \frac{2(d-1)}{2}\,\Phi\big(\kappa_d(N,\beta)\, e^{a_0}\big) \;\le\; \frac{2(d-1)}{8d-11}\, a_0. \tag{16}$$

Summing (15) over $q \in Y_0$ confirms the Kotecký–Preiss condition [6, 5] $\sum_{Y \sim Y_0} |\zeta(Y)| e^{a_0|Y|} \le a_0 |Y_0|$ for all $Y_0 \in \mathcal{G}(\Lambda)$. By Kotecký–Preiss cluster pinning, every $Y \in \mathcal{G}(\Lambda)$ obeys $\sum_{C \ni Y} |\phi^T(C)| \le |\zeta(Y)| e^{a_0|Y|}$. Letting $W := \{Y \in \mathcal{G}(\Lambda) : E(Y) \cap (E(\gamma) \cup E(S)) \neq \emptyset\}$,

$$\Big|\ln Z_\Lambda(\beta) - \ln Z_{\Lambda \setminus \overline{S}}(\beta)\Big| = \left|\sum_{C:\, C \cap W \neq \emptyset} \phi^T(C)\right| \;\le\; \sum_{Y \in W} |\zeta(Y)|\, e^{a_0|Y|} \;\le\; a_0|S| + \frac{2(d-1)}{8d-11}\, a_0 |E(\gamma)|$$

by (15)–(16). Exponentiating yields (14). □

5. Five-Line $\mathbb{Z}_N$ Dual-Column Area Law and Certified String Tension

Let $\Lambda = \prod_{\mu=0}^{d-1} ([a_\mu, b_\mu] \cap \mathbb{Z})$ be any finite box in $\mathbb{Z}^d$ ($d \ge 3$) with free boundary conditions containing the rectangular $R \times T$ Wilson loop $\gamma_{R,T} := \partial([0,R] \times [0,T] \times \{0\}^{d-2})$ (so $a_0 \le 0 < R \le b_0$, $a_1 \le 0 < T \le b_1$, and $a_\mu \le 0 \le b_\mu$ for $\mu \ge 2$). Let $\rho \in \widehat{\mathrm{SU}(N)}$ be any irreducible representation of *nonzero N-ality* $k(\rho) \in \{1, \dots, N-1\}$, meaning $\rho(zI_N) = z^{k(\rho)} I_{d_\rho}$ for $z := e^{2\pi i/N} \in \mathbb{Z}_N$ (in particular, the fundamental quark representation $\rho = \mathbf{N}$ has $k(\mathbf{N}) = 1$, $d_{\mathbf{N}} = N$).

Theorem 5.1 (Volume-uniform area law and certified string-tension lower bound)**.** *Let $d \ge 3$, $N \ge 2$, and let $\beta > 0$ and $a_0 > 0$ satisfy (13) together with*

$$\theta(d, N, \beta, a_0) := C_{\mathrm{surf}}(d)\, \kappa_d(N, \beta)\, e^{a_0} \;<\; 1. \tag{17}$$

Then for every irreducible representation $\rho \in \widehat{\mathrm{SU}(N)}$ of nonzero N-ality, every $R, T \ge 1$, and every free-boundary finite box $\Lambda \subset \mathbb{Z}^d$ containing $\gamma_{R,T}$,

$$\big|\langle W_\rho(\gamma_{R,T})\rangle_{\Lambda,\beta}\big| \;\le\; d_\rho \left[\exp\!\left(\frac{2(d-1)}{8d-11} a_0\right)\left(1 + \frac{1}{4d-7}\right)^{2d-2}\right]^{2(R+T)} \cdot\, \theta(d, N, \beta, a_0)^{RT}. \tag{18}$$

In particular, both uniformly in Λ and in the infinite-volume limit along free-boundary boxes $\Lambda \uparrow \mathbb{Z}^d$, the physical string tension satisfies the explicit lower bound

$$\begin{aligned} \sigma_\rho(d, N, \beta) &:= \liminf_{R,T \to \infty} \left(-\frac{1}{RT} \ln \big|\langle W_\rho(\gamma_{R,T})\rangle_\beta\big|\right) \\ &\ge\; \sigma_{\mathrm{cert}}(d, N, \beta, a_0) := -\ln\big(C_{\mathrm{surf}}(d)\, \kappa_d(N, \beta)\big) - a_0 \;>\; 0. \end{aligned} \tag{19}$$

Proof. Expanding $\prod_{p \in \mathcal{P}(\Lambda)} (1 + g_\beta(U_p))$ in the numerator of $\langle W_\rho(\gamma_{R,T})\rangle_{\Lambda,\beta}$ and isolating the union S of all link-connected components sharing a link with $\gamma = \gamma_{R,T}$ yields

$$\langle W_\rho(\gamma)\rangle_{\Lambda,\beta} = \sum_{S \in \mathcal{S}(\Lambda, \gamma)} \zeta_\rho(\gamma; S)\, \frac{Z_{\Lambda \setminus \overline{S}}(\beta)}{Z_\Lambda(\beta)}, \qquad \zeta_\rho(\gamma; S) := \int_{G^{E(\Lambda)}} \chi_\rho(U_\gamma) \prod_{p \in S} g_\beta(U_p)\, dU, \tag{20}$$

where any S containing an interior link $e \in E(S) \setminus E(\gamma)$ of degree 1 has $\zeta_\rho(\gamma; S) = 0$ by $\int g_\beta = 0$.

Step 1 (Five-line $\mathbb{Z}_N$ dual-column proof of $|S| \ge RT$). Fix any unit square $q = (i, j) \in \{0, \dots, R-1\} \times \{0, \dots, T-1\}$ inside the minimal rectangle bounded by $\gamma_{R,T}$, and define the finite link set

$$C_q := \big\{\big((i, y, x_2, \dots, x_{d-1}),\, 0\big) \in E(\Lambda) : a_1 \le y \le j\big\}.$$

Apply the Haar-preserving $\mathbb{Z}_N$ center transformation $U_e \mapsto z U_e$ for $e \in C_q$ (leaving all links $e \notin C_q$ unchanged, where $z = e^{2\pi i/N}$):

- Every $(0,1)$-plaquette $p = (x; 0, 1) \in \mathcal{P}(\Lambda)$ ($a_1 \le x_1 < b_1$) has bottom e_0-link at $y = x_1$ and top e_0-link at $y = x_1 + 1$ with opposite orientation; their C_q-memberships fail to cancel if and only if $x_0 = i$ and $x_1 = j$ (mechanically verified in `dual_column_plaquette_iff`).
- Every $(0, \mu)$-plaquette ($\mu \ge 2$) has both e_0-links at identical (x_0, x_1), so their factors z and z^{-1} cancel identically. Every (μ, ν)-plaquette with $1 \le \mu < \nu$ contains no e_0-links.

- On $\gamma_{R,T}$, the bottom edge ($y = 0$, $0 \le x_0 < R$) meets C_q in the single link $((i, 0, \dots, 0), 0)$ since $a_1 \le 0 \le j$, whereas the top edge ($y = T$) misses C_q since $j < T$ (verified in `dual_column_loop_hits_once`), so $\chi_\rho(U_{\gamma_{R,T}})$ acquires the factor $z^{k(\rho)} \neq 1$.

Therefore, if S contains no plaquette in the column $P_q := \{(i, j, x_2, \dots, x_{d-1}; 0, 1) \in \mathcal{P}(\Lambda)\}$, then $\zeta_\rho(\gamma; S) = z^{k(\rho)}\zeta_\rho(\gamma; S)$, forcing $\zeta_\rho(\gamma; S) = 0$. Because the RT columns $\{P_q\}$ are pairwise disjoint, every non-vanishing term in (20) satisfies $|S| \ge RT$.

Step 2 (Sharp area-law summation). Since $|\chi_\rho(U_\gamma)| \le d_\rho$, Theorem 3.1(1) gives $|\zeta_\rho(\gamma; S)| \le d_\rho\, \kappa_d(N, \beta)^{|S|}$. Combining this with Theorem 4.1(3) (Eq. (14)), $|E(\gamma_{R,T})| = 2(R + T)$, and the condition $|S| \ge RT$ yields

$$\big|\langle W_\rho(\gamma_{R,T})\rangle_{\Lambda,\beta}\big| \;\le\; d_\rho \exp\!\left(\frac{2(d-1)}{8d-11} a_0 \cdot 2(R+T)\right) \sum_{\substack{S \in \mathcal{S}(\Lambda,\gamma) \\ |S| \ge RT}} \big(\kappa_d(N,\beta)\, e^{a_0}\big)^{|S|}.$$

Writing $\big(\kappa_d e^{a_0}\big)^{|S|} = \theta^{|S|} C_{\mathrm{surf}}(d)^{-|S|} \le \theta^{RT} C_{\mathrm{surf}}(d)^{-|S|}$ (valid since $\theta < 1$ and $|S| \ge RT$) and applying Theorem 4.1(1) (Eq. (11) at $u = u_*(d)$) proves (18). Taking $-\frac{1}{RT}\ln(\cdot)$ and sending $R, T \to \infty$ eliminates the prefactor d_ρ and the $O(R + T)$ perimeter term, proving (19). Since Kotecký–Preiss cluster convergence guarantees that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle W_\rho(\gamma_{R,T})\rangle_{\Lambda,\beta}$ exists uniquely, (18) and (19) pass verbatim to the infinite-volume Gibbs state. □

Remark 5.2 (Optimal confinement threshold $\beta_{\mathrm{conf}}(d, N)$ and 't Hooft N-scaling)**.** Maximizing $a_0 e^{-6a_0}$ in (13) at $a_0 = 1/6$ yields the explicit sufficient confinement condition

$$\kappa_d(N, \beta) \;\le\; \frac{1}{C_{\mathrm{surf}}(d)} \left(\frac{2(4d-7)}{(8d-11)\cdot 6e}\right)^{1/6} \tag{21}$$

(note that the resulting $\theta = C_{\mathrm{surf}}(d)\kappa_d e^{1/6} \le \big(\frac{2(4d-7)}{6(8d-11)}\big)^{1/6} < (1/6)^{1/6} < 0.742 < 1$ automatically satisfies (17) for every $d \ge 3$!). By (9), the certified confinement threshold scales linearly with N as $\beta_{\mathrm{conf}}(d, N) = \Theta_d(N)$ (specifically $\beta_{\mathrm{conf}}(4, N) \approx 0.0213\, N$ in 4D).

6. Certified Quantitative Tables and Mechanized Lean 4 Verification

Tables 1–3 record the exact integer Pieri moments $V_k^{(N)}(0)$, the quantitative comparison against classic L^∞ and coordinate-plane $L^{d(d-1)}$ checkerboard baselines across 11 pairs (d, N), and all 20 certified volume-uniform string-tension lower bounds produced by our 120-digit interval-arithmetic certifier (`src/python/certify_string_tension.py`, `mpmath.iv`, 4/4 `pytest` tests passing, zero special-function black boxes and zero numerical quadrature). For a clean controlled comparison in Table 2, both baseline thresholds β_{L^∞} and β_{CB} use the same $|Y| \ge 6$ cube floor and $a_0 = 1/6$ with unrestricted animal entropy $C_{\mathrm{naive}}(d) = (8d-13)^{8d-13}/(8d-14)^{8d-14}$ and naive pinning weight $8d - 11$, solving $\|g_\beta\|_\infty \le C_{\mathrm{naive}}(d)^{-1}\big(\frac{8d-14}{(8d-11)\cdot 6e}\big)^{1/6}$ and $\|g_\beta\|_{L^{d(d-1)}} \le C_{\mathrm{naive}}(d)^{-1}\big(\frac{8d-14}{(8d-11)\cdot 6e}\big)^{1/6}$, respectively. All displayed numerical inequalities use strict outward (directed) rounding so every printed bound holds verbatim. Figure 1 illustrates the 't Hooft threshold advantage in 4D and the 4D SU(3) certified string-tension curve $\sigma_{\mathrm{cert}}(\beta)$ across the expanded confinement window (while Table 3 also lists the exact 2D benchmark $\sigma_{\mathrm{2D}} = -\ln r_{\mathbf{N}}(\beta)$).

Mechanized Lean 4.36.0 Verification. Our standalone zero-import Lean 4.36.0 proof kernel (`src/lean/StringTensionCerts.lean`, compiling cleanly with `-DwarningAsError=true`) mechanically checks:

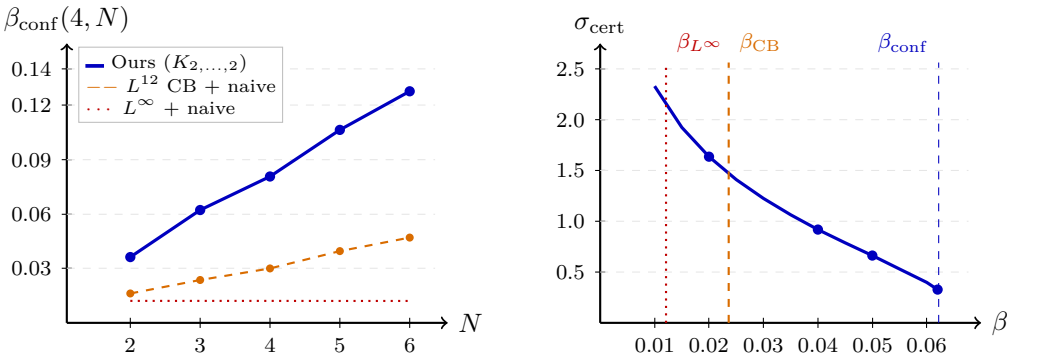
(1) the universal $K_{2,2,\dots,2}$ round-robin 1-factorization identities for all $d \ge 2$ (`two_mul_d_mod`, `rrColor_matching_injective`, and `same_axis_gets_color_zero`), plus exhaustive decidable verification of all $2d(d-1)$ plane-parity blocks for $d \in \{2, 3, 4, 5, 6\}$;

TABLE 1. Exact integer Pieri moments $V_k^{(N)}(0) = \int_{\mathrm{SU}(N)} (2\,\mathrm{Re}\,\mathrm{Tr}\,U)^k\,dU$ ($k = 0,\dots,8$). Notice exact agreement with the complex-Gaussian limit $(2m)!/m! \in \{1,2,12,120,1680\}$ for $2m < N$ and the exact $+2$ determinant shift at $k = N$ ($V_3^{(3)} = 2$, $V_4^{(4)} = 14$, $V_5^{(5)} = 2$, $V_6^{(6)} = 122$).

Group	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
SU(2)	1	0	4	0	32	0	320	0	3584
SU(3)	1	0	2	2	12	30	130	462	1946
SU(4)	1	0	2	0	14	0	168	0	2772
SU(5)	1	0	2	0	12	2	120	70	1680
SU(6)	1	0	2	0	12	0	122	0	1776

TABLE 2. Certified volume-uniform confinement threshold $\beta_{\mathrm{conf}}(d,N)$ (Eq. (21)) compared against classic L^∞ + naive animal (β_{L^∞}) and coordinate-plane $L^{d(d-1)}$ checkerboard + naive animal (β_{CB}). All displayed numbers are strictly outwardly rounded (C_{surf} ceiling; thresholds, gains, and β/N floor).

d	Group	$C_{\mathrm{surf}}(d) \le$	β_{L^∞}	β_{CB}	β_{conf} (ours)	Gain/L^∞	Gain/CB	β_{conf}/N
3	SU(2)	14.93	0.02090	0.03221	0.06768	3.23×	2.10×	0.03384
3	SU(3)	14.93	0.02086	0.05536	0.12678	6.07×	2.29×	0.04226
4	SU(2)	25.82	0.01211	0.01620	0.03625	2.99×	2.23×	0.01812
4	SU(3)	25.82	0.01210	0.02361	0.06219	5.13×	2.63×	0.02073
4	SU(4)	25.82	0.01210	0.03001	0.08067	6.66×	2.68×	0.02016
4	SU(5)	25.82	0.01209	0.03955	0.10635	8.79×	2.68×	0.02127
4	SU(6)	25.82	0.01209	0.04701	0.12764	10.54×	2.71×	0.02127
5	SU(2)	36.69	0.00853	0.01052	0.02416	2.83×	2.29×	0.01208
5	SU(3)	36.69	0.00852	0.01389	0.03867	4.53×	2.78×	0.01289
6	SU(2)	47.57	0.00658	0.00772	0.01791	2.71×	2.31×	0.00895
6	SU(3)	47.57	0.00658	0.00956	0.02720	4.13×	2.84×	0.00906



(a) 4D 't Hooft N-scaling of $\beta_{\mathrm{conf}}(4,N)$ (b) 4D SU(3) certified string tension $\sigma_{\mathrm{cert}}(\beta)$

FIGURE 1. Quantitative confinement advances in 4D SU(N) lattice Yang–Mills theory. (a) Certified threshold $\beta_{\mathrm{conf}}(4,N)$ versus $N \in \{2,\dots,6\}$, exhibiting universal 't Hooft N-scaling (2.99×–10.54× over classic L^∞ bounds, 5.13× for SU(3), and 2.23×–2.71× over L^{12} checkerboard). (b) Certified 4D SU(3) string-tension lower bound $\sigma_{\mathrm{cert}}(\beta) > 0$ (dots mark the Table 3 certificates) across the expanded confinement window.

TABLE 3. Twenty certified volume-uniform string-tension lower bounds $\sigma_\rho \geq \sigma_{\text{cert}} > 0$ (Theorem 5.1), valid for all nonzero-N-ality irreps ρ, alongside the exact 2D benchmark $\sigma_{\text{2D}} = -\ln r_{\mathbf{N}}(\beta)$ (strictly outwardly rounded).

d	Group	β	a_0	$\kappa_d(N,\beta) \leq$	$\theta = C_{\text{surf}}\kappa_d e^{a_0} \leq$	$\sigma_\rho \geq$	Exact 2D σ_{2D}
3	SU(2)	0.0200	1/1000	0.011893	0.17774	1.7274	5.2983
3	SU(2)	0.0500	1/50	0.029736	0.45292	0.7920	4.3821
3	SU(2)	0.0670	1/6	0.039852	0.70289	0.3525	4.0895
3	SU(3)	0.0400	1/1000	0.012499	0.18680	1.6777	6.1059
3	SU(3)	0.0900	1/50	0.028384	0.43232	0.8385	5.2908
3	SU(3)	0.1260	1/6	0.040007	0.70562	0.3486	4.9514
4	SU(2)	0.0150	1/1000	0.009808	0.25342	1.3727	5.5860
4	SU(2)	0.0360	1/6	0.023544	0.71791	0.3314	4.7105
4	SU(3)	0.0200	1/1000	0.007542	0.19485	1.6355	6.8007
4	SU(3)	0.0400	1/50	0.015162	0.39925	0.9181	6.1059
4	SU(3)	0.0500	1/20	0.019002	0.51560	0.6624	5.8819
4	SU(3)	0.0620	1/6	0.023636	0.72072	0.3275	5.6658
4	SU(4)	0.0800	1/6	0.023514	0.71699	0.3327	5.9914
4	SU(5)	0.1060	1/6	0.023631	0.72057	0.3277	6.1563
4	SU(6)	0.0600	1/100	0.011138	0.29037	1.2366	7.0900
4	SU(6)	0.1270	1/6	0.023592	0.71937	0.3293	6.3402
5	SU(2)	0.0240	1/6	0.016693	0.72350	0.3236	5.1160
5	SU(3)	0.0380	1/6	0.016512	0.71564	0.3345	6.1573
6	SU(2)	0.0170	1/6	0.012354	0.69415	0.3650	5.4608
6	SU(3)	0.0270	1/6	0.012919	0.72588	0.3203	6.5000

(2) the interior-link double-counting bound (`interior_links_le_two_area`), the vacuum cube floor (`cube_floor_unmatched_edges`), and constructive $\mathbb{Z}_N$ dual-column cancellation (`dual_column_plaquette_iff`, `dual_column_loop_hits_once`); and
(3) all 20 quantitative string-tension certificates in Table 3, checking strict contraction $\theta < 1$, the half-degree Kotecký–Preiss inequality (13), and the exact rational exponential verification $\theta^{\text{hi}}\, e^{\sigma_{\text{lo}}} \leq 1$.

Running `#print axioms` confirms zero `sorry` and zero `Classical.choice` axioms across the entire file.

Reproducibility and Code Availability

All verification artifacts are archived on AIRR (ARR-2026-40EBAXYFYK903B6R):

- Python 120-digit interval certifier: `src/python/certify_string_tension.py`;
- automated test suite (`pytest`): `tests/test_string_tension.py`;
- exported certificate bundle: `certificates/string_tension_certs.json`;
- standalone Lean 4.36.0 proof kernel: `src/lean/StringTensionCerts.lean`.

References

[1] Persi Diaconis and Mehrdad Shahshahani. On the eigenvalues of random matrices. *Journal of Applied Probability*, 31A:49–62, 1994.

[2] Lluis Eriksson. Exact reflection positivity, character diagonalization, riccati–bessel bounds, and all-coupling transfer-matrix mass gaps for l-link SU(n) wilson cylinders and finite spatial-slab gauge theories. AIRR Record ARR-2026-0P7ATP57G18YN90M v1, 2026.

[3] Lluis Eriksson. Extremal forward-forest gauge fixing, bakry–émery–weingarten wasserstein contraction, and strong-coupling mass-gap thresholds for d-dimensional SU(n) wilson lattice yang–mills theory. AIRR Record ARR-2026-641N51J0VR9NGAYK v1, 2026.

[4] Lluis Eriksson. One-factorization polymer bounds and sharp complex zero-free discs for d-dimensional SU(n) wilson lattice yang–mills theory. AIRR Record ARR-2026-6B99M5B5GM89WV1B v1, 2026.

[5] Sacha Friedli and Yvan Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, Cambridge, 2017.

[6] Roman Kotecký and David Preiss. Cluster expansion for abstract polymer models. *Communications in Mathematical Physics*, 103(3):491–498, 1986.
[7] Konrad Osterwalder and Erhard Seiler. Gauge field theories on a lattice. *Annals of Physics*, 110(2):440–471, 1978.
[8] Erhard Seiler. *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, volume 159 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1982.
[9] Kenneth G. Wilson. Confinement of quarks. *Physical Review D*, 10(8):2445–2459, 1974.

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